Pseudo-amenability of Brandt semigroup algebras

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Abstract. In this paper it is shown that for a Brandt semigroup $S$ over a group $G$ with an arbitrary index set $I$, if $G$ is amenable, then the Banach semigroup algebra $\ell^1(S)$ is pseudo-amenable.

Keywords: pseudo-amenability, Brandt semigroup algebra, amenable group

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1. Introduction

The concept of amenability for Banach algebras was introduced by Johnson in 1972 [6]. Several modifications of this notion, such as approximate amenability and pseudo-amenability, were introduced in [2] and [4]. In the current paper we investigate the pseudo-amenability of Brandt semigroup algebras. It was shown in [2] and [4] that for the group algebra $L^1(G)$, amenability, approximate amenability and pseudo-amenability coincide and are equivalent to the amenability of locally compact group $G$. In the semigroup case we know that, if $S$ is a discrete semigroup, then amenability of $\ell^1(S)$ implies that $S$ is regular and amenable [1]. Ghahramani et al. [3] have shown that, if $\ell^1(S)$ is approximately amenable, then $S$ is regular and amenable. The present author and Pourabbas in [9] have shown that for a Brandt semigroup $S$ over a group $G$ with an index set $I$, the following are equivalent.

(i) $\ell^1(S)$ is amenable.
(ii) $\ell^1(S)$ is approximately amenable.
(iii) $I$ is finite and $G$ is amenable.

This result corrects [7, Theorem 1.8]. In the present paper we show that for a Brandt semigroup $S$ over a group $G$ with an arbitrary (finite or infinite) index set $I$, amenability of $G$ implies pseudo-amenability of $\ell^1(S)$.

2. Preliminaries

Throughout $\hat{\otimes}$ denotes the completed projective tensor product. For an element $x$ of a set $X$, $\delta_x$ is its point mass measure in $\ell^1(X)$. Also, we frequently use the identification $\ell^1(X \times Y) = \ell^1(X) \hat{\otimes} \ell^1(Y)$ for the sets $X$ and $Y$.

A Banach algebra $A$ is called (approximately) amenable, if for any dual Banach $A$-bimodule $E$, every bounded derivation from $A$ to $E$ is (approximately) inner. It is well known that amenability of $A$ is equivalent to existence of a bounded approximate diagonal, that is a bounded net $(m_i) \in A \hat{\otimes} A$ such that for every
A Banach algebra $A$ is called pseudo-amenable ([4]) if there is a net $(n_i) \in A \hat{\otimes} A$, called an approximate diagonal for $A$, such that $a \cdot n_i - n_i \cdot a \to 0$ and $\pi(n_i)a \to a$ for each $a \in A$.

Let $I$ be a nonempty set and let $G$ be a discrete group. Consider the set $T := I \times G \times I$, add a null element $\emptyset$ to $T$, and define a semigroup multiplication on $S := T \cup \{\emptyset\}$, as follows. For $i, i', j, j' \in I$ and $g, g' \in G$, let

$$(i, g, j)(i', g', j') = \begin{cases} (i, gg', j') & \text{if } j = i', \\ \emptyset & \text{if } j \neq i', \end{cases}$$

also let $\emptyset(i, g, j) = (i, g, j)\emptyset = \emptyset$ and $\emptyset \emptyset = \emptyset$. Then $S$ becomes a semigroup that is called Brandt semigroup over $G$ with index $I$, and usually denoted by $B(I, G)$. For more details see [5].

The Banach space $\ell^1(T)$, with the convolution product,

$$(ab)(i, g, j) = \sum_{k \in I, h \in G} a(i, gh^{-1}, k)b(k, h, j),$$

for $a, b \in \ell^1(T)$, $i, j \in I$, $g \in G$, becomes a Banach algebra. (Note that if $G$ is the one point group, and $I$ is finite, then $\ell^1(T)$ is an ordinary matrix algebra.) We have a closed relation between the Banach algebra $\ell^1(T)$ and the Banach semigroup algebra $\ell^1(S)$:

**Lemma 1.** There exists a homeomorphic isomorphism $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$ of Banach algebras, where the multiplication of $\ell^1(T) \oplus \mathbb{C}$ is coordinatewise.

