The trace theorem

\[ W_p^{2,1}(\Omega_T) \ni f \mapsto \nabla_x f \in W_p^{1-1/p, 1/2-1/2p}(\partial \Omega_T) \]
revisited

PETER WEIDEMAIER

Abstract. Filling a possible gap in the literature, we give a complete and readable proof of this trace theorem, which also shows that the imbedding constant is uniformly bounded for \( T \downarrow 0 \). The proof is based on a version of Hardy’s inequality (cp. Appendix).

Keywords: trace theory, anisotropic Sobolev spaces

Classification: 46E35

Introduction.

The imbedding theorem described in the title can be found in LADYSHENSKAYA et al. [L/S/U, Chapter II, Lemma 3.4]. However, none of the references cited there seems to contain a complete proof. The theorem is also stated in IL’IN [I, Theorem 8.4]; but there too, no proof is given. Things look even worse, if we ask for the dependence of the imbedding constant \( c(T) \) on the height \( T \) of the space-time cylinder (for small \( T \)). In some applications of this trace theorem to nonlinear problems, one needs \( c(T) \leq c_0 \) for all \( T \) small (cf. WEIDEMAIER [W], particularly the Appendix). However, the formulation in IL’IN [I, Theorem 8.4], exhibits an explosion of \( c(T) \) for \( T \downarrow 0 \). To settle things, we shall give in this note a detailed proof for the imbedding, which also shows the uniformity of \( c(T) \) for \( T \downarrow 0 \).

The paper is organized as follows: in Chapter 1 we deduce an integral representation for \( \nabla_x f \) in terms of \( \partial_t f, \partial_x^2 f \), which is the basis for the estimates in Chapter 2.

Let us fix the notation: \( \Omega_T := \Omega \times (0, T) \) with the typical point \( (x,t) \in \Omega_T \); here \( \Omega \subset \mathbb{R}^n \). The prime characterizes \((n-1)\)-dimensional quantities: thus we write \( x \in \mathbb{R}^n \) as \( x = (x', x_n), x' \in \mathbb{R}^{n-1}; Q^{n-1}_1(a', b') \) is the open parallelepiped \( \prod_{i=1}^{n-1}(a_i, b_i) \), when \( a' = (a_1, \ldots, a_{n-1}), b' = (b_1, \ldots, b_{n-1}); Q^{n-1}_1(\lambda) := Q^{n-1}(-\lambda 1', \lambda 1') \) for \( \lambda \in \mathbb{R} \); here \( 1' := (1, \ldots, 1) \in \mathbb{N}^{n-1}; Q^n_+(\lambda) := Q^{n-1}(\lambda) \times (0, \lambda) \); the superscript ‘ always indicates the deletion of a coordinate (the n-th. one, if not further specified), e.g. \( \hat{y} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \) (\( 1 \leq i \leq n \)) and \( \hat{Q}^{n+1}_+(a, b) := \prod_{i=1}^{n+1}(a_i, b_i) \).

\[ W_p^{2,1}(\Omega_T) := \{ u | \partial_x^\alpha u, \partial_t u \ (\text{distr. sense}) \in L_p(\Omega_T) \ \forall |\alpha| \leq 2 \} \]
with the obvious norm.

I thank Prof. V.A. Solonnikov, Leningrad, for valuable hints.
For a bounded domain \( \Omega \subset \mathbb{R}^n \), \( \partial \Omega \in C^2 \) means that \( \partial \Omega \) is a \( C^2 \)-hypersurface. The spaces \( W_p^{\alpha,\beta}(\partial \Omega_T) \) \( (\alpha, \beta \in (0, 1)) \) are defined as usual, via a partition of unity on \( \partial \Omega \), and using local charts. We use the notation \( c^* \) to emphasize the non-dependence of the constant \( c \) on the quantity \( T \) (for \( T \) small).

