Sets invariant under projections onto
two dimensional subspaces

SIMON FITZPATRICK, BRUCE CALVERT

Abstract. The Blaschke–Kakutani result characterizes inner product spaces \( E \), among normed spaces of dimension at least 3, by the property that for every 2 dimensional subspace \( F \) there is a norm 1 linear projection onto \( F \). In this paper, we determine which closed neighborhoods \( B \) of zero in a real locally convex space \( E \) of dimension at least 3 have the property that for every 2 dimensional subspace \( F \) there is a continuous linear projection \( P \) onto \( F \) with \( P(B) \subseteq B \).

Keywords: inner product space, two dimensional subspace, projection

Classification: 46C05, 52A15

1. Introduction.

As mentioned in the summary, if \( B \) is the closed unit ball in a normed space \( E \) and for every 2 dimensional subspace \( F \) there is a linear projection \( P \) of \( E \) onto \( F \) with \( P(B) \subseteq B \), then the norm is given by inner product, as explained in Chapter 12 of Amir’s book [1]. A natural question is to see, if there are other sets \( B \) such that for every 2 dimensional \( F \) there is a linear projection onto \( F \) under which \( B \) is invariant, or whether we characterize the ball in an inner product space by this property, among a wider class of sets \( B \).

Restricting ourselves to closed neighborhoods of zero, we find \( B \) is the inverse image under a continuous linear map of: a closed neighborhood of 0 in \( \mathbb{R} \), a unit ball in \( \mathbb{R}^2 \), or a unit ball in an inner product space.

The reader will note that a similar problem motivates the paper [3].

2. Two dimensional results.

The following result appears as Theorem 8 of [3].

Lemma 2.1. Let \( B \) be a closed nonempty subset of \( \mathbb{R}^2 \) and suppose there is \( w \in \mathbb{R}^2, w \neq 0 \) and \( \lambda_n \to \infty \), such that \( \lambda_n^{-1}w \in B \) or \( \lambda_n w \notin B \). For every one dimensional subspace \( m \), there exists a linear projection \( P : \mathbb{R}^2 \to m \) with \( P(B) \subseteq B \) iff \( B \) is one of:

(a) a subset, containing 0, of a line through 0,
(b) a union of parallel lines, containing 0,
(c) a bounded convex symmetric neighborhood of 0.
**Lemma 2.2.** Let $B$ be a closed subset of $\mathbb{R}^2$ such that for any vertical line $x = c$ there is a $v \in \mathbb{R}^2$ such that projecting affinely onto $x = c$ along $\mathbb{R}v$ takes $B$ to $B$. Then $B$ is either

(a) a union of lines, all parallel, or

(b) the epigraph of a convex function $h : \mathbb{R} \to \mathbb{R}$, or the negative of such a set.

**Proof:** One possibility is that $B$ is empty. Otherwise, we consider two cases, depending on whether $\text{cocl}(B)$ is equal to $\mathbb{R}^2$ or not.

(a) $K = \text{cocl}(B) \neq \mathbb{R}^2$. Suppose $u$ is an extreme point of $K$. We claim $u \in B$. For if not, take $B(u, r) \subseteq B'$, $r > 0$, with $\partial B(u, r)$ intersecting $\partial K$ in two points $u$ and $w$, noting $K \neq \{u\}$ since $B$ intersects every vertical line. Now $u \notin \text{aff} v, w$, since it is extreme, so $u$ is in the open half space given by $\text{aff}\{v, w\}$ which does not intersect $B$. This contradicts $u \in \text{cocl}(B)$.

Suppose $(a, b) \in \mathbb{R}^2$ is a point in $\partial K$. To fix ideas, suppose $c < b$ implies $(a, c) \notin K$, by relabelling the $y$ axis. Suppose there is a nonempty open interval $(e, f) \subseteq (b, \infty)$ with $(a, g) \notin B$, if $g \in (e, f)$. Then projecting onto $\{(x, y) : x = a\}$ along a line of slope $\alpha(a)$ gives the open strip $\{(x, y) \in \mathbb{R}^2 : y \in (e, f) + \alpha(a)(\alpha - a)\} \subseteq B'$.