**Proof:** Consider the following short exact sequence of Banach algebras and continuous algebra homomorphisms:

$$0 \to \ell^1(T) \to \ell^1(S) \to \mathbb{C} \to 0,$$

where the second arrow $\Psi : \ell^1(T) \to \ell^1(S)$ is defined by $\Psi(b)(t) := b(t)$ and $\Psi(b)(\emptyset) := -\sum_{s \in T} b(s)$, for $b \in \ell^1(T)$ and $t \in T \subseteq S$, and the third arrow $\Phi : \ell^1(S) \to \mathbb{C}$ is the integral functional, $\Phi(a) := \sum_{s \in S} a(s)$ ($a \in \ell^1(S)$). Now, let $\Theta : \ell^1(S) \to \ell^1(T)$ be the restriction map, $\Theta(a) := a |_T$. Then $\Theta$ is a continuous algebra homomorphism and $\Theta \Psi = \text{Id}_{\ell^1(T)}$. Thus the exact sequence splits and we have $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$. \hfill \Box

**Lemma 2.** If $\ell^1(T)$ is pseudo-amenable, then so is $\ell^1(S)$.

**Proof:** Suppose that $\ell^1(T)$ is pseudo-amenable. Then by Lemma 1 and [4, Proposition 2.1], $\ell^1(S)$ is pseudo-amenable. \hfill \Box
3. The main result

Let $S, T, G$ and $I$ be as above. We need some other notations and computations:

For $a \in \ell^1(T)$ and every $u, v \in I$, let $a_{(u,v)}$ be an element of $\ell^1(G)$ defined by $a_{(u,v)}(g) := a(u, g, v)$ ($g \in G$). Note that

$$\|a\|_{\ell^1(T)} = \sum_{u,v \in I} \|a_{(u,v)}\|_{\ell^1(G)}.$$  

For $b \in \ell^1(G \times G)$, $c \in \ell^1(G)$ and any $i, j, i', j' \in I$, let $E^b_{(i,j,i',j')}$ and $H^c_{(i,j)}$ be elements of $\ell^1(T \times T)$ and $\ell^1(T)$ respectively, defined by

$$E^b_{(i,j,i',j')}(u, g, v, u', g', v') = \begin{cases} b(g, g') & \text{if } u = i, v = j, u' = i', v' = j', \\ 0 & \text{otherwise}, \end{cases}$$

$$H^c_{(i,j)}(u, g, v) = \begin{cases} c(g) & \text{if } u = i, v = j, \\ 0 & \text{otherwise}, \end{cases}$$

where $u, v, u', v' \in I$ and $g, g' \in G$. Also note that

$$\|E^b_{(i,j,i',j')}\|_{\ell^1(T \times T)} = \|b\|_{\ell^1(G \times G)}, \quad \|H^c_{(i,j)}\|_{\ell^1(T)} = \|c\|_{\ell^1(G)}.$$

For $u, v \in I$ and $g \in G$, the module action of $\ell^1(T)$ on $\ell^1(T \times T)$ becomes

$$\delta_{(u,g,v)} \cdot E^b_{(i,j,i',j')} = \begin{cases} E^b_{(u,j,i',j')} & \text{if } i = v, \\ 0 & \text{if } i \neq v, \end{cases}$$

$$E^b_{(i,j,i',j')} \cdot \delta_{(u,g,v)} = \begin{cases} E^b_{(i,j,i',v)} & \text{if } j' = u, \\ 0 & \text{if } j' \neq u. \end{cases}$$

For the multiplication of $\ell^1(T)$ we have

$$\delta_{(u,g,v)} H^c_{(i,j)} = \begin{cases} H^c_{(u,j)} & \text{if } i = v, \\ H^c_{(u,j)} \delta_{(u,g,v)} & \text{if } i \neq v, \end{cases}$$

$$H^c_{(i,j)} \delta_{(u,g,v)} = \begin{cases} H^c_{(i,v)} & \text{if } j = u, \\ 0 & \text{if } j \neq u. \end{cases}$$

And finally, the diagonal maps $\pi : \ell^1(T \times T) \rightarrow \ell^1(T)$ and $\pi : \ell^1(G \times G) \rightarrow \ell^1(G)$ have the relation

$$\pi(E^b_{(i,j,i',j')}) = \begin{cases} H^c_{(i,j')} & \text{if } j = i', \\ 0 & \text{if } j \neq i'. \end{cases}$$

We are now ready to prove our main result:

**Theorem 3.** Suppose that $G$ is amenable. Then $\ell^1(S)$ is pseudo-amenable.
Proof: Let \((m_\lambda)_{\lambda \in \Lambda} \in \ell^1(G \times G)\) be a bounded approximate diagonal for the amenable Banach algebra \(\ell^1(G)\). For any finite nonempty subset \(F\) of \(I\) and \(\lambda \in \Lambda\), let

\[ W_{F,\lambda} := \frac{1}{\# F} \sum_{i,j \in F} E_{(i,j,j,i)}^{m_\lambda}, \]

where \(\# F\) denotes the cardinal of \(F\). We show that the net \((W_{F,\lambda}) \in \ell^1(T \times T)\) over the directed set \(\Gamma \times \Lambda\), where \(\Gamma\) is the directed set of finite subsets of \(I\) ordered by inclusion, is an approximate diagonal for \(\ell^1(T)\).

