Some results on the product of distributions and the change of variable

EMIN ÖZCAĞ, BRIAN FISHER

Abstract. Let $F$ and $G$ be distributions in $\mathcal{D}'$ and let $f$ be an infinitely differentiable function with $f'(x) > 0$, (or $< 0$). It is proved that if the neutrix product $F \circ G$ exists and equals $H$, then the neutrix product $F(f) \circ G(f)$ exists and equals $H(f)$.

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In the following, we let $N$ be the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^n n : \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as $n$ tends to infinity.

We will use $n$ or $m$ to denote a general term in $N'$ so that if \{$a_n$\} is a sequence of real numbers, then $N - \lim_{n \to \infty} a_n$ means exactly the same thing as $N - \lim_{m \to \infty} a_m$.

Note that if \{$a_n$\} is a sequence of real numbers which converges to $a$ in the normal sense as $n$ tends to infinity, then the sequence \{$a_n$\} converges to $a$ in the neutrix sense as $n$ tends to infinity and

$$\lim_{n \to \infty} a_n = N - \lim_{n \to \infty} a_n$$

We now let $\rho(x)$ be a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) \, dx = 1$.

Putting $\delta_n(x) = n \rho(nx)$ for $n = 1, 2, \ldots$, it follows that \{$\delta_n(x)$\} is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then, if $F$ is an arbitrary distribution in $\mathcal{D}'$, we define

$$F_n(x) = (F * \delta_n)(x) = \langle F(t), \delta_n(x-t) \rangle$$
for \( n = 1, 2, \ldots \). It follows that \( \{F_n(x)\} \) is a regular sequence of infinitely differentiable functions converging to the distribution \( F(x) \).

The following definition for the product of two distributions was given in [2].

**Definition 1.** Let \( F \) and \( G \) be distributions in \( \mathcal{D}' \) and let \( G_n = G * \delta_n \). We say that the neutrix product \( F \circ G \) of \( F \) and \( G \) exists and is equal to the distribution \( H \) on the interval \( (a, b) \) if

\[
\lim_{n \to \infty} \langle F G_n, \phi \rangle = \langle H, \phi \rangle
\]

for all functions \( \phi \) in \( \mathcal{D} \) with support contained in the interval \( (a, b) \). If

\[
\lim_{n \to \infty} \langle F G_n, \phi \rangle = \langle H, \phi \rangle,
\]

we simply say that the product \( F.G \) exists and equals \( H \).

Note that if we put \( F_m = F * \delta_m \), we have

\[
\langle F G_n, \phi \rangle = \lim_{m \to \infty} \langle F_m G_n, \phi \rangle
\]

and so the equation (1) could be replaced by the equation

\[
\lim_{n \to \infty} \left[ \lim_{m \to \infty} \langle F_m G_n, \phi \rangle \right] = \langle H, \phi \rangle.
\]

The next definition for the change of variable in distributions was given in [3].

**Definition 2.** Let \( F \) be a distribution in \( \mathcal{D}' \) and let \( f \) be a locally summable function. We say that the distribution \( F(f(x)) \) exists and is equal to the distribution \( H \) on the interval \( (a, b) \) if

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x)) \phi(x) \, dx = \langle H, \phi \rangle
\]

for all test functions \( \phi \) in \( \mathcal{D} \) with support contained in the interval \( (a, b) \), where

\[
F_n(x) = (F * \delta_n)(x).
\]

The following theorem was proved in [5].

**Theorem 1.** Let \( F \) be a distribution in \( \mathcal{D}' \) and let \( f \) be an infinitely differentiable function with \( f'(x) > 0 \), (or < 0), for all \( x \) in the interval \( (a, b) \). Then the distribution \( F(f(x)) \) exists on the interval \( (a, b) \).

Further, if \( F \) is the \( p \)-th derivative of a locally summable function \( F^{(-p)} \) on the interval \( (f(a), f(b)) \), (or \( f(b), f(a) \), \( g \) inverse of \( f \)), then

\[
\langle F(f(x)), \phi(x) \rangle = (-1)^p \int_{f(a)}^{f(b)} F^{(-p)}(x)[g'(x)\phi(g(x))]^{(p)} \, dx = \langle F^{(-p)}(f(x))f'(x) \left[ \frac{1}{f'(x)} \frac{d}{dx} \right]^p \phi(x), \phi(x) \rangle \, dx
\]
for all $\phi$ in $\mathcal{D}$ with support contained in the interval $(a, b)$.