1. Integral representation.

Our starting point is an integral representation for \( \partial \mathcal{L} f \) in terms of \( f \): if \( f \) is smooth and defined on \( Q_{n-1}^{-1}(-\lambda, 1') \times [0, 2\lambda] \times [0, 3T] \), then we have (cf. IL’IN/ SOLONNIKOV [I/S, p. 70, (6)] with \( m_i = 0, k_i = l_i \))

\[
\partial \mathcal{L} f(x, t) = \frac{A}{T^r} \int_{Q_n+1(0, Tz)} \cdots \int f((x, t) + y) \Pi(y, T) \, dy + \\
+ \sum_{i=1}^{n+1} B_i \int_0^T v^{-(1+r)} \int_{Q_n+1(0, vz)} \cdots \int f((x, t) + y) \Pi_i(\tilde{y}, v) \partial_i^l \psi_i(y_i, v) \, dy \, dv
\]

for \( (x, t) \in \overline{Q_n^+}(\lambda) \times [0, T] \), \( T \leq T_0(\lambda) \) and \( \nu_j \leq l_j - 1 \), where (cp. [I/S, pp. 69–70])

\[
\Pi(y, T) := \prod_{j=1}^{n+1} \partial_j^{l_j} \chi_j(y_j, T)
\]

\[
\chi_j(y_j, T) := y_j^{l_j-\nu_j-1} \int_{y_j}^{T^{\kappa_j}} (T^{\kappa_j} - s)^{\mu_j} s^{\lambda_j} \, ds,
\]

\[
\Pi_i(\tilde{y}, v) := \prod_{j=1}^{n+1} \partial_j^{l_j} \chi_j(y_j, v),
\]

\[
\psi_i(y_i, v) := (\sum_{j \neq i} y_i^{l_i+\lambda_i-\nu_i} (y_i - y_i)\mu_i
\]

with certain parameters \( \mu_j, \lambda_j \in \mathbb{N} \) and certain \( A, B_i \in \mathbb{R} \); here \( Tz := (T^{\kappa_1}, \ldots, T^{\kappa_{n+1}}) \), \( r := \kappa \cdot (1 + \lambda + \mu) \), \( \frac{1}{r} := (1, \ldots, 1) \in \mathbb{N}^{n+1} \).

In the sequel we fix \( l := (2, \ldots, 2, 1) \in \mathbb{N}^{n+1}, \kappa := (\kappa', \kappa, \kappa_{n+1}) := \frac{1}{2} = (\frac{1}{2}, \ldots, \frac{1}{2}, 1) \) and choose the parameters \( \mu_j, \lambda_j \) so large that \( \partial_j^k \psi_j(y_j, v) \) vanishes for \( y_j = 0, \nu_j = T^{\kappa_j}, 1 \leq k \leq l_j \). Hence, integrating by parts and introducing \( K_i(y, v) := \Pi_i(\tilde{y}, v) \psi_i(y_i, v) (0 \leq y_i \leq v^{\kappa_i}) \), we have shown that

\[
(1.1) \quad \partial \mathcal{L} f(x, t) = \frac{A}{T^r} \int_{Q_n+1(0, Tz)} \cdots \int f((x, t) + y) \Pi(y, T) \, dy + \\
+ \sum_{i=1}^{n+1} \tilde{B}_i \int_0^T v^{-(1+r)} \int_{Q_n+1(0, vz)} \cdots \int \partial_i^{l_i} f((x, t) + y) K_i(y, v) \, dy \, dv.
\]
The kernels \( \Pi, K_i \) in this representation satisfy (uniformly w.r.t. \( y \in Q^{n+1}(0, i\mathbb{Q}) \))

\[
\begin{align*}
|\partial_y^\alpha \Pi(y, v)| & \leq c \cdot v^{r-k(1+\nu+\alpha)} \quad \forall |\alpha| \leq 2 \\
|\partial_{n+1}^s K_i(y, v)| & \leq c \cdot y_n^\varepsilon \cdot v^{r+1-k(1+\nu)-\varepsilon \kappa_n-s} \\
(\partial_{n+1} := \partial_{y_{n+1}}, 0 \leq s \leq 1, 1 \leq i \leq n+1, \varepsilon \in [0, 1]).
\end{align*}
\]