Suppose for the purpose of obtaining a contradiction that this intersects $\partial K$. Points in the intersection must be nonextreme points, giving a nonempty open line interval in $\partial K \cap B'$, having slope $\beta$ say. Taking $(p, q) \in \mathbb{R}^2$ in this interval, a projection onto $x = p$ taking $B$ to $B$ must be along the line with slope $\beta$. But there is an end of the closed line segment in $\partial K$ with slope $\beta$ which must be an extreme point, hence in $B$, and which projects onto $(p, q)$, a contradiction.

Hence either $\partial K$ has slope $\alpha(a)$, or $(a, c) \in B$ for all $c > b$. In the first case, projecting onto any line $x = c$, taking $B$ to $B$, must take $\partial K$ to $\partial K$ and be along the line slope $\alpha(a)$, giving $B$ as the union of lines with slope $\alpha(a)$. In the second case, $B$ being closed is equal to $K$, which is the epigraph of a convex function from $\mathbb{R}$ to $\mathbb{R}$. Without our assumption that the lower half of $x = a$ was in $B'$ we could reverse the direction of the $y$ axis to give $B$ as the negative of such an epigraph.

(b) $\text{cocl}(B) = \mathbb{R}^2$. If a whole vertical line is in $B$, then $B = \mathbb{R}^2$. Suppose now that for all $c \in \mathbb{R}$, if $S(c) = \{y : (c, y) \in B\}$ then $S(c) \neq \mathbb{R}$. Note for all $c$, $S(c)$ is not bounded above or below. We have for all $c, \alpha(c)$ such that for all $d$,

\[(1) \quad S(d) + \alpha(c)(c - d) \subseteq S(c).\]

We take two cases, depending on whether $\alpha$ is either nondecreasing or nonincreasing, or not. If $\alpha$ is nonincreasing, by renaming we may assume it is nondecreasing.

(b1) $\alpha$ is nondecreasing. We define $p(x) = \int_0^x \alpha(x) \, dx$, which gives the epigraph $H$ of $p$ of a closed convex set such that for all $c$ and $d$, $S(d) + \alpha(c)(c - d) \subseteq S(c)$.

Since $S(c) \neq \mathbb{R}$ and $S(c)$ is not bounded above or below for all $c$, $S(c)$ has more than one component, so that there is a bounded open interval $(d, e)$ in $S(c)'$, with the points $(c, d)$ and $(c, e)$ in $B$. Let $H_b$ be a vertical translate of $H$ with $(c, d) \in H_b$. Now $H_b \cap B$ is invariant under projections onto lines $x = c$ along lines with slope $\alpha(c)$, and by (a), since $(c, (d + e)/2) \notin B$, $H_b \cap B$ is a union of lines, with slope $\alpha$.
say. Thus the line through \((c, d)\) with slope \(\alpha\) is in \(\partial K\), and so \(\alpha(d) = \alpha\) for all \(d\). Hence, by (1), since \(S(d) + \alpha(c - d) \subseteq S(c)\) and \(S(c) + \alpha(d - c) \subseteq S(d)\), we have \(S(d) + \alpha(c - d) = S(c)\) and \(B\) is a union of lines with slope \(\alpha\).

(b2) There are \(z, y, w \in \mathbb{R}, z < y < w\), such that \(\alpha(z) > \alpha(y) < \alpha(w)\). (If we had \(\alpha(z) < \alpha(y) > \alpha(w)\), we could relabel the \(y\) axis to obtain this assumption.) By (1), \(S(w) + \alpha(y)(y - w) \subseteq S(y)\), and \(S(y) + \alpha(w)(w - y) \subseteq S(w)\), so \(S(y) + (\alpha(w) - \alpha(y))(w - y) \subseteq S(y)\). Let \(x_1 = (\alpha(w) - \alpha(y))(w - y) > 0\). Let \(x_2 = (\alpha(z) - \alpha(y))(z - y) > 0\). We have two cases; \(x_1/x_2\) is rational or irrational.