Using the equation (3), it was proved that if $f$ had a single simple zero at the point $x = x_1$ in the interval $(a, b)$, then

$$
\delta^{(s)}(f(x)) = \frac{1}{|f'(x_1)|} \left[ \frac{1}{f'(x)} \frac{d}{dx} \right]^s \delta(x - x_1)
$$

on the interval $(a, b)$ for $s = 0, 1, 2, \ldots$, showing that the Definition 2 is in agreement with the definition of $\delta^{(s)}(f(x))$ given by Gel’fand and Shilov [6].

The problem of defining the product $F(f) \circ G(g)$ was considered in [4]. Putting $F(f) = F_1$ and $G(g) = G_1$, the product $F_1 \circ G_1 = H_1$ is of course defined by the equation

$$
\mathcal{N} \lim_{n \to \infty} \left[ \mathcal{N} \lim_{m \to \infty} \langle F_{1m}G_{1n}, \phi \rangle \right] = \langle H_1, \phi \rangle,
$$

for all $\phi$ in $\mathcal{D}$, where $F_{1m} = F_1 \ast \delta_m$ and $G_{1n} = G_1 \ast \delta_n$.

However, it was pointed out that since the distributions $F(f)$ and $G(g)$ were defined by the sequences $\{F_m\}$ and $\{G_n\}$, the product $F(f) \circ G(g)$ should be defined by these sequences, leading to the following definition.

**Definition 3.** Let $F$ and $G$ be distributions in $\mathcal{D}'$, let $f$ and $g$ be locally summable functions and let $F_m = F \ast \delta_m$ and $G_n = G \ast \delta_n$. We say that the neutrix product $F(f) \circ G(g)$ of $F(f)$ and $G(g)$ exists and is equal to the distribution $H$ on the interval $(a, b)$ if $F_m(f)G_n(g)$ is a locally summable function on the interval $(a, b)$ and

$$
\mathcal{N} \lim_{n \to \infty} \left[ \mathcal{N} \lim_{m \to \infty} \langle F_m(f)G_n(g), \phi \rangle \right] = \langle H_1, \phi \rangle,
$$

for all $\phi$ in $\mathcal{D}$ with support contained in the interval $(a, b)$.

The following two examples were given in [4] and show that the neutrix product $F(f) \circ G(g)$ can be equal to, but is not necessarily equal to the neutrix product $F_1 \circ G_1$.

**Example 1.** Let $F = x_+^{1/2}$, $G = \delta'(x)$, $f = x_+^2$ and $g = x_+$. Then

$$
F(f) = F_1 = x_+, \quad G(g) = G_1 = \frac{1}{2} \delta'(x)
$$

and

$$
F(f) \circ G(g) = -\frac{1}{2} \delta(x) = F_1 \circ G_1.
$$

**Example 2.** Let $F = x_+^{-1/2}$, $G = \delta(x)$, $f = x$ and $g = x_+^{1/2}$. Then

$$
F(f) = F_1 = x_+^{-1/2}, \quad G(g) = G_1 = 0
$$

and

$$
F(f) \circ G(g) = \delta(x) \neq 0 = F_1 \circ G_1.
$$

The following theorem was, however, proved in [4].
**Theorem 2.** Let $F$ and $G$ be distributions in $\mathcal{D}'$, let $f$ be a locally summable function and let $g$ be an infinitely differentiable function. If the distributions $F(f) = F_1$ and $G(g) = G_1$ exist and the neutrix product $F(f) \circ G(g)$ exists on the interval $(a, b)$, then 

$$F(f) \circ G(g) = F_1 \circ G_1$$

on the interval $(a, b)$. In particular, if $g(x) = x$, then

$$F(f) \circ G(g) = F_1 \circ G_1$$

on the interval $(a, b)$.

In this theorem, $F_1 \circ G(g)$ was used to denote the distribution defined by

$$\lim_{n \to \infty} \langle F_1 G_n, (g), \phi \rangle.$$

We now prove the following theorem.

**Theorem 3.** Let $F$ and $G$ be distributions in $\mathcal{D}'$ and let $f$ be an infinitely differentiable function with $f'(x) > 0$, (or $< 0$), for all $x$ in the interval $(a, b)$. If the neutrix product $F \circ G$ exists and is equal to $H$ on the interval $(f(a), f(b))$, (or $(f(b), f(a))$), then

$$F(f) \circ G(f) = H(f)$$

on the interval $(a, b)$.

**Proof:** Note first of all that the distributions $F(f)$ and $G(f)$ exist on the interval $(f(a), f(b))$, (or $(f(b), f(a))$), by Theorem 1.