For the proof of these two inequalities, we first note that \( \partial_j^{l_j+\alpha_j} \chi_j(y_j, v) \) is a linear combination of terms of the form \( (v^{\kappa_j} - y_j)^{\rho_1} y_j^{\rho_2} \) with \( \rho_1 + \rho_2 = \mu_j + \lambda_j - \nu_j - \alpha_j \), \( \rho_2 > 0 \) (for \( \lambda_j \) large) and consequently

\[
|\partial_j^{l_j+\alpha_j} \chi_j(y_j, v)| \leq c \cdot y_n^\varepsilon \cdot v^{-\kappa_j(\varepsilon + \alpha_j)} \cdot v^{\kappa_j(\mu_j + \lambda_j - \nu_j)} \quad (0 \leq y_j \leq v^{\kappa_j})
\]

for arbitrary \( \varepsilon \in [0, 1[; \) this implies (for \( 1 \leq k \leq n-1 \))

\[
\begin{align*}
|\partial_{n+1}^k \Pi_k(y, v)| & \leq c \cdot y_n^\varepsilon \cdot v^{-\kappa_n \varepsilon - \kappa_n+1 \cdot s} \cdot v^{\kappa_n(\mu + \lambda - \nu) - \kappa_n \delta_k} \\
|\partial_{n+1}^s \Pi_n(y, v)| & \leq c \cdot v^{-\kappa_n+1 \cdot s} \cdot v^{\kappa_n(\mu + \lambda - \nu) - \kappa_n \delta_n} \\
|\Pi_{n+1}(y, v)| & \leq c \cdot y_n^\varepsilon \cdot v^{-\kappa_n \varepsilon} \cdot v^{\kappa_n(\mu + \lambda - \nu) - \kappa_n+1 \cdot \delta_{n+1}},
\end{align*}
\]

where \( \delta_j := \mu_j + \lambda_j - \nu_j \). The definition of \( \psi_i \) easily implies

\[
\begin{align*}
|\psi_k(y_k, v)| & \leq c \cdot v^{\kappa_n \cdot (l_k + \delta_k)} \\
|\psi_n(y_n, v)| & \leq c \cdot y_n^\varepsilon \cdot v^{-\kappa_n \varepsilon} \cdot v^{\kappa_n \cdot (l_n + \delta_n)} \\
|\partial_{n+1}^s \psi_{n+1}(y_{n+1}, v)| & \leq c \cdot v^{-\kappa_n \cdot (l_{n+1} + \delta_{n+1})} ;
\end{align*}
\]

since \( K_i(y, v) = \Pi_i(y, v) \psi_i(y_i, v) \), \( \kappa_i l_i = 1 \) (1 \( \leq i \leq n+1 \), \( \kappa_n+1 = 1 \), \( r = \kappa \cdot (1 + \lambda + \mu) \)), these formulas yield (1.3). For (1.2) compare IL’IN/ SOLONNIKOV [I/S, p. 72].

2. Estimates.

Our aim in this chapter is to prove the imbedding \( W_p^{2,1}(\Omega_T) \ni f \mapsto \nabla_x f \in W_p^{1-1/p, 1/p - 1/2p}(\partial \Omega_T) \) with the imbedding constant \( c^* \) independent of \( T \) (for \( T \) small); here we let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with boundary of the class \( C^2 \). Flattening the boundary locally, it is no restriction to assume that \( \Omega \) is a cube i.e. \( \Omega = Q^n_+(\lambda) \).

Since \( C^2(Q^n_+(\lambda) \times [0, T]) \) is dense in \( W_p^{2,1}(Q^n_+(\lambda) \times (0, T]) \) (cf. RÁKOSNÍK [R, Theorem 3]) and since the Hestenes-Whitney extension method (cf. ADAMS [A, p. 83]) yields a linear continuous extension operator \( E_T : W_p^{2,1}(Q^n_+(\lambda) \times (0, T)) \to W_p^{2,1}(Q^n_+(2\lambda) \times (0, 2T)) \) with

\[
E_T(C^2(Q^n_+(\lambda) \times [0, T])) \subset C^2(Q^n_+(2\lambda) \times [0, 2T]) \]

and
\[ \|E_T\|_{W_{p}^{2,1}(Q_{+}^{+}(\lambda) \times (0,T))} \leq B^* \text{ uniformly for all small } T, \text{ it is sufficient to prove} \]
\[ \|\nabla_x f\|_{W_{p}^{1-rac{1}{p}, \frac{1}{p}(1-rac{1}{p})}(Q_{n-1}(\lambda) \times (0,T))} \leq c^* \cdot \|f\|_{W_{p}^{2,1}(Q_{+}^{+}(\lambda) \times (0,T))} \]

for all \( f \in C^{2}(Q_{+}^{+}(\lambda) \times [0,2T]) \). The most difficult part in this inequality is the estimate for the time-regularity of the trace, i.e.

