On a class of locally Butler groups

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Abstract. A torsionfree abelian group $B$ is called a Butler group if $B_{\text{ext}}(B,T) = 0$ for any torsion group $T$. It has been shown in [DHR] that under $CH$ any countable pure subgroup of a Butler group of cardinality not exceeding $\aleph_\omega$ is again Butler. The purpose of this note is to show that this property has any Butler group which can be expressed as a smooth union $\bigcup_{\alpha<\mu} B_\alpha$ of pure subgroups $B_\alpha$ having countable typesets.

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All groups in this paper are abelian. If $p$ is a prime and $x$ an element of a torsionfree group $G$ then $h_p^G(x)$ is the $p$-height of $x$ in $G$ and $t^G(x) = t(x)$ is the type of $x$ in $G$. The typeset $t(G)$ of $G$ is the set of types of all non-zero elements of $G$. The corank of a pure subgroup $H$ of $G$ is the rank of $G/H$. If $\Pi$ is a set of primes and $T$ is a torsion group then we say that $T$ is $\Pi$-primary if $T_p = 0$ for all $p \notin \Pi$.

If $S$ is a subset of a torsionfree group $G$, then $\langle S \rangle^G_s$ denotes the pure subgroup generated by $S$. A subgroup $H$ of $G$ is said to be a generalized regular subgroup of $G$ if $G/H$ is torsion and for each rank one pure subgroup $J$ of $G$, $(J/J \cap H)_p = 0$ for almost all primes $p$. A torsionfree group $G$ is said to be locally completely decomposable if, for each prime $p$, the localization $G_p = Z_p \otimes G$ is completely decomposable. For the unexplained terminology and notations see [F1].

A torsionfree group $B$ is said to be a Butler group if $B_{\text{ext}}(B,T) = 0$ for all torsion groups $T$, where $B_{\text{ext}}$ is the subfunctor of $\text{Ext}$ consisting of all balanced-exact extensions. It is known [BS] that this definition coincides with the familiar one if $B$ has finite rank, i.e., a pure subgroup of a completely decomposable group, or, equivalently [B], a torsionfree homomorphic image of a completely decomposable group of finite rank.

Following [FV] we shall call a torsionfree group locally Butler if any its pure subgroup of finite rank is Butler. Dugas [D] proved that any Butler group, the cardinality of which does not exceed $\aleph_1$ is locally Butler. In this paper we are going to generalize this result by showing that the same property has any Butler group $B$ expressible as a smooth union $\bigcup_{\alpha<\mu} B_\alpha$ of pure subgroups $B_\alpha$ with countable typesets. Doing this we also give for this class of groups an affirmative answer concerning the problems (1) and (2) formulated in [A].

Lemma 1. Let $X$ be a subgroup of a torsionfree group $G$ with $G/X$ torsion and $J \leq G$ be of rank one. If $H$ is a subgroup of $G$ such that $(X + J) \cap H/X \cap H$
is \(\Pi\)-primary for some set of primes \(\Pi\), then there is a subgroup \(K\) of \(J\) such that \(J/K\) is \(\Pi\)-primary and \((X + K) \cap H = X \cap H\).

**Proof:** Decompose \(J/X \cap J\) into \(L/X \cap J \oplus K/X \cap J\), where \(L/X \cap J\) is the \(\Pi\)-primary part of the torsion group \(J/X \cap J\). Now consider the homomorphism \(\psi : (X + J) \cap H \to J/X \cap J\) given for \(h = x + j\) by the formula \(\psi h = j + X \cap J\). Obviously, \(\psi\) is well-defined and it naturally induces the monomorphism \(\phi : (X + J) \cap H / X \cap H \to J/X \cap J\). By hypothesis, \(\text{Im} \psi = \text{Im} \phi \leq L/X \cap J\) and so the results follow easily from the inclusion \(\psi((X + K) \cap H) \leq K/X \cap J\). \(\square\)

**Lemma 2.** Let \(H\) be a corank one pure subgroup of a torsionfree group \(G\) with countable typeset. If \(K\) is a generalized regular subgroup of \(H\), then there is a generalized regular subgroup \(L\) of \(G\) such that \(L \cap H = K\).