We will suppose that $f'(x) > 0$ and that $g$ is the inverse of $f$ on the interval $(a, b)$. Letting $\phi$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $(a, b)$ and making the substitution $t = f(x)$, we have

$$\int_{-\infty}^{\infty} F_m(f(x))G_n(f(x))\phi(x) \, dx = \int_{-\infty}^{\infty} F_m(t)G_n(t)\phi(g(t))g'(t) \, dt = \int_{-\infty}^{\infty} F_m(t)G_n(t)\psi(t) \, dt,$$

where $\psi(t) = \phi(g(t))g'(t)$ is a function in $\mathcal{D}$ with support contained in the interval $(f(a), f(b))$. It follows that

$$\lim_{n \to \infty} \left[ \lim_{m \to \infty} \langle F_m(f)G_n(f), \phi \rangle \right] = \langle H, \psi \rangle$$

for all $\phi$ or $\psi$.

Further, on making the substitution $t = f(x)$, we have

$$\int_{-\infty}^{\infty} H_n(t)\psi(t) \, dt = \int_{-\infty}^{\infty} H_n(t)\phi(g(t))g'(t) \, dt = \int_{-\infty}^{\infty} H_n(f(x))\phi(x) \, dx$$

and so

$$\lim_{n \to \infty} \langle H_n, \psi \rangle = \langle H(f), \phi \rangle.$$

The result of the theorem follows. \qed
Theorem 4. Let $F$ and $G$ be distributions in $\mathcal{D}'$ and let $f$ be an infinitely differentiable function with $f'(x) > 0$, (or $< 0$), for all $x$ in the interval $(a,b)$. If the neutrix products $F \circ G$ and $F \circ G'$, (or $F' \circ G$), exist on the interval $(f(a), f(b))$, (or $(f(b), f(a))$), then
\[ [F(f) \circ G(f)]' = [F(f)]' \circ G(f) + F(f) \circ [G(f)]' \]
on the interval $(a,b)$.

Proof: The usual law
\[ (F \circ G)' = F' \circ G + F \circ G' \]
for the differentiation of a product holds, see [2], and so the result of the theorem follows immediately from Theorem 3. □

Theorem 5. Let $f$ be an infinitely differentiable function with $f'(x) > 0$, (or $< 0$), for all $x$ in the interval $(a,b)$ and having a simple zero at the point $x = x_1$ in the interval $(a,b)$. Then the neutrix products $(f(x))^r_+ \circ \delta(s)(f(x))$ and $\delta(s)(f(x)) \circ (f(x))^r_+$ exist and
\[ (f(x))^r_+ \cdot \delta(s)(f(x)) = \delta(s)(f(x)) \cdot (f(x))^r_+ = 0 \]
for $s = 0, 1, \ldots, r - 1$ and $r = 1, 2, \ldots$ and
\[ (f(x))^r_+ \circ \delta(s)(f(x)) = \delta(s)(f(x)) \circ (f(x))^r_+ = \]
\[ \frac{(-1)^r s!}{2(s-r)!} \frac{1}{|f'(x_1)|} \left[ \frac{1}{f'(x)} \frac{d}{dx} \right]^{s-r} \delta(x - x_1), \]
for $r = 0, 1, \ldots, s$ and $s = r, r + 1, r + 2, \ldots$ on the interval $(a,b)$.

Proof: If $g$ is an $s$ times continuously differentiable function at the origin, then the product $g \cdot \delta(s) = \delta(s) \cdot g$ is given by
\[ g(x) \cdot \delta(s)(x) = \delta(s)(x) \cdot g(x) = \sum_{i=0}^{s} (-1)^{s+i} \binom{s}{i} g^{s-i}(0) \delta(i)(x). \]
It follows that
\[ x^r_+ \cdot \delta(s)(x) = \delta(s)(x) \cdot x^r_+ = 0 \]
for $s = 1, 2, \ldots, r - 1$ and $r = 1, 2, \ldots$ and the equation (6) follows immediately on using Theorem 3. It was proved in [2] that
\[ x^r_+ \circ \delta(s)(x) = \delta(s)(x) \circ x^r_+ = \frac{(-1)^r s!}{2(s-r)!} \delta(s-r)(x), \]
for $r, s = 0, 1, 2, \ldots, s \geq r$, and it follows on using Theorem 3 that
\[ (f(x))^r_+ \circ \delta(s)(f(x)) = \delta(s)(f(x)) \circ (f(x))^r_+ = \frac{(-1)^r s!}{2(s-r)!} \delta(s-r)(f(x)), \]
for $r, s = 0, 1, 2, \ldots$. The equation (7) follows immediately on using equation (5). □
Example 3.