\[ \|\Delta_t h\|_{p,Q^{n+1}(\lambda) \times (0,T-h)} \leq h \cdot \|\partial_t (\gamma H_1)\|_{p,Q^{n+1}(\lambda) \times (0,T)} \]

(2.1) \[ |\nabla_x f|_{L_{p}^{0, \frac{1}{2}(1-\frac{1}{p})}(Q_{n-1}(\lambda) \times (0,T))} \leq c^* \cdot \|f\|_{W_{p}^{2,1}(Q_{+}^{+}(\lambda) \times (0,T))} \],

where \( |g|^{p}_{L_{p}^{0,\beta}(Q_{n-1}(\lambda) \times (0,T))} := \int_{0}^{T} h^{-(1+p\beta)} \|\Delta_t h g\|^{p}_{p,Q^{n+1}(\lambda) \times (0,T-h)} dh \)

for \( \beta \in (0,1) \), when \( (\Delta_t h, g)(x', t) := g(x', t+h) - g(x', t) \) and \( \|\cdot\|_{p, X} := \|\cdot\|_{L_{p}(X)} \).

The estimate for the spatial regularity follows from the more elementary trace theorem \( W_{p}^{1}(\Omega) \rightarrow W_{p}^{1-rac{1}{p}, \frac{1}{p}(1-rac{1}{p})}(\partial\Omega) \) (cp. KUFNER et al. [K/J/F, 6.8.13 Theorem, p. 337]) by an easy scaling argument (in \( t \)). In the sequel, we shall prove (2.1). For this purpose, we start from the representation (1.1) for \( \partial_j f \ (1 \leq j \leq n) \): splitting \( \int_{0}^{T} (\cdots) dv = \int_{h}^{T} (\cdots) dv + \int_{h}^{T} (\cdots) dv \) in the sum in the second line in (1.1) we get

\[ \partial_j f(\cdot) = H_1(\cdot) + \sum_{i=1}^{n+1} \tilde{B}_i\{H_2^{(i)}(\cdot) + H_3^{(i)}(\cdot)\}, \]

where

\[ H_1(\cdot) := \frac{A}{T^r} \int_{Q^{n+1}(0,T\mathbb{Z})} \cdots \int f(\cdot + y)\Pi(y,T) dy, \]

\[ H_2^{(i)}(\cdot) := \int_{0}^{h} v^{-(1+r)} \int_{Q^{n+1}(0,v\mathbb{Z})} \cdots \int \partial_i^l f(\cdot + y) \cdot K_i(y,v) dy dv, \]

(2.2) \[ H_3^{(i)}(\cdot) := \int_{h}^{T} v^{-(1+r)} \int_{Q^{n+1}(0,v\mathbb{Z})} \cdots \int \partial_i^l f(\cdot + y) \cdot K_i(y,v) dy dv. \]

In the sequel, we set \( (\gamma H_1)(x', t) := H_1(x', 0, t) \); we find

\[ \|\Delta_t h(\gamma H_1)\|_{p,Q^{n+1}(\lambda) \times (0,T-h)} \leq h \cdot \|\partial_t (\gamma H_1)\|_{p,Q^{n+1}(\lambda) \times (0,T)} \]

(2.3) \[ |\partial_t (\gamma H_1)(x', t)| \leq \frac{A}{T^r} \cdot \|\Pi(\cdot, T)\|_{\infty,Q^{n+1}(0,T\mathbb{Z})} \cdot |Q^{n+1}(0,T\mathbb{Z})|^{1/p'} \]

(\text{use } |\Delta_t h f(\tau)| \leq \int_{0}^{h} |f'(\tau + s)| ds \text{ and Minkowski's integral inequality (cp. WHEELEN/ ZYGMUND [W/Z, p. 143]))}; now

\[ |\partial_t (\gamma H_1)(x', t)| \leq \frac{A}{T^r} \cdot \|\Pi(\cdot, T)\|_{\infty,Q^{n+1}(0,T\mathbb{Z})} \cdot |Q^{n+1}(0,T\mathbb{Z})|^{1/p'} \]

\[ \cdot \|\partial_t f((x', 0, t) + \cdot)\|_{p,Q^{n+1}(0,T\mathbb{Z})} \]
by (2.2) and Hölder’s inequality; hence

