Fixed points of asymptotically regular mappings in spaces with uniformly normal structure

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Abstract. It is proved that: for every Banach space $X$ which has uniformly normal structure there exists a $k > 1$ with the property: if $A$ is a nonempty bounded closed convex subset of $X$ and $T : A \to A$ is an asymptotically regular mapping such that

$$\liminf_{n \to \infty} \|T^n\| < k,$$

where $\|T\|$ is the Lipschitz constant (norm) of $T$, then $T$ has a fixed point in $A$.

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1. Introduction.

The concept of uniformly normal structure is due to A.A. Gillespie and B.B. Williams [7]. A Banach space $X$ has uniformly normal structure if

$$N(X) = \sup \{ r_A(A) : A \subset X, \text{ convex, diam } A = 1 \} < 1,$$

where

$$r_A(A) = \inf \{ \sup \{ \|x - y\| : y \in A \} : x \in A \}.$$

It was proved in [4], [2] that $N(X) \leq 1 - \delta_X(1)$; thus $\varepsilon_0(X) < 1$ implies uniformly normal structure. In the paper [11] X.T. Yu proved that if $X$ is a uniformly smooth space (or more generally, $\lim_{t \downarrow 0} \rho_X(t)t^{-1} < \frac{1}{2}$), then $X$ has a uniformly normal structure. Also, in [12] it was proved that uniformly normal structure does not necessarily imply that the space has good geometric properties.

The concept of asymptotic regularity is due to F. Browder and V. Petryshyn [1]. A mapping $T : X \to X$ is said to be asymptotically regular if

$$\lim_{n \to \infty} \|T^{n+1}x - T^nx\| = 0$$

for all $x \in X$.

If $T$ is nonexpansive, then $T_\lambda := \lambda \cdot I + (1 - \lambda) \cdot T$ is asymptotically regular for all $0 < \lambda < 1$ (see [6]).
Recently P.K. Lin in \cite{10} has constructed a uniformly asymptotically regular Lipschitzian mapping acting on a weakly compact subset of $l^2$ which has no fixed point.

E.A. Lifshitz (see \cite{5}) associated with each metric space $(M, d)$ a constant $\kappa(M) \geq 1$. Define Lifshitz characteristic $\kappa_0(X)$ to be the infimum of $\kappa(C)$ where $C$ ranges over all nonempty closed bounded convex subsets of the Banach space $X$. D.J. Downing and B. Turett \cite{5} proved the following

**Theorem 1.** Let $X$ be a Banach space.

(1) Then $\varepsilon_0(X) < 1$ if and only if $\kappa_0(X) > 1$.

(2) If $\gamma > 1$ satisfies $\gamma(1 - \delta_X(\gamma^{-1})) = 1$, then $\gamma \leq \kappa_0(X)$.

In \cite{8} the present author proved the following

**Theorem 2.** Let $X$ be a Banach space with the Lifshitz characteristic $\kappa_0(X) > 1$ and let $C$ be a nonempty bounded closed convex subset of $X$. If $T : C \to C$ is an asymptotically regular mapping such that

$$\liminf_{n \to \infty} ||T^n|| < \kappa_0(X),$$

then $T$ has a fixed point in $C$.

2. Main result.

The main result of this paper is interesting in the Banach spaces $X$ which satisfy the conditions: $\varepsilon_0(X) \geq 1$ and $N(X) < 1$ (cf. \cite{3}).

We start with the following

**Lemma 1** \cite{3}. Let $X$ be a Banach space with $N(X) < 1$. Then for every bounded sequence $\{x_n\}$ there exists a point $z \in \text{conv}\{x_n\}$, such that:

(i) $\limsup_{n \to \infty} ||z - x_n|| \leq N(X) \cdot \limsup_{s \to \infty} \{||x_n - x_m|| : n, m \geq s\},$

(ii) for every $y \in X$, $||z - y|| \leq \limsup_{n \to \infty} ||y - x_n||$.

**Lemma 2** \cite{9}. Let $A$ be a nonempty closed convex subset of a Banach space $X$ and let $\{n_i\}$ be an increasing sequence of natural numbers. Assume that $T : A \to A$ is an asymptotically regular mapping such that for some $m \in \mathbb{N}$, $T^m$ is continuous. If

$$\hat{r}(x) = \limsup_{i \to \infty} ||x - T^{n_i}u|| = 0$$

for some $u \in A$ and $x \in A$, then $Tx = x$.

**Theorem 3.** Let $A$ be a nonempty bounded closed convex subset of a Banach space $X$ which has uniformly normal structure, i.e. $N(X) < 1$. If $T : A \to A$ is an asymptotically regular mapping such that

$$\liminf_{n \to \infty} ||T^n|| = k < [N(X)]^{-1/2},$$

then $T$ has a fixed point in $C$. 

In [8] the present author proved the following
then $T$ has a fixed point in $A$.