**Proof:** Obviously, there is an ordinal \(\lambda \leq \omega\) such that \(\{ t^G(g) \mid g \in G \setminus H \} = \{ t_i \mid i < \lambda \}\). For each \(i < \lambda\) take a rank one pure subgroup \(J_i\) of \(G\) such that \(t(J_i) = t_i\) and \(J_i \cap H = 0\). Using the induction, we are going to show that for each \(i < \lambda\) there is a generalized regular subgroup \(K_i\) of \(J_i\) such that \(L_i = K + K_1 + ... + K_i\) meets \(H\) in \(K\).

For \(n = 1\) we have \((K \oplus J_1) \cap H = K \oplus (J_1 \cap H) = K\) and so we can set \(K_1 = J_1\). Assume that for some \(1 < n < \lambda\) the subgroup \(L_{n-1} = K + K_1 + ... + K_{n-1}\) with \(L_{n-1} \cap H = K\) has been defined. Denoting \(X_n = K + ... + K_{n-1} + J_n\) we have \((L_{n-1} + J_n) \cap H / L_{n-1} \cap H = (K + X_n) \cap H / K = K + (X_n \cap H) / K \simeq (X_n \cap H) / X_n \cap K\).

Now \(X_n / X_n \cap H \simeq (X_n + H) / H\) is torsionfree, \(H\) being pure in \(G\), and consequently \(X_n \cap H\) is a finite rank Butler group. Moreover, for \(0 \neq x \in X_n \cap K\), the natural embedding induces the monomorphism \(\langle x \rangle_{X_n \cap H} \langle x \rangle_{X_n \cap K} \to \langle x \rangle^H \langle x \rangle^K\) and so [B1] gives that the factor-group \(X_n \cap H / X_n \cap K\) has a finite number of non-zero primary components, only. A simple application of Lemma 1 gives the existence of \(K_n \leq J_n\) with the desired properties.

Setting \(L = K + \sum_{i < \lambda} K_i = \cup_{i < \lambda} L_i\) we have \(L \cap H = (\cup_{i < \lambda} L_i) \cap H = \cup_{i < \lambda} (L_i \cap H) = K\) and it remains to show that \(L\) is generalized regular in \(G\).

Take \(0 \neq g \in L\) arbitrarily. For \(g \in H\), it is \(g \in L \cap H = K\) and consequently the factor-group \(\langle g \rangle^G / \langle g \rangle^L = \langle g \rangle^H / \langle g \rangle^K\) has a finite number of non-zero primary components, only.

So, let \(g \notin H\). There is \(n < \lambda\) such that \(t^G(g) = t_n = t(J_n)\). Since \(r(G/H) = 1\), we have \(m g = x + h\) for some \(0 \neq m \in \mathbb{Z}, x \in K_n, h \in K, H/K\) being torsion. The set \(\Pi = \{ p \mid h^G_p (mg) > h^G_p(x) \} \cup \{ p \mid p|m \} \cup \{ p \mid (J_n/K_n) \notin \} \cup \{ p \mid (h^H_p / h^K_p) \neq 0\} \cup \{ p \mid (h^L_p / h^G_p) \neq 0\}\) of primes is obviously finite and for each prime \(p \notin \Pi\) we have \(h^G_p(x) = h^G_p(x) > h^G_p(mg)\), therefore \(h^G_p(mg) < h^G_p(h) = h^K_p(h) = h^L_p(h)\) and consequently \(h^G_p(g) = h^L_p(mg) \geq h^L_p(x+h) \geq h^L_p(x) \cap h^L_p(h) \geq h^G_p(mg) = h^G_p(g)\) showing that \(\langle g \rangle^* / \langle g \rangle^*\) is \(\Pi\)-primary and finishing therefore the proof. \(\square\)