\[
\leq c^* \cdot T^{-\frac{|\kappa|}{2}} (1 - \frac{1}{p'}) - \kappa_j \cdot \|\partial_t f((x',0,t) + \cdot)\|_{p,Q^{n+1}(0,T\mathbb{R})}
\]

by the kernel-estimate (1.2). Now observe that

\[
\|\partial_t f((x',0,t) + \cdot)\|_{p,Q^{n+1}(0,T\mathbb{R})}^p = \int_0^{T^{\kappa n}} \|\partial_t f(x' + \cdot, y_n, t + \cdot)\|_{p,Q^{n+1}(0,T\mathbb{R})}^p \, dy_n,
\]

which easily implies via Fubini’s theorem

\[
\left(\int \ldots \int_{Q^{n-1}(\lambda) \times (0,T)} \|\partial_t f((x',0,t) + \cdot)\|_{p,Q^{n+1}(0,T\mathbb{R})}^p \, dx' \, dt\right)^{1/p} \leq |\bar{Q}^{n+1}(0,T\mathbb{R})|^{1/p} \|\partial_t f\|_{p,Q^n((-\lambda 1',0), (\lambda 1' + T\mathbb{R}', T^{\kappa n})) \times (0,2T)}.
\]

Hence, by the last inequality, (2.4) and since \(|\bar{Q}^{n+1}(0,T\mathbb{R})| = T|\kappa| - \frac{T}{2}\) and \(\kappa_j = \frac{1}{2}\);

r.h. side in (2.3)

\[
\leq c^* \cdot h \cdot T^{-\frac{1}{2}(1 + \frac{1}{p}) - \kappa_j} \cdot \|\partial_t f\|_{p,Q^n((-\lambda 1',0), (\lambda 1' + T\mathbb{R}', T^{1/2})) \times (0,2T)}
\]

so that, abbreviating \(\rho = \rho(p) := \frac{1}{2}(1 - \frac{1}{p})\),

\[
|\gamma H_1|_{L^{0,\rho}_p(Q^{n-1}(\lambda) \times (0,T))} \leq c^* \cdot T^{-\frac{1}{2}(1 + \frac{1}{p})} \left(\int_0^T h^{-1 + p(1 - \rho)} \, dh\right)^{1/p} \|\partial_t f\|_{p,Q^n(\mathfrak{a}, \mathfrak{b}) \times (0,2T)}
\]

with \(\mathfrak{a} := (-\lambda 1',0)\) and \(\mathfrak{b} := (\lambda 1' + T\mathbb{R}', T^{1/2})\); now \(1 - \rho = \frac{1}{2}(1 + \frac{1}{p})\) and the \(T\) factors in the last inequality cancelled, as desired.

Let us turn our attention to \(H_2^{(i)}\) : trivially, for \(h \leq T\),

\[
\|\Delta_{t,h} (\gamma H_2^{(i)})\|_{p,Q^{n-1}(\lambda) \times (0,T-h)} \leq 2 \cdot \|\gamma H_2^{(i)}\|_{p,Q^{n-1}(\lambda) \times (0,T)};
\]

furthermore, using the kernel estimate (1.3) (with \(s = 0\), we get

\[
|\gamma H_2^{(i)}(x',t)| \leq c^* \cdot \int_0^h \nu^{-1 + |\kappa| + |\kappa_n|} + \frac{1}{2} \int \ldots \int_{Q^{n+1}(0,v\mathbb{R})} y_n^\varepsilon \cdot |\partial_t f((x',0,t) + y)| \, dy \, dv;
\]
we now represent the integrand as
\[
\left\{ v^{-\frac{1}{p'}\left(1+|\kappa|\right)+\frac{1}{2}(\rho-\varepsilon,\kappa_n)} \right\} \cdot \left\{ v^{-\frac{1}{p'}\left(1+|\kappa|\right)+\frac{1}{2}(\rho-\varepsilon,\kappa_n)} \cdot y_n^{\varepsilon} \cdot |\partial_i^l f((x',0,t) + y)| \right\}
\]
(note that $1/2 = \rho + 1/2p$); we choose $\varepsilon \in (0, \rho/\kappa_n)$; Hölder’s inequality (with $p', p$) in $y-v$ space then yields

