Orthomodular lattices with fully nontrivial commutators

MILAN MATOUŠEK

Abstract. An orthomodular lattice $L$ is said to have fully nontrivial commutator if the commutator of any pair $x, y \in L$ is different from zero. In this note we consider the class of all orthomodular lattices with fully nontrivial commutators. We show that this class forms a quasivariety, we describe it in terms of quasiidentities and situate important types of orthomodular lattices (free lattices, Hilbertian lattices, etc.) within this class. We also show that the quasivariety in question is not a variety answering thus the question implicitly posed in [4].

Keywords: orthomodular lattice, commutator, quasivariety

Classification: 06C15, 08C15

0. Preliminaries.

Let us first recall basic notions as we shall use them in the sequel (see e.g. [1], [6], [7], [9], etc.). Let us consider the orthomodular lattices as algebras of the type $\Delta = \{\land, \lor, \perp, 0, 1\}$, where the operations are subject to the standard axioms. Further, let us denote by $OML$ the class of all orthomodular lattices, by $I$ the class of all one-point OMLs, and by $BA$ the class of all Boolean algebras. Thus, $I \subset BA \subset OML$.

Let $V = \{x_1, x_2, x_3, \ldots\}$ be the set of variables. As usual, instead of $x_1, x_2$ we shall sometimes write $x, y$. Further, let $T$ denote the set of all $\Delta$-terms. Supposing $t \in T$, let us denote by $\text{var}(t)$ the set of all variables occurring in $t$. If $n \in \mathbb{N}$ is a natural number, we shall use the notation $T_n = \{t \in T; \text{var}(t) \subseteq \{x_1, x_2, \ldots, x_n\}\}$. For the sake of transparency, let us sometimes write $t(x_1, x_2, \ldots, x_n)$ instead of $t \in T_n$. If $t \in T_n$ and $t_1, t_2, \ldots, t_n \in T$, the formula $t[x_1 \rightarrow t_1, x_2 \rightarrow t_2, \ldots, x_n \rightarrow t_n]$ will denote the term which comes into existence by substituting every $x_i$ in $t$ by $t_i$ $(i = 1, 2, \ldots, n)$.

Suppose that $L \in OML$. Then a mapping $v : V \rightarrow L$ will be called a valuation in $L$. Obviously, the mapping $v$ can be naturally extended over $T$. Thus, we obtain $v_T : T \rightarrow L$. If $t \in T_n$ and $v(x_i) = a_i$ $(i = 1, 2, \ldots, n)$, then the element $v_T(t)$ will be denoted by $t_L(a_1, a_2, \ldots, a_n)$. So, every term $t \in T_n$ determines a mapping $t_L : L^n \rightarrow L$.

The identities are words of the type $s \cong t$, where $s, t \in T$ (the sign $\cong$ stands for the relation which transforms in a given $L \in OML$ to an equality). The
quasiidentities are words of the type \((s_1 \cong t_1 & \ldots & s_n \cong t_n) \Rightarrow s \cong t\), where \(s_1, \ldots, s_n, t_1, \ldots, t_n, s, t \in T\).

The semantic validity will be denoted by the symbol \(\models\). If \(s, t \in T\) and \(OML \models s \cong t\), then we shall write \(s \sim t\). If \(OML \models s \leq t\), then we shall write \(s \leq t\).

If \(\alpha_1\) (resp. \(\alpha_2\)) is an identity or a quasiidentity, we say that \(\alpha_1\) is equivalent to \(\alpha_2\) if the following logical equivalence is valid in any \(L \in OML : L \models \alpha_1 \iff L \models \alpha_2\).

The next simple proposition shows that in \(OML\) we may restrict ourselves to identities and quasiidentities of a very special type.

**Proposition 0.1.**

(a) Every identity is equivalent to an identity of the form \(s \cong 0\).

(b) Every identity is equivalent to a quasiidentity.

(c) Every quasiidentity is equivalent to a quasiidentity of the form \(s \cong 0 \Rightarrow t \cong 0\).