**Proof:** Let $T: A \to A$ and let $\{n_i\}$ be a sequence of natural numbers such that

$$\liminf_{n \to \infty} \|T^n\| = \lim_{i \to \infty} \|T^{n_i}\| = k < [N(X)]^{-1/2}.$$  

Consider the sequence $\{T^{n_i}x\}$ for an $x \in A$. Let $z(x)$ be a point satisfying Lemma 1 for $\{T^{n_i}x\}$. Let $r(x) = \limsup_{i \to \infty} \|T^{n_i}x - x\|$. By the condition (i) of Lemma 1, we have

$$\limsup_{i \to \infty} \|T^{n_i}x - z\| \leq N(X) \cdot \limsup_{s \to \infty} \{\|T^{n_i}x - T^{n_j}x\| : n_i, n_j \geq s\} \leq N(X) \cdot \limsup_{j \to \infty} (\limsup_{i \to \infty} \|T^{n_i}x - T^{n_j}x\|) \leq N(X) \cdot \limsup_{j \to \infty} \left(\limsup_{i \to \infty} \left(\|T^{n_i}x - T^{n_i+n_j}x\| + \|T^{n_i+n_j}x - T^{n_j}x\|\right)\right) \leq N(X) \cdot \limsup_{j \to \infty} (\limsup_{i \to \infty} \|T^{n_i}\| \cdot \limsup_{j \to \infty} \|x - T^{n_j}x\|) = k \cdot N(X) \cdot \limsup_{j \to \infty} \|x - T^{n_j}x\|.

Moreover, for $i > 1$, we have

$$\|T^{n_i}z - z\| \leq \limsup_{j \to \infty} \|T^{n_i}z - T^{n_j}x\| \leq \limsup_{j \to \infty} \left(\|T^{n_i}z - T^{n_i+n_j}x\| + \|T^{n_i+n_j}x - T^{n_j}x\|\right) \leq \limsup_{j \to \infty} \left(\|T^{n_i}\| \cdot \|z - T^{n_j}x\| + \sum_{v=0}^{n_i-1} \|T^{n_j+v+1}x - T^{n_j+v}x\|\right) \leq \|T^{n_i}\| \cdot \limsup_{j \to \infty} \|z - T^{n_j}x\|.

By (1) and (2)

$$r(z) \leq k^2 \cdot N(X) \cdot r(x) = a \cdot r(x), \quad \text{with} \quad a < 1.$$ 

Define a sequence $\{x_m\}$ in the following way: $x_1$ is an arbitrarily chosen point of $A$, $x_{m+1} = z(x_m)$. Then $\{x_m\}$ is a Cauchy sequence. In fact, we have

$$\|x_{m+1} - x_m\| \leq \|x_{m+1} - T^{n_i}x_m\| + \|T^{n_i}x_m - x_m\| \leq \|x_{m+1} - T^{n_i}x_m\| + r(x_m).$$
Taking the limit superior as \( i \to +\infty \),
\[
\|x_{m+1} - x_m\| \leq \limsup_{i \to \infty} \|x_{m+1} - T^{n_i}x_m\| + r(x_m) \leq k \cdot N(X) \cdot r(x_m) + r(x_m) = [1 + k \cdot N(X)] \cdot r(x_m).
\]
Hence, by (3)
\[
\|x_{m+1} - x_m\| \leq [1 + k \cdot N(X)] \cdot r(x_m) \leq [1 + k \cdot N(X)] \cdot a^m \cdot r(x_1) \to 0
\]
as \( m \to +\infty \). Let \( x_0 = \lim_{m \to \infty} x_m \). Finally
\[
\|x_0 - T^{n_i}x_0\| \leq \|x_0 - x_m\| + \|x_m - T^{n_i}x_m\| + \|T^{n_i}x_m - T^{n_i}x_0\| \leq
\]
\[
(1 + \|T^{n_i}\|) \cdot \|x_0 - x_m\| + \|x_m - T^{n_i}x_m\|.
\]
Taking the limit superior as \( i \to +\infty \) on both sides we get
\[
\limsup_{i \to \infty} \|x_0 - T^{n_i}x_0\| \leq (1 + k) \cdot \|x_0 - x_m\| + a^m \cdot r(x_1) \to 0
\]
as \( m \to +\infty \). Therefore, by Lemma 2, \( T x_0 = x_0 \). \( \square \)

For James spaces \( X_M = (l^2, |\cdot|_M) \), where \( |\cdot|_M = \max\{\|\cdot\|_2, M \cdot \|\cdot\|_\infty\} \), \( M \geq 1 \) we have
1) \( \varepsilon_0(X_M) = \begin{cases} 2 \cdot (M^2 - 1)^{1/2} & \text{for } M < \sqrt{2}, \\ 2 & \text{for } M > \sqrt{2}, \end{cases} \)
and \( \varepsilon_0(X_M) < 1 \) if and only if \( M < \frac{\sqrt{5}}{2} \);
2) for \( 1 \leq M < \frac{\sqrt{5}}{2} \), the condition \( \gamma < [N(X_M)]^{-1/2} \) is weaker than \( \gamma < \gamma_0 \), where \( \gamma_0 \) is the unique solution of \( x(1 - \delta_{X_M}(\frac{1}{x})) = 1 \);
and
\[
N(X_M) = \frac{M}{\sqrt{2}} \text{ for } 1 \leq M \leq \sqrt{2}, [3].
\]
Combining these results we get the following

**Corollary 1.** Let \( A \) be a nonempty bounded closed convex subset of a James space \( X_M, 1 \leq M < \sqrt{2} \). If \( T : A \to A \) is an asymptotically regular mapping such that
\[
\liminf_{n \to \infty} \|T^n\| < \frac{2^{1/4}}{\sqrt{M}},
\]
then \( T \) has a fixed point in \( A \).
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REFERENCES


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