(b2a) \(x_1/x_2 \in \mathbb{Q}\). Let \(x_1 = kd, x_2 = hd, k, h \in \mathbb{N}, d > 0\). Then \(s(y) - khd \subseteq S(y)\) and \(S(y) + khd \subseteq S(y)\). Hence the map \(x \to x + khd\) is onto \(S(y)\), since \(x \in S(y)\) gives \(x = (x - khd) + (khd)\). Now let \(g : S(y) \to S(w)\) be given by \(z = g(z) + \alpha(y)(w - y)\), and let \(f : S(w) \to S(y)\) be given by \(x = f(x) + \alpha(w)(y - w)\). The map \(x \to x + khd\) is the composite \((f \circ g)^k\), so \(g\) and \(f\) are bijections,

\[
S(w) = S(y) + \alpha(y)(w - y).
\]

(b2b) \(x_1/x_2 = \alpha \notin \mathbb{Q}\). There are sequences \(n_i, m_i\) in \(\mathbb{N}\) with \(|n_i \alpha - m_i| \leq \frac{1}{n_i}\). So \(y \to y + \alpha x_2\) and \(y \to y - x_2\) take \(S(y)\) to \(S(y)\). Hence for \(y \in S(y), y_i = y - m_i x_2 + (n_i - 1)x_1 \in S(y)\) and \(y_i \to y - x_1\), giving \(y - x_1 \in S(y)\) since \(S(y)\) is closed. Hence, as in (b2a), the map \(g : S(y) \to S(w)\) is a bijection, or \(S(w) = S(y) + \alpha(y)(w - y)\), so (2) holds for all \(x_1\) and \(x_2\). We either have: (c) for all \(z < y, \alpha(z) > \alpha(y)\), or (d) there is \(z_0 < y, \alpha(z_0) < \alpha(y)\). In case (d) we have for all \(w > y\), (2) holds, by using \(z\) above, if \(\alpha(w) > \alpha(y)\) and \(z_0\), if \(\alpha(w) < \alpha(y)\), and noting (2) holds, if \(\alpha(w) = \alpha(y)\). And in case (c), we replace (2) by \(S(z) = S(y) + \alpha(y)(w - y)\) for all \(z < y\). In case (d), we have \(\alpha(w) = \alpha(y)\) for \(w > y\) and in case (c) we have \(\alpha(z) = \alpha(y)\) for all \(z < y\), a contradiction to (b2).

3. Three dimensional results.

**Lemma 3.1.** Let \(B\) be a closed subset of \(\mathbb{R}^3\), \(N\) a two dimensional subspace, \(d \in \mathbb{N}, d \neq 0\). Suppose any plane \(M\) containing 0 but not \(\mathbb{R}d\) is the range of a projection \(P\) with \(P(B) \subseteq B\) and \(P(N) \subseteq N\). Then \(B\) is a union of translates of \(\mathbb{R}d\), or \(B \subseteq N\).

**Proof:** Let \(b \in B \setminus N\). Any line \(m\) in \(b + N\) not parallel to \(\mathbb{R}d\) is the range of an affine projection in \(b + N\). By Lemma 1.2, \(B \cap (b + N)\) is a union of parallel lines or a convex set \(K_b \neq b + B\) intersecting every translate of \(m\) in \(b + N\). Supposing the latter and not the former, we have a contradiction by taking \(m\) to be a supporting line to \(K_b\) not parallel to \(\mathbb{R}d\). Hence \(B \cap (b + N)\) is a union of translates of a line \(k\) in \(b + N\). If \(k\) is not parallel to \(\mathbb{R}d\) and \(B \cap (b + N) \neq b + N\), we may take a translate of \(k\) contained in the complement of \(B\) in \(b + N\), to obtain a contradiction.

The following result of Blaschke is proved simply in [2, Lemma 1] except that \(p\) is assumed to be a norm.
Lemma 3.2. Let $X$ be a real three dimensional normed space with the basis \{\(e_1, e_2, e_3\)\}, where \(e_i\) is a unit vector. Suppose every two dimensional subspace which contains \(e_1\) is the range of a nonexpansive projection along a vector in span \{\(e_2, e_3\)\}. Then there is a function \(F : \mathbb{R}^2 \to \mathbb{R}\) such that for all \(x_i \in \mathbb{R}\), \(\|x_1 e_1 + x_2 e_2 + x_3 e_3\| = F(x_1, \|x_2 e_2 + x_3 e_3\|)\).