The proof is elementary. \(\square\)

Suppose that \(\Omega\) denotes a set of quasiidentities in \(OML\). Put \(\text{Mod}(\Omega) = \{L \in OML; L \models \alpha\text{ for any }\alpha \in \Omega\}\). Further, if \(A \subseteq OML\) then \(\text{Id}(A)\) will stand for the set of all identities which are valid in any \(L \in A\).

By a variety (resp. by a quasivariety) in \(OML\) we call any class of \(OML\) of the type \(\text{Mod}(\Omega)\), where \(\Omega\) is a set of identities (resp. a set of quasiidentities). A quasivariety which is not a variety will be called a proper quasivariety.

The following statement recalls a famous result of universal algebras (see \([6]\)).

**Proposition 0.2.** Suppose that \(Q\) is a subclass of \(OML\). Then \(Q\) is a quasivariety if and only if it is closed under the formation of subalgebras, products, ultraproducts, isomorphic algebras and it contains a trivial algebra.

A quasivariety \(Q\) is a variety if and only if it is closed under the formation of epimorphic images.

Let us now recall a central notion of our concern in this note. Put

\[ c(x, y) = (x \land y) \lor (x \land y^\perp) \lor (x^\perp \land y) \lor (x^\perp \land y^\perp). \]

Then the term \(c(x, y)\) is called the commutator of \(x, y\).

Let us first observe the following useful property of the commutator (see \([1]\)). (Following the language of the lattice theory, a set \(P \subseteq L\) is called a \(p\)-filter in \(L\) if \(P\) is filter in \(L\) and \(a \in P\) implies \(x \lor (x^\perp \land a) \in P\) for all \(x \in L\) (see \([1, p.182]\), \([7, p.75]\)). A \(p\)-ideal in \(L\) is defined dually.)

**Proposition 0.3.** Suppose that \(L \in OML\). Put

\[ P = \{z \in L; \exists a_1, \ldots, a_n, b_1, \ldots, b_n \in L : \land_{i=1}^n c(a_i, b_i) \leq z\}. \]

Then \(P\) is a \(p\)-filter in \(L\).

The proof is simple. Let us call the latter \(p\)-filter \(P\) the commutator \(p\)-filter in \(L\). \(\square\)
In the conclusion of preliminaries, let us agree to reserve the symbol $F_2$ for the free orthomodular lattice over the set $\{x, y\}$. It should be noted that $F_2$ has 96 elements and that its representation is treated in detail in [1, p. 82]. Obviously, for any term $s \in T_2$ there exists exactly one term $t \in F_2$ such that $s \sim t$.

1. The quasivariety $NC$.

Put

$$NC = \{L \in OML; \forall x, y \in L : c_L(x, y) \neq 0_L\} \cup I,$$

where $I$ is the class of all one-point $OML$s and the indices indicate the belonging to the respective $L$.

We shall now examine the class $NC$. (It should be noted that the notion of commutator has already proved its significance in the study of $OML$s — see e.g. [2], [3], [9], etc. Here we add some universal algebra aspects.)

**Proposition 1.1.**

(a) Every free orthomodular lattice belongs to $NC$.

(b) $BA$ is a proper subclass of $NC$.

**Proof:** (a) Let $F$ be a free orthomodular lattice over a set $M$ ($M \neq \emptyset$). Let $P$ be the commutator $p$-filter in $F$. The factor morphism $\delta : F \to F/P$ is the reflection of $F$ into $BA$ (see [7, p. 299]). Then $F/P$ is a free Boolean algebra over $\delta(M)$ ($\delta(M) \neq \emptyset$). If there were two elements $x, y \in F$ such that $c_F(x, y) = 0_F$, then $P = F$ and therefore $F/P$ is a singleton. This is a contradiction since $F/P$ was a free Boolean algebra. Thus, for all elements $x, y \in F$ we have $c_F(x, y) \neq 0_F$. It follows that $F \in NC$.