\[
(x + x^2)^r_+ \circ \delta(r)(x + x^2) = \delta(r)(x + x^2) \circ (x + x^2)^r_+ = \\
= \frac{1}{2}(-1)^r r! \delta(x + \delta(x + 1)),
\]

\[
(x + x^2)^r_+ \circ \delta(r+1)(x + x^2) = \delta(r+1)(x + x^2) \circ (x + x^2)^r_+ = \\
= \frac{1}{2}(-1)^r (r + 1)! \left[\delta'(x) + 2\delta(x) + \delta'(x + 1) + 2\delta(x + 1)\right]
\]

for \( r = 0, 1, 2, \ldots \) on the real line.

**Proof:** The function \( f(x) = x + x^2 \) has simple zeros at the points \( x = 0, -1 \). It follows from the equations (5) and (7) that

\[
(x + x^2)^r_+ \circ \delta(r)(x + x^2) = \delta(r)(x + x^2) \circ (x + x^2)^r_+ = \\
= \frac{1}{2}(-1)^r r! \delta(x + x^2) = \\
= \frac{1}{2}(-1)^r r! \left[\delta(x) + \delta(x + 1)\right],
\]

proving the equation (8) for \( r = 0, 1, 2, \ldots \).

It again follows from the equations (5) and (7) that

\[
(x + x^2)^r_+ \circ \delta(r+1)(x + x^2) = \delta(r+1)(x + x^2) \circ (x + x^2)^r_+ = \\
= \frac{1}{2}(-1)^r (r + 1)! \left[\delta'(x) + 2\delta(x) + \delta'(x + 1) + 2\delta(x + 1)\right],
\]

proving the equation (9) for \( r = 0, 1, 2, \ldots \). \( \square \)

**Theorem 6.** Let \( f \) be an infinitely differentiable function with \( f'(x) > 0 \), (or \( < 0 \)), for all \( x \) in the interval \((a, b)\) and having a simple zero at the point \( x = x_1 \) in the interval \((a, b)\). Then the neutrix products \((f(x))^{-r} \circ \delta(s)(f(x))\) and \(\delta(s)(f(x)) \circ (f(x))^{-r}\) exist and

\[
(f(x))^{-r} \circ \delta(s)(f(x)) = \frac{(-1)^r s!}{(r + s)! [f'(x_1)]} \left[\frac{1}{f'(x)} \frac{d}{dx}\right]^{r+s} \delta(x - x_1),
\]

\[
\delta(s)(f(x)) \circ (f(x))^{-r} = 0,
\]

for \( r = 1, 2, \ldots \) and \( s = 0, 1, 2, \ldots \) on the interval \((a, b)\).

**Proof:** It was proved in [2] that

\[
x^{-r} \circ \delta(s)(x) = \frac{(-1)^r s!}{(r + s)!} \delta(r+s)(x),
\]

\[
\delta(s)(x) \circ x^{-r} = 0
\]

for \( r = 1, 2, \ldots \) and \( s = 0, 1, 2, \ldots \). Equations (10) and (11) follow immediately as in the proof of Theorem 6. \( \square \)
Example 4.

(12) \((x^2 - 1)^{-1} \circ \delta(x^2 - 1) = -\frac{1}{4} [\delta'(x - 1) + \delta(x - 1) - \delta'(x + 1) + \delta(x + 1)],\)

(13) 
\[
\delta^{(s)}(x^2 - 1) \circ (x^2 - 1)^{-r} = 0,
\]

for \(r = 1, 2, \ldots\) and \(s = 0, 1, 2, \ldots\) on the real line.

**Proof:** The function \(f(x) = x^2 - 1\) has simple zeros at the points \(x = \pm 1\). It follows from the equations (5) and (10) that

\[
(x^2 - 1)^{-1} \circ \delta(x^2 - 1) = -\frac{1}{4x} [\delta'(x - 1) + \delta'(x + 1)] =
\]

\[
= -\frac{1}{4} [\delta'(x - 1) + \delta(x - 1) - \delta'(x + 1) + \delta(x + 1)]
\]

proving equation (12). \(\Box\)

The equation (13) follows immediately from the equations (5) and (11) for \(r = 1, 2, \ldots\) and \(s = 0, 1, 2, \ldots\).