Theorem 3.3. Let $B$ be a closed neighborhood of 0 in $\mathbb{R}^3$. For all planes $M$ through 0, there exists a linear projection $P$ of $\mathbb{R}^3$ onto $M$ with $P(B) \subseteq B$ iff $B$ is one of:

(a) the closed unit ball given by an inner product,
(b) a union of parallel planes,
(c) $K + \mathbb{R}v$, where $K$ is a bounded convex symmetric neighborhood of 0 in a plane $M$ through 0 and $\mathbb{R}v$ is a line not in $M$.

Proof: We let $C = \text{cocl}(B)$ and consider four distinct cases:

(i) $C$ contains no lines,
(ii) $C$ contains a line but no planes,
(iii) $C$ contains a plane by not $\mathbb{R}^3$,
(iv) $C = \mathbb{R}^3$.

(i) Let $D = C \cap -C$. Then $D$ is a closed convex bounded symmetric neighborhood of 0, invariant under projections onto all 2 dimensional subspaces, and hence the unit ball given by an inner product, by the Blaschke–Kakutani theorem.

Take any 2 dimensional subspace $M$, and consider $\partial D \cap M$ and $\partial C \cap M$. Let $\mathbb{R}e$ be perpendicular to $M$ under the inner product. Any plane through $\mathbb{R}e$ is the range of a projection taking $C$ to $C$, hence $D$ to $D$, hence is along a direction in $M$. We can parametrize $\partial D \cap M$ and $\partial C \cap M$ to give radius $d(\theta)$ and $c(\theta)$ say as functions of angle $\theta$; these functions are absolutely continuous and their derivative is equal for angles, where $d(\theta)$ and $c(\theta)$ have a unique tangent, i.e. almost everywhere. Hence, if $d(\theta)$ and $c(\theta)$ are equal to $\theta_0$, they are equal near $\theta_0$, and $M \cap \partial C \cap \partial D$ is open in $M \cap \partial D$. Since $M \cap \partial C \cap \partial D$ is also closed in $M \cap \partial D$, and nonempty, and $M \cap \partial D$ is connected, $M \cap \partial C = M \cap \partial D$. Hence $C = D$.

We claim $B = D$. If $x \in \partial D$, but $x \notin B$, then $x \notin \text{cocl}(B)$, a contradiction, giving $\partial D \subseteq B$. If $x \in \text{int}(D)$, take $P$ a projection onto $M$, a 2 dimensional subspace containing $x$, with $P(B) \subseteq B$. Then $x \in P(\partial D) \subseteq B$. Hence $D \subseteq B$, giving $B = D$.

(ii) $C$ may be represented as $K + \mathbb{R}v$, where $K$ is a closed convex set, not containing a line, in a plane $M$, and $v \notin M$. All projections onto planes not containing $\mathbb{R}v$ are along $\mathbb{R}v$, so $B \setminus \mathbb{R}v$ is a union of lines parallel to $\mathbb{R}v$. Let $B_1 = B \cup \mathbb{R}v$. Now in $\mathbb{R}^3/\mathbb{R}v$, we have all lines through 0 being the range of a projection taking the quotient $B_1/\mathbb{R}v$ to itself.

By Lemma 1.1 and our hypotheses, it must be a closed bounded convex symmetric neighborhood of 0. Hence $B_1 = K + \mathbb{R}v$, with $K$ a closed bounded symmetric convex neighborhood of 0 in $M$, $v \notin M$. Hence, $\mathbb{R}v \subseteq B$, and $B = K + \mathbb{R}v$.

(iii) Let $N$ be a plane through $O$ with a translate of $N$ contained in $c$. Now any plane $M$ through $O$, $M \neq N$, is the range of projection along a direction in $N$. 


Hence for \( b \in B \setminus N \), any line \( b + N \) is the range of an affine projection in \( b + N \) taking \( B \) to \( B \).

By Lemma 1.2, \( B \cap (b + N) \) is a convex set not equal to \( b + N \) but meeting all lines, which is impossible, or is a union of parallel lines. Hence \( b + N \subseteq B \).

(iv) We assume \( B \) is not a union of parallel planes.