(b) Suppose that $B \in BA$ with $\text{card}(B) > 1$. Then for any couple $x, y \in B$ we have $c_B(x, y) = 1_B$. Thus, $BA \subseteq NC$. On the other hand, the free orthomodular lattice $F_2$ belongs to $NC$ but it does not belong to $BA$. □

**Corollary 1.2.** The class $NC$ is a proper quasivariety of $OML$.

**Proof:** By our definition of $NC$, we immediately see that

$$NC = \text{Mod}(\{c(x, y) \cong 0 \Rightarrow 1 \cong 0\}).$$

It follows that $NC$ is a quasivariety. We have to show that $NC$ is not a variety. Take the lattice $MO_2$ of [7] (recall that $MO_2$ is the horizontal sum of two 4-point Boolean algebras). One easily checks that $MO_2 \notin NC$. Since $MO_2$ has two generators, it is a homomorphic image of $F_2$. But $F_2$ belongs to $NC$ and therefore $NC$ is not a variety. □

An important class of $OML$s is formed by set-representable $OML'$s (see e.g. [9]). Since $F_2$ is set-representable, which does not seem to be explicitly known but can be established easily, and since every Boolean algebra is set-representable, Proposition 1.1 naturally suggests whether all set-representable $OML$s belong to $NC$. This is however not the case as the following example shows.
Example 1.3. Let $X$ be a four-point set and let $L$ denote the collection of all subsets of $X$ with an even number of points. Then $L \in OML$ but $L \notin NC$. (The proof is simple.) □

Another type of $OML$s are Hilbertian lattices. Let $H$ be a Hilbert space over reals and let $L(H)$ denote the (orthomodular) lattice of all closed subspaces of $H$. Since the lattice $L(H)$ plays an important role both in the lattice theory and the applications (see [5], [9], etc.) it may be desirable to situate $L(H)$ within the class $NC$.

Theorem 1.4. The lattice $L(H)$ belongs to $NC$ if and only if $\dim(H)$ is a (finite) odd number.

Proof: Suppose first that $\dim H < \infty$ and $\dim H$ is an odd number. Assume that $A, B \in L(H)$. Since $\dim A + \dim A^\perp = \dim H$, we have either $\dim A > \dim A^\perp$ or $\dim A < \dim A^\perp$, and we have the analogous situation for $B$, too. Obviously, $c(A, B) = c(A, B^\perp) = c(A^\perp, B) = c(A^\perp, B^\perp)$ (where commutator is considered in $L(H)$). We have to show that each of the latter commutators is different from 0.

Without any loss of generality, we may assume that $\dim A > \dim A^\perp$ and $\dim B > \dim B^\perp$. Then we obtain $\dim A + \dim B > \dim H$. Since

$$\dim(A \lor B) = \dim A + \dim B - \dim(A \land B) \quad \text{and} \quad \dim(A \lor B) \leq \dim H,$$

we infer that $\dim(A \land B) > 0$. Thus, $A \land B \neq \{0_H\}$ and therefore $c(A, B) \neq 0_{L(H)}$.

Suppose now that $\dim H < \infty$ and $\dim H$ is an even number. Thus, $\dim H = 2n$. Let \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\} be an orthonormal base of $H$. Let $A$ (resp., $B$) be the linear subspace of $H$ generated by $a_1, a_2, \ldots, a_n$ (resp., generated by $a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n$). One can check easily that $A^\perp$ is generated by $b_1, b_2, \ldots, b_n$ and $B^\perp$ is generated by $a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n$. It implies that $A \land B = A^\perp \land B^\perp = \{0_H\}$.

Finally, if $\dim H$ is infinite, we can write $H$ as a direct sum of (infinitely many) copies of $R^2$. Since $L(R^2) \in NC$ and the commutator of $L(H)$ is formed “coordinatewise”, we immediately obtain that $L(H) \notin NC$. This completes the proof of Theorem 1.4. □

2. Special terms in $OML$ (the set $T_{ex}$).

Let us denote by $T_{ex}$ the set of all terms $t(x_1, x_2, \ldots, x_n) (n \geq 1)$ with the following property: Supposing that $f : K \to L$ is a surjective morphism in $OML$, then the equality $t_L(b_1, b_2, \ldots, b_n) = 0_L$ implies the existence of elements $a_1, a_2, \ldots, a_n \in K$ such that $f(a_i) = b_i$ (i = 1, 2, \ldots, n) and $t_K(a_1, a_2, \ldots, a_n) = 0_K$.