\[
(2.8) \quad \text{l.h.s. in (2.7)} \leq c^* \cdot \left( \int_0^h v^{-1+\frac{p'}{2}(\rho-\varepsilon,\kappa_n)} dv \right)^{1/p'} \cdot I^{1/p}
\]
with
\[
I := \int_0^h \int \ldots \int_{Q^{n+1}(0, v\xi)} v^{-\left(1+|\kappa|\right)+\frac{p}{2}(\rho-\varepsilon,\kappa_n)} \cdot y_n^{\varepsilon p} \cdot |\partial_i^l f((x',0,t) + y)|^p dy \, dv,
\]
where in the first integral we took into account that $|Q^{n+1}(0, v\xi)| = v^{|\kappa|}$; the first integral is clearly proportional to $h^{1/2}(\rho-\varepsilon,\kappa_n)$. Thus, after a computation as in (2.5), we get

\[
(2.9) \quad \| \gamma H_2^{(i)} \|_{p, Q^{-}\lambda(0,T)} \leq c^* \cdot h^{1/2}(\rho-\varepsilon,\kappa_n) \cdot \tilde{I}^{1/p}
\]
with
\[
\tilde{I} := \int_0^h \int \ldots \int_{Q^{n+1}(0, v\xi)} v^{-\left(1+|\kappa|\right)+\frac{p}{2}(\rho-\varepsilon,\kappa_n)} |\tilde{Q}^{n+1}(0, v\xi)| \cdot \int \ldots \int_{Q^{n+1}(a, b(v))} z_n^{\varepsilon p} \cdot |\partial_i^l f(z)|^p \, dz \, dv,
\]
where $a := (-\lambda \mathbf{1}', 0, 0)$, $b(v) := (\lambda \mathbf{1}', v\kappa, v\kappa, T + v)$; since $b(v) \leq b(h)$, we can continue

\[
\tilde{I} \leq \int_0^h v^{-1+h} \cdot \frac{p}{2}(\rho-\varepsilon,\kappa_n) \, dv \int \ldots \int_{Q^{n+1}(a, b(h))} z_n^{\varepsilon p} \cdot |\partial_i^l f(z)|^p \, dz
\]
\[
\leq c^* \cdot h^{(\rho-\varepsilon,\kappa_n)\cdot p/2} \int_0^{h\kappa n} z_n^{\varepsilon p} \cdot \varphi(z_n) \, dz_n
\]
with $\varphi(z_n) := \|\partial_i^l f(\cdot, z_n, \cdot)\|^p_{p, Q^{-\lambda(0,T)}}$ by Fubini’s theorem and since $h \leq T$; consequently, by (2.6), (2.9) and the last line

\[
(2.10) \quad |\gamma H_2|_{L^{0,\rho}_{p}(Q^{-\lambda(0,T))}} \leq
\leq c^* \cdot \int_0^T h^{-\left(1+P\rho-\varepsilon,\kappa_n\right)} \int_0^{h\kappa n} z_n^{\varepsilon p} \cdot \varphi(z_n) \, dz_n \, dh
\]
The trace theorem $W^{2,1}_p(\Omega_T) \ni f \mapsto \nabla_x f \in W^{1-1/p,1/2-1/2p}_{p'}(\partial \Omega_T)$ revisited

and by the version of Hardy’s inequality from Lemma, (i) in the Appendix

\[ \leq c^* \cdot (p \cdot \varepsilon \cdot \kappa_n)^{-1} \cdot \int_0^{T \cdot \varepsilon} \varphi(z_n) \, dz_n \]

\[ = c^* \cdot (p \cdot \varepsilon \cdot \kappa_n)^{-1} \cdot \| \partial_i^l f \|^p_{p', Q^n((\lambda T', 0), (\lambda T'+TZ', T^{1/2}))} \times (0,2T) \]

which is the desired result for $H^2_{(i)}$.