(a) We claim that for any line \( \mathbb{R}w, w \neq 0 \), and any \( M \in \mathbb{R}, B \) intersects \( (M, \infty)w \).

For, take a plane \( \mathbb{R}w + \mathbb{R}v \), and project onto it along \( u \). Suppose we project onto \( \mathbb{R}w + \mathbb{R}u \) along \( y \). \( B \) intersects \( (M, \infty)w + \mathbb{R}y + \mathbb{R}u \). Projecting onto \( \mathbb{R}w + \mathbb{R}u \) gives \( (M, \infty)w \) intersecting \( B \).

(b) Since \( B \neq \mathbb{R}^3 \), take \( a \in B', a \neq 0 \). Take a plane \( N \) through \( \mathbb{R}a \), and project along \( b \), so \( B(a, \delta) + \mathbb{R}b \subseteq B' \). Take the plane \( \mathbb{R}a + \mathbb{R}b \) and project along \( c \) onto it. For \( \delta > 0 \) small, \( B(a, \delta) + \mathbb{R}b + \mathbb{R}c \subseteq B' \). Let us call the set between two parallel planes a "slice".

(c) We claim there is a basis \((f_1, f_2, f_3)\) and a nonempty open ball \( \mathbb{B}(c, \delta) \) with the three slices \( \mathbb{B}(c, \delta) + \mathbb{R}f_1 + \mathbb{R}f_2, \mathbb{B}(c, \delta) + \mathbb{R}f_2 + \mathbb{R}f_3, \mathbb{B}(c, \delta) + \mathbb{R}f_1 + \mathbb{R}f_3 \) all contained in \( B' \). Since we are assuming \( B \) is not a union of parallel lines, take the slice \( B(a, \delta) + M \subseteq B', \delta > 0, M \) on a plane through \( 0 \) and by Lemma 3.1 take \( N \neq M \) a plane through \( 0 \) with projection along \( r \notin M \). By Lemma 3.1, take \( Q \) another plane through \( 0 \), not containing \( N \cap M \), with projection along \( s \notin M \). Let \( c \) be the point of intersection of \( a + M, N \) and \( Q \). We take the three planes through \( c: c + M, c + \mathbb{R}r + (M \cap N), c + \mathbb{R}s + (M \cap Q) \). These are all contained in \( B' \), together with slices containing them, and the intersection is \( \{c\} \). Together they give \( f_i \) as required.

(d) We claim there is a sequence of projections \( P_n \) onto planes through \( 0 \) with \( \|P_n\| \to \infty \). Assume by renaming that \( c \) is the positive octant. For \( \delta > 0 \), let \( f_\delta = f_3^* - \delta(f_1^* + f_2^*) \), where \((f_1^*, f_2^*, f_3^*)\) is the dual basis to \((f_1, f_2, f_3)\).

Suppose there is \( \delta > 0 \) with \( \{x = (f_\delta, x) \geq 0\} \cap B \cap \{x: x_1 \geq c_1, x_2 \geq c_2, x_3 \leq c_3\} \) nonempty. Then by compactness there is an maximal such \( \delta \), \( d(\max) \), and an \( e \in B \) with \( (f_\delta(\max), e) = 0, e_1 \geq c_1, e_2 \geq c_2, e_3 \leq c_3 \). For \( \delta > d(\max) \) there is no such \( e \). If there is no \( \delta > 0 \), take \( d(\max) = 0 \) and in this case by (a) there is \( e \in B \) with \( e_1 \geq c_1, e_2 \geq c_2 \) and \( e_3 = 0 \).

Let \( \delta(n) = d(\max) + \) and let \( P_n \) be a projection on \( N(f_\delta(n)) \). If \( P_{n(m)} \) is a bounded subsequence, then \( P_{n(m)}e \to e \), giving \( P_{n(m)}e \) in \( B \), with \( (P_{n(m)}e_1) \geq c_1, (P_{n(m)}e_2) \geq c_2, (P_{n(m)}e_3) \leq c_3 \), contradicting the maximality of \( d(\max) \). Hence \( \|P_n\| \to \infty \).