The following proposition indicates the significance of the class $T_{ex}$:

Proposition 2.1. Suppose that $s(x_1, x_2, \ldots, x_n) \in T_{ex}$ and suppose that $t(x_1, x_2, \ldots, x_n)$ is an arbitrary term. Put $Q = Mod(\{s \cong 0 \iff t \cong 0\})$. Then $Q$ is variety.

Proof: Since $s \in T_{ex}$, the class $Q$ is closed under the formation of epimorphic images. The rest follows from Proposition 0.2. □
In what follows in this paragraph we shall concentrate on the study of when a term belongs to $\mathcal{T}^{ex}$. We shall succeed in clarifying this question for the case of two variables.

We shall first prove some auxiliary results.

Lemma 2.2.

(a) Suppose that $s, t \in \mathcal{T}$ and suppose that $s \sim t$. If $s \in \mathcal{T}^{ex}$, then $t \in \mathcal{T}^{ex}$.

(b) Suppose that $t(x_1, x_2, \ldots, x_n) \in \mathcal{T}$. Suppose that we are given natural numbers $i_1, i_2, \ldots, i_m$ ($m \in \mathbb{N}$) such that $1 \leq i_1 < i_2 < \ldots < i_m \leq n$. Put

$s = t[x_{i_1} \rightarrow x_{i_1}^{⊥}, x_{i_2} \rightarrow x_{i_2}^{⊥}, \ldots, x_{i_m} \rightarrow x_{i_m}^{⊥}]$. Then $s \in \mathcal{T}^{ex}$ if and only if $t \in \mathcal{T}^{ex}$.

Proof: (a) Evident.

(b) Without any loss of generality, we may suppose that $s = t[x_1 \rightarrow x_1^{⊥}]$ (the general case can be obtained by induction). Suppose first that $t \in \mathcal{T}^{ex}$. Suppose further that we are given an epimorphism $f : K \rightarrow L$ and we have $s_L(b_1, b_2, \ldots, b_n) = 0_L$ for some elements $b_1, b_2, \ldots, b_n \in L$. Then $t_L(b_1^{⊥}, b_2, \ldots, b_n) = s_L(b_1, b_2, \ldots, b_n) = 0_L$. Since $t \in \mathcal{T}^{ex}$, there exist elements $a_1^{⊥}, a_2, \ldots, a_n \in K$ such that $f(a_i) = b_i$ ($i = 1, 2, \ldots, n$) and $t_K(a_1^{⊥}, a_2, \ldots, a_n) = 0_K$. It follows that $s_K(a_1, a_2, \ldots, a_n) = t_K(a_1^{⊥}, a_2, \ldots, a_n) = 0_K$.

Conversely, suppose that $s \in \mathcal{T}^{ex}$. Put $t_1 = s[x_1 \rightarrow x_1^{⊥}]$. According to the argument above, we have $t_1 \in \mathcal{T}^{ex}$. Since $t \sim t_1$, we infer that $t \in \mathcal{T}^{ex}$. This completes the proof of Lemma 2.2. □

Lemma 2.3. Suppose that $s(x_1, x_2, \ldots, x_n) \in \mathcal{T}$. Then the following three conditions are equivalent:

(i) $BA \models s \cong 1$,

(ii) Put

$$c_n = \vee_{\epsilon \in \{1, \perp\}^{\{1, \ldots, n\}}} (x_1^{\epsilon(1)} \land x_2^{\epsilon(2)} \land \ldots \land x_n^{\epsilon(n)})$$

where $x_i^{⊥} = x_i$. Then $OML \models c_n(x_1, x_2, \ldots, x_n) \leq s(x_1, x_2, \ldots, x_n)$.

(iii) If $B_1 = \{0, 1\}$ is a two-point Boolean algebra and $v : \{x_1, x_2, \ldots, x_n\} \rightarrow B_1$ is a mapping, then $s_{B_1}(v(x_1), v(x_2), \ldots, v(x_n)) = 1$.