**Theorem 7.** Let \(f\) be an infinitely differentiable function with \(f'(x) > 0\), (or \(< 0\)), for all \(x\) in the interval \((a, b)\) and having a simple zero at the point \(x = x_1\) in the interval \((a, b)\). Then the neutrix products \((f(x))^\lambda_+ \circ (f(x))_{-r}^{- \lambda} - r\) and \((f(x))_{-r}^{- \lambda} \circ (f(x))^\lambda_+\) exist and

\[
(f(x))^\lambda_+ \circ (f(x))_{-r}^{- \lambda} - r = (f(x))_{-r}^{- \lambda} \circ (f(x))^\lambda_+ =
\]

\[
= -\frac{\pi \cosec(\pi \lambda)}{2(r - 1)!} \frac{1}{|f'(x_1)|} \left[ \frac{1}{f'(x_1)} \frac{d}{dx} \right]^{r-1} \delta(x - x_1),
\]

for \(\lambda \neq 0, \pm 1, \pm 2, \ldots\) and \(r = 1, 2, \ldots\) on the interval \((a, b)\).

**Proof:** It was proved in [2] that

\[
x_\lambda^+ \circ x_{-r}^{- \lambda} = x_{-r}^{- \lambda} \circ x_+^\lambda = -\frac{\pi \cosec(\pi \lambda)}{2(r - 1)!} \delta^{(r-1)}(x),
\]

for \(\lambda \neq 0, \pm 1, \pm 2, \ldots\) and \(r = 1, 2, \ldots\). Equation (14) follows immediately as in the proof of Theorem 6. \(\Box\)

**Example 5.** Let \(f(x) = t\) be the inverse of the function \(g(t) = t + t^3 = x\). Then

\[
(f(x))^\lambda_+ \circ (f(x))_{-r}^{- \lambda - 1} = (f(x))_{-r}^{- \lambda - 1} \circ (f(x))^\lambda_+ =
\]

\[
= -\frac{1}{2} \pi \cosec(\pi \lambda) \delta(x),
\]

(16) 
\[
(f(x))^\lambda_+ \circ (f(x))_{-r}^{- \lambda - 2} = (f(x))_{-r}^{- \lambda - 2} \circ (f(x))^\lambda_+ =
\]

\[
= -\frac{1}{2} \pi \cosec(\pi \lambda) [\delta'(x) + \delta(x)].
\]
for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ on the real line.

**Proof:**

$$g'(t) = 1 + 3t^2 > 0$$

for all $t$, it follows that $f'(x) > 0$ for all $x$ and so on using the equation (3) with $p = 1$, we have for all $\phi$ in $D$

$$\langle \delta(f(x)), \phi(x) \rangle = - \int_{-\infty}^{\infty} H(x)d[(1 + 3x^2)\phi(x + x^3)] =$$

$$= - \int_{-\infty}^{\infty} d[(1 + 3x^2)\phi(x + x^3)] = \phi(0).$$

It follows that

$$\delta(f(x)) = \delta(x).$$

Using the equation (3) again with $p = 2$, we have for all $x$ in $D$

$$\langle \delta'(f(x)), \phi(x) \rangle = \int_{0}^{\infty} d[(1 + 3x^2)\phi(x + x^3)]' =$$

$$= - \phi'(0) - \int_{0}^{\infty} d[(1 + 3x^2)\phi(x + x^3)] =$$

$$= - \phi'(0) + \phi(0).$$

It follows that

$$\delta'(f(x)) = \delta'(x) + \delta(x).$$

It now follows from the equations (15) and (17) that

$$(f(x))^\lambda_+ \circ (f(x))_{-\lambda-1} = (f(x))_{-\lambda-1} \circ (f(x))^\lambda_+ =$$

$$= - \frac{1}{2} \pi \csc(\pi \lambda) \delta(f(x)) =$$

$$= - \frac{1}{2} \pi \csc(\pi \lambda) \delta(x),$$

proving the equation (15) for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

It again follows from the equations (14) and (18) that

$$(f(x))^\lambda_+ \circ (f(x))_{-\lambda-2} = (f(x))_{-\lambda-2} \circ (f(x))^\lambda_+ =$$

$$= - \frac{1}{2} \pi \csc(\pi \lambda) \delta'(f(x)) =$$

$$= - \frac{1}{2} \pi \csc(\pi \lambda)[\delta'(x) + \delta(x)],$$

proving the equation (16) for $\lambda \neq 0, \pm 1, \pm 2, \ldots$. \qed
Some results on the product of distributions and the change of variable

References


Department of Mathematics, The University, Leicester, LE1 7RH, Great Britain

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