(e) We derive a contradiction, showing \( B \) is a union of parallel lines. Since \( \|P_n\| \to \infty \), and \( P_n(B) \) contains the symmetric convex set \( P_nB(0, \varepsilon) \) for some \( \varepsilon > 0 \), we have \( P_n(B) \) intersecting \( c + \mathbb{R}f_i + \mathbb{R}f_j \) for \( n \) large, for some \( i \) and \( j \).

(f) We claim \( B \) is a union of parallel planes. Since \( B \) is a union of parallel lines, there is \( q \neq 0 \), so \( B \) is a union of translates of \( \mathbb{R}q \). By 2.1 applied to \( \mathbb{R}^3/\mathbb{R}q \), we have \( B/\mathbb{R}q \) a union of parallel lines, since its convex closure is \( \mathbb{R}^3/\mathbb{R}q \), and it is a neighborhood of \( 0 \). This gives \( B \) a union of parallel planes. \( \square \)
4. Higher dimensions.

**Theorem 4.1.** Suppose $B$ is a closed neighborhood of 0 in a real locally convex topological vector space $X$ of dimension $\geq 3$. For all two dimensional subspaces $M$ there is a continuous linear projection $P$ of $X$ onto $M$ with $P(B) \subseteq B$, iff $B$ is the inverse image under a continuous linear map $T$ of:

(a) the closed unit ball in an inner product $H$,
(b) the closed unit ball given by a norm on $\mathbb{R}^2$, or
(c) a closed neighborhood of 0 in $\mathbb{R}$.

**Proof:** $\implies$ (1) We suppose that for all 3 dimensional subspaces $F$ of $X$, $F \cap B$ is a union of parallel planes. We claim $B$ is a union of parallel closed hyperplanes, so (c) holds.

For $H$ a closed subspace of codimension $\geq 2$ with $H \subseteq B$, we claim there is a closed subspace $H_{+1}$ with $H_{+1} \subseteq B$ and $H$ of codimension 1 in $H_{+1}$. Let $H_{-1}$ be a closed subspace of $H$ of codimension 1 and let $E$ be a three dimensional subspace of $X$ with $E \cap H_{-1} = \{0\}$. Let $M$ be a two dimensional subspace of $E$ contained in $B$. Given $h \in H_{-1}$, $h \neq 0$, $(\mathbb{R}h + M) \cap B$ is a union of translates of $M$, so $h + M \subseteq B$, giving $H_{-1} + M \subseteq B$. Take $H_{+1} = H_{-1} + M$. By Zorn’s lemma, a closed subspace $H$ of codimension $\leq 1$ in $X$ is contained in $B$. If $x \in B \setminus H$, and $h \in H$, let $E$ be a three dimensional space containing $x, h$ and a two dimensional subspace $M$ of $H$. Then $x + M \subseteq B$, giving $x + h \in B$. Thus for $x \in B, x + H \subseteq B$, and the claim is proved, $B = \cup \{x + H : x \in B\}$.

(2) We now suppose there exists a 3 dimensional subspace $F_0$ such that $F_0 \cap B$ contains no plane, and we suppose that for all three dimensional subspaces $F, F \cap B$ contains a line. We claim $B$ is convex, contains a 2 codimensional closed subspace $E$, and with $E_2$ a complementary subspace, $B \cap E_2$ is a bounded symmetric neighborhood of 0 in $E_2$. We take $E_2 \subseteq F_0$ with $E_2 \cap B$ a bounded symmetric neighborhood $K$ of 0. Let $e \in E_2, e \neq 0, B \cap \mathbb{R}e = \{\lambda e : |\lambda| \leq 1\}$.

$B$ is convex since if $a, b \in B$, we take a 3 dimensional space $G$, containing $a, b$ and $e$, and note that if $B \cap G$ is a union of planes, it is of the form $M + \lambda e, |\lambda| \leq 1$, and hence $B \cap G$ is convex.

Let $H$ be a closed subspace of $X$, of codimension $> 2$, with $H \subseteq B$. Take $f \notin E_2 + H$. Now $(E_2 + \mathbb{R}f) \cap B$ contains a line $Re$ say, giving $B \supseteq H + Re$ since it is closed and convex. Hence, by Zorn’s lemma there is a closed subspace $E$ of codimension 2 with $E \subseteq B$.