Finally, let us turn to $H^3_{(i)}$; we again use (2.3) and observe that the correct expression for $\partial_t (\gamma H^3_{(i)})$ is obtained just by replacing $K_i$ (in the definition of $H^3_{(i)}$) by $\partial_{n+1} K_i$ (integrate by parts); after estimating $|\partial_{n+1} K_i|$ according to (1.3), we arrive at

\[ (2.11) \quad |\partial_t (\gamma H^3_{(i)})(x', t)| \leq \]

\[ \leq c^* \cdot \int_h^T v^{-(1+|\kappa|+\frac{1}{2}+\varepsilon \cdot \kappa_n)} \left( \prod_{n+1} y_n^{\varepsilon \cdot \kappa_n} \cdot |\partial_i^{l_i} f((x', 0, t) + y)| \right) \, dy \, dv \]

(cp. (2.7); here the $v$-exponent is smaller by one, since $\partial_{n+1} K_i$ entails (in (1.3)) the additional factor $v^{-1}$); in the last integral we write the integrand in the form

\[ \{ v^{-\frac{1}{p'}}(1+|\kappa|)-(1-\rho-\delta) \} \cdot \{ v^{-\frac{1}{p}}(1+|\kappa|)-\varepsilon \cdot \kappa_n \cdot \} y_n^{\varepsilon \cdot \kappa_n} \cdot |\partial_i^{l_i} f((x', 0, t) + y)| \}

(note that $-\frac{1}{2} = \frac{1}{2p} + \rho - 1$), where we introduced $\delta \in (0, 1 - \rho)$. Now apply Hölder’s inequality (with $p', p$) in $y$-$v$ space and get

r.h.s. in (2.11) $\leq c^* \cdot \left( \int_h^T v^{-1-p'(1-\rho-\delta)} \, dv \right)^{1/p'} \cdot J^{1/p}$

with

\[ J := \int_h^T v^{-(1+|\kappa|+\frac{1}{2})-p(\varepsilon \cdot \kappa_n+\delta)} \left( \prod_{n+1} y_n^{\varepsilon \cdot \kappa_n} \cdot |\partial_i^{l_i} f((x', 0, t) + y)| \right) \, dy \, dv ; \]

proceeding as in the argument leading from (2.8) to (2.9), the last estimate allows us to conclude

$\| \partial_t (\gamma H^3_{(i)}) \|^p_{p', Q^{n-1}(\lambda) \times (0,T)} \leq$

\[ \leq c^* \cdot h^{-1(1-\rho-\delta)} \cdot \left( \int_h^T v^{-1-p(\varepsilon \cdot \kappa_n+\delta)} \left( \int_0^{r \cdot \varepsilon} \varphi(z_n) \, dz_n \right)^{1/p} \right) \]

with $\varphi(\cdot)$ as before (since $v \leq T$); by (2.3)

\[ |\gamma H^3_{(i)}|^p_{L^0_p(Q^{n-1}(\lambda) \times (0,T))} \leq c^* \cdot \int_0^T h^{-1+\rho\delta} \left( \int_h^T v^{-1-p(\varepsilon \cdot \kappa_n+\delta)} \left( \int_0^{r \cdot \varepsilon} \varphi(z_n) \, dz_n \right) \right) \, dh \]

\[ \leq c^* \cdot (p \cdot \delta)^{-1} \cdot \int_0^T v^{-1-p \cdot \varepsilon \cdot \kappa_n} \left( \int_0^{r \cdot \varepsilon} \varphi(z_n) \, dz_n \right) \, dv \]

by Appendix, Lemma (ii); now we may continue as after (2.10) and the desired result for $H^3_{(i)}$ follows.

Thus (2.1) is proved for all $T \leq T_0(\lambda) = \lambda^2$. 
Appendix.

We note a version of Hardy’s inequality.

Lemma. Suppose that \( f \in L^1(0,T^\gamma) \) is nonnegative, \( 0 < T \leq \infty ; \varepsilon, \gamma > 0 \). Then

\[
\text{(i)} \int_0^T x^{-1-\varepsilon-\gamma} f^\gamma dy dx \leq (\gamma \cdot \varepsilon)^{-1} \int_0^T f^\gamma dy,
\]

\[
\text{(ii)} \int_0^T x^{-1+\varepsilon-\gamma} \int_x^y y^{-\varepsilon} f(y) dy dx \leq (\gamma \cdot \varepsilon)^{-1} \int_0^T f(y) dy.
\]

Proof: These inequalities are proved in BESOV/ IL’IN/ NIKOL’SKII [B/I/N, 2.15, p. 28] (even in a more general form) for \( T = \infty \). For \( T \) finite they follow easily by applying the version for \( T = \infty \) to the extension by zero of \( f \) to \( \mathbb{R}^+ \). □

References


University of Bayreuth, Faculty of Mathematics and Physics, P.O.Box 101251, 8580 Bayreuth, Federal Republic of Germany

(Received August 6, 1990)