Proof: (i) $\iff$ (ii) See [1, Ch. 7, Th. 2.19].

(iii) $\Rightarrow$ (i) The assumption (iii) means that $B_1 \models s \cong 1$. It is well known that $Id(B_1) = Id(BA)$. Thus, $s \cong 1 \in Id(BA)$ which gives $BA \models s \cong 1$. □

Let us now recall that the center of an orthomodular lattice $L$, $C(L)$, is the set of all absolutely commutative elements in $L$ (see e.g. [1], [7]). As known, $C(L)$ is a Boolean subalgebra of $L$ (in fact, $C(L)$ is the intersection of all maximal Boolean subalgebras of $L$).
Lemma 2.4. Suppose \( s(x_1, x_2) \in T \). Then there exist a term \( t(x_1, x_2) \in C(F_2) \) such that the following two conditions are satisfied:

(i) \( s(x_1, x_2) \leq t(x_1, x_2) \),
(ii) \( OML \models s(x_1, x_2) \cong 0 \Rightarrow t(x_1, x_2) \cong 0 \).

Proof: All the terms \( u(x_1, x_2) \in F_2 \) such that \( OML \models s(x_1, x_2) \cong 0 \Rightarrow u(x_1, x_2) \cong 0 \), form a \( p \)-ideal, \( I \), in \( F_2 \). As \( F_2 \) has only finite Boolean subalgebras, \( I \) has a maximal element, some \( t(x_1, x_2) \). According to [7, Exercise 8, p. 117], this element belongs to \( C(F_2) \).

We are now ready to state one of the main results of this paper.

Theorem 2.5. Suppose that \( s(x, y) \in T \). Then \( s \in T^{ex} \) if and only if \( BA \not\models s \cong 1 \).

Proof: Suppose first that \( s \in T^{ex} \). Consider a homomorphism of a two-point Boolean algebra \( B_1 \) onto a one-point Boolean algebra \( B_0 \). Then there is a mapping \( v : \{x, y\} \rightarrow \{0, 1\} \) such that \( s_{B_1}(v(x), v(y)) = 0 \). Since \( 0 \neq 1 \), we see that \( BA \not\models s \cong 1 \) (Lemma 2.3).

Suppose now that \( BA \not\models s \cong 1 \). Then there is a valuation \( v : \{x, y\} \rightarrow \{0, 1\} \) such that \( s_{B_1}(v(x), v(y)) = 0 \). Without any loss of generality, let us assume that \( v(x) = v(y) = 0 \). (Obviously, if \( v(x) = 1 \), then we could consider the term \( s_1 = s[x \rightarrow x^\bot] \).)

Suppose that \( t \in C(F_2) \) with \( s \leq t \), \( OML \models s \cong 0 \Rightarrow t \cong 0 \). We are going to show that \( s[x \rightarrow x \land t^\bot, y \rightarrow y \land t^\bot] \sim 0 \).

We have \( t \in C(F_2) \) and therefore \( F_2 = [t] \oplus (t') \). Let us write \( L_1 = [t], L_2 = (t') \). Suppose that \( \pi_i : F_2 \rightarrow L_i \) \((i = 1, 2) \) are the respective projections onto \( L_i \) (thus, \( \pi_1(p) = p \land t, \pi_2(p) = p \land t' \)). Put \( \sigma = s[x \rightarrow x \land t^\bot, y \rightarrow y \land t^\bot] \). We can (and shall) assume that \( \sigma \in F_2 \). We shall now prove that \( \pi_1(\sigma) = \pi_2(\sigma) = 0 \). Making use of \( s(x, y) \leq t(x, y) \), we have

\[ \pi_1(\sigma) = \pi_1(\sigma(x, y)) = \pi_{L_1}(\pi_1(x), \pi_1(y)) = \pi_{L_1}(x \land t, y \land t) = s_{L_1}((x \land t) \land t', (y \land t) \land t^\bot) = s_{L_1}(0, 0) = 0, \]

and

\[ \pi_2(\sigma) = \pi_{L_2}(x \land t', y \land t^\bot) = s_{L_2}((x \land t') \land t^\bot, (y \land t') \land t^\bot) = s_{L_2}(\pi_2(x), \pi_2(y)) = \pi_2(s(x, y)) = s(x, y) \land t^\bot(x, y) = 0, \]

while \( s(x, y) \leq t(x, y) \).