Since $B$ is closed and convex, $K + E \subseteq B$. Let $b \in B$, with $b = b_2 + b_e, b_2 \in E_2, b_e \in E$. We claim $b_2 \in K$. If $b_E \neq 0, B \cap (E_2 + \mathbb{R}b_E)$ is projected onto $E_2$ taking $B$ to $B$, hence along $b_E$, and $b_2 \in K = B \cap E_2$. Thus $B = K + E$, giving (b).

(3) We suppose there exists a three dimensional subspace $E_0$ such that $E_0 \cap B$ contains no line. Now as in (2) we find $B$ is convex, and the same idea gives $B$ symmetric. By Zorn’s lemma, there is a maximal closed subspace $E \subseteq B$. Let $Q : X \to X/E$ be the projection. We see $Q(B)$ is convex, symmetric, and radial. If $p$ is its Minkowski functional, by maximality of $E$, if $p(Qx) = 0$, then $Rx \in B$ and $x \in E$, so $p$ is a norm.
We claim $p$ is given by an inner product, by the Blaschke–Kakutani theorem. Let $M$ be a 2 dimensional subspace of $x/E$ and take $N$ a two dimensional subspace of $X$ with $QN = M$. Let $R$ be a continuous projection of $X$ onto $N$ with $R(B) \subseteq B$. We define $P : X/E \to M$ by $P(Qx) = QR(x)$; this is well defined for if $Qx = 0$, then $x \in E$ giving $Rx \in E$ and $QRx = 0$. We see $P$ maps $X/E \to M$ and is the identity on $M$ and maps $Q(B)$ to $Q(B)$. Hence $Q(B)$ is the closed unit ball in an inner product space, $Q : X \to X$ is continuous and linear, and $B = Q^{-1}(Q(B))$ giving (a).

$\leftarrow$ (a) Suppose (a) holds. Let $M$ be a 2 dimensional subspace of $X$.

(i) Let $TM$ be a 2 dimensional subspace of $H$. Let $R$ be the projection on $TM$ under which the unit ball $B[0,1]$ in $H$ is invariant. Let $T|_M$ be the restriction, and define $P = (T|_M)^{-1}RT$. One checks $P$ takes $X$ to $M$, is the identity on $M$, is a continuous linear map and maps $B = T^{-1}(B[0,1])$ to itself.

(ii) Let $TM$ be a 1 dimensional subspace of $H$. Take $(e_1, e_2)$ a basis of $M, Te_1 = 0$. Let $S : X \to M$ be a continuous projection, $Sx = x_1(x)e_1 + x_2(x)e_2$. Define $Px = (T|_{Re_2})^{-1}RTx + x_1(x)e_1$, where $R$ is the projection on $TM$ leaving $B[0,1]$ invariant.

(iii) Let $TM$ be 0 dimensional. Let $S : X \to M$ be as in (ii) and take $P = S$.

(b) Suppose (b) holds. Let $M$ be a 2 dimensional subspace of $X$. Let $T : X \to \mathbb{R}^2$ be given, $B[0,1]$ the unit ball in $\mathbb{R}^2$, and $B = T^{-1}B[0,1]$.

(i) Let $TM = \mathbb{R}^2$. Define $P = (T|_M)^{-1}T$.

(ii) Let $TM$ be 1 dimensional. Let $R$ be the projection on $\mathbb{R}^2$ of $TM$ leaving $B[0,1]$ invariant, and define $P$ as in (a)(ii).

(iii) Let $TM$ be 0 dimensional. Define $P$ as in (a)(iii).

(c) Suppose (c) holds. Let $M$ be a two dimensional subspace of $X$. Let $T : X \to \mathbb{R}^2$ be given, $A$ a closed neighborhood of 0 in $\mathbb{R}$ and $B = T^{-1}(A)$.

(i) Let $T(Rm) = \mathbb{R}, m \in M$. Define $P = (T|_{Rm})^{-1}T$.

(ii) Let $T(M) = 0$. Define $P$ as in (a)(ii).

\[ \square \]

References


Department of Mathematics and Statistics, University of Auckland, Auckland, New Zealand

(Received January 14, 1991)