**Lemma 3.** Let \(G = \cup_{\alpha < \mu} G_\alpha\) be a smooth union of pure subgroups of a torsionfree group \(G\) where \(\mu\) is a limit ordinal. If, for each \(\alpha < \mu\), \(L_\alpha\) is a generalized regular subgroup of \(G_\alpha\) such that \(L_\alpha \leq L_\beta\) and \(L_\alpha \cap G_0 = L_0\) whenever \(\alpha \leq \beta < \mu\), then \(L = \cup_{\alpha < \mu} L_\alpha\) is a generalized regular subgroup of \(G\) satisfying \(L \cap G_0 = L_0\).
Proof: If $0 \neq g \in L$ is arbitrary, then $g \in L_\alpha$ for some $\alpha < \mu$ and the inclusion $(g)^*_{L_\alpha} \subseteq (g)^*_{L}$ induces the epimorphism $(g)^{G_\alpha}_{*}/(g)^{L_\alpha}_{*} \to (g)^{G}_{*}/(g)^{L}_{*}$, from which the assertion follows easily. □

Theorem 4. Let $G = \cup_{\alpha<\mu} G_\alpha$ be a smooth union of pure subgroups $G_\alpha$ of a torsionfree group $G$ having countable typesets. If $K$ is a generalized regular subgroup of $G_0$ then there is a generalized regular subgroup $L$ of $G$ such that $L \cap G_0 = K$.

Proof: By transfinite induction based on Lemmas 2 and 3. □

Corollary 5. Let $H$ be a pure subgroup of a torsionfree group $G$ with countable typeset. If $K$ is a generalized regular subgroup of $H$ then there exists a generalized regular subgroup $L$ of $G$ such that $L \cap H = K$.

Corollary 6 [D]. Let $H$ be a countable pure subgroup of a torsionfree group $G$ of cardinality $\aleph_1$. If $K$ is a generalized regular subgroup of $H$ then there is a generalized regular subgroup $L$ of $G$ such that $L \cap H = K$.

Now we are prepared to prove the main result giving a partial solution of the problems (1) and (2) stated in [A].

Theorem 7. Let a torsionfree group $G$ be a smooth union $G = \cup_{\alpha<\mu} G_\alpha$ of pure subgroups $G_\alpha$ with countable typesets. The following conditions are equivalent:

(i) $G$ is locally completely decomposable and if $L$ is a generalized regular subgroup of $G$ and $H$ is a pure finite rank subgroup of $G$, then $(H/H \cap L)_p = 0$ for almost all primes $p$;

(ii) $G$ is locally completely decomposable and locally Butler.

Proof: Assume (i) and let $H$ be a rank finite pure subgroup of $G$. There is $\alpha < \mu$ such that $H \leq G_\alpha$ and consequently if $K$ is a generalized regular subgroup of $H$, Corollary 5 gives the existence of a generalized regular subgroup $M$ of $G_\alpha$ with $M \cap H = K$. A simple application of Theorem 4 leads to the existence of a generalized regular subgroup $L$ of $G$ satisfying $L \cap G_\alpha = M$ and hence $L \cap H = K$. By hypothesis $H/H \cap L = H/K$ has only a finite number of non-zero primary components and since $H$ is locally completely decomposable by [F1, Th. 86.6], it is Butler by [B1]. For the converse see [A]. □

Theorem 8. Any Butler group $G$ expressible as a smooth union $G = \cup_{\alpha<\mu} G_\alpha$ of pure subgroups $G_\alpha$ with countable typesets is locally Butler.

Proof: By [A], any Butler group satisfies the condition (i) from Theorem 7. □

Corollary 9. Any Butler group with countable typeset is locally Butler.

Corollary 10 [D]. Any Butler group of cardinality $\aleph_1$ is locally Butler.

References


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