We shall now complete the proof by showing that \( s \in T^{ex} \). Suppose that \( K, L \in OML \) and suppose that \( f : K \rightarrow L \) is a surjective homomorphism. Suppose further that we have \( s_L(b_1, b_2) = 0_L \) for \( b_1, b_2 \in L \). Since \( OML \models s \cong 0 \Rightarrow t \cong 0 \), we obtain \( t_L(b_1, b_2) = 0_L \).

Take elements \( d_1, d_2 \in K \) such that \( f(d_i) = b_i \) \((i = 1, 2) \). Let \( K_1 \) be the orthomodular lattice generated in \( K \) by \( \{d_1, d_2\} \). Then there is an epimorphism \( g : F_2 \rightarrow K_1 \) such that \( g(x) = d_1 \) and \( g(y) = d_2 \). Put \( a_i = d_i \land t_{K_1}^\bot(d_1, d_2) \) \((i = 1, 2) \). Obviously \( a_1, a_2 \in K_1 \). Moreover, we obtain

\[ f(a_i) = f(d_i) \land (f(t_{K_1}^\bot(d_1, d_2))) = b_i \land (t_L(f(d_1), f(d_2))) \] 

\[ = b_i \land (t_L(b_1, b_2)) \land b_i \land \mathbf{1}_L = b_i \land 1_L \] 

Utilizing the above part of the proof, we finally have
Orthomodular lattices with fully nontrivial commutators

\[ s_K(a_1, a_2) = s_K(d_1 \wedge t_1(d_1, d_2), d_2 \wedge t_1(d_1, d_2)) = \sigma_K(d_1, d_2) = g(\sigma) = g(0) = 0_K. \]

This completes the proof of Theorem 2.5.

\[ \square \]

Remark. In the paper [8], R. Mayet obtained the following result which is related to Theorem 2.5: Suppose that \( t(x_1, x_2, \ldots, x_n) \in T \) is a term with the following properties:

(i) the term \( t \) can be constructed from the terms of the set \( \{x_1, x_1^\perp, \ldots, x_n, x_n^\perp\} \) applying only the operation symbols \( \wedge, \vee, \)

(ii) at most one of the terms \( x_i, x_i^\perp (i = 1, 2, \ldots, n) \) is a subterm of \( t \).

Then \( t \in T^{ex} \).

Theorem 2.6. The class \( NC \) is the smallest proper quasivariety in \( OML \) which can be described by means of a single quasiidentity in the variables \( x, y. \)

Proof: Put \( Q = \text{Mod}(s(x, y) \equiv 0 \Rightarrow t(x, y) \equiv 0). \) Suppose that \( Q \) is proper quasivariety. Since \( Q \) is not a variety, we have \( s \not\in T^{ex}. \) By Proposition 2.5 and Lemma 2.3, we infer that \( c(x, y) \leq s(x, y). \) Suppose that \( L \in NC. \) Further, suppose that \( s_L(a, b) = 0_L \) for some elements \( a, b \in L. \) Since \( c \leq s, \) we see that \( c_L(a, b) = 0_L. \) It forces \( 1_L = 0_L \) and therefore \( L \) is a one-point algebra. It implies that \( t_L(a, b) = 0_L. \) Thus, \( L \in Q \) and the proof is complete.

3. Related open questions.

In the conclusion of this note we would like to formulate two open questions.

1. Suppose that \( F \) is a finitely generated orthomodular lattice. Suppose that \( x \in F \) and \( I_x \) is the smallest \( p \)-ideal in \( F \) containing \( x. \) When is \( I_x \) a principal ideal?

2. Can we generalize Theorem 2.5 for the case of the terms with an arbitrary numbers of variables?

References


Department of Logic, Philosophical Faculty of the Charles University, 116 42 Prague, Czechoslovakia

(Received June 17, 1991)