For any \(u, v \in I\) and \(g \in G\), by equations (2) and (3), we have,

\[ \delta_{(u,g,v)} \cdot W_{F,\lambda} = \begin{cases} \frac{1}{\# F} \sum_{j \in F} E_{(u,j,j,u)}^{\delta_g \cdot m_\lambda} & \text{if } v \in F, \\ 0 & \text{if } v \notin F, \end{cases} \]

\[ W_{F,\lambda} \cdot \delta_{(u,g,v)} = \begin{cases} \frac{1}{\# F} \sum_{j \in F} E_{(u,j,j,u)}^{m_\lambda \cdot \delta_g} & \text{if } u \in F, \\ 0 & \text{if } u \notin F, \end{cases} \]

and thus,

\[ \delta_{(u,g,v)} \cdot W_{F,\lambda} - W_{F,\lambda} \cdot \delta_{(u,g,v)} = \begin{cases} \frac{1}{\# F} \sum_{j \in F} E_{(u,j,j,u)}^{\delta_g \cdot m_\lambda - m_\lambda \cdot \delta_g} & \text{if } u, v \in F, \\ -\frac{1}{\# F} \sum_{j \in F} E_{(u,j,j,u)}^{m_\lambda \cdot \delta_g} & \text{if } v \in F, u \notin F, \\ -\frac{1}{\# F} \sum_{j \in F} E_{(u,j,j,u)}^{\delta_g \cdot m_\lambda} & \text{if } u \in F, v \notin F, \\ 0 & \text{if } v \notin F, u \notin F. \end{cases} \]

Then, for \(a = \sum_{u,v \in I, g \in G} a(u,g,v)\delta_{(u,g,v)}\) in \(\ell^1(T)\) we have

\[ a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a = \frac{1}{\# F} \sum_{j,u,v \in F} E_{(u,j,j,u)}^{a(u,v) \cdot m_\lambda - m_\lambda \cdot a(u,v)} \]

\[ + \frac{1}{\# F} \sum_{j,v \in F, u \in I-F} E_{(u,j,j,v)}^{a(u,v) \cdot m_\lambda} \]

\[ - \frac{1}{\# F} \sum_{j,u \in F, v \in I-F} E_{(u,j,j,v)}^{m_\lambda \cdot a(u,v)}, \]

and thus, by (1),

\[ \|a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a\| \leq \sum_{u,v \in F} \|a(u,v) \cdot m_\lambda - m_\lambda \cdot a(u,v)\| 

+ \sum_{v \in F, u \in I-F} \|a(u,v) \cdot m_\lambda\| 

+ \sum_{u \in F, v \in I-F} \|m_\lambda \cdot a(u,v)\|. \]
Now, suppose that $M > 0$ is a bound for the norms of $m_\lambda$’s. Let $\epsilon > 0$ be arbitrary, and let $F_0$ be an element of $\Gamma$ such that

$$\sum_{(u,v) \in J_0} |a(u, g, v)| = \sum_{(u,v) \in J_0} \|a(u,v)\| < \epsilon,$$

where $J_0 = (I \times (I - F_0)) \cup ((I - F_0) \times I)$. And choose a $\lambda_0 \in \Lambda$ such that for every $\lambda \geq \lambda_0$,

$$\sum_{u,v \in F_0} \|a(u,v) \cdot m_\lambda - m_\lambda \cdot a(u,v)\| < \epsilon.$$

Now, if $(F, \lambda) \in \Gamma \times \Lambda$ such that $F_0 \subseteq F$, $\lambda \geq \lambda_0$, then we have,

$$\sum_{u,v \in F} \|a(u,v) \cdot m_\lambda - m_\lambda \cdot a(u,v)\| \leq \sum_{u,v \in F_0} \|a(u,v) \cdot m_\lambda - m_\lambda \cdot a(u,v)\|$$

$$+ \sum_{(u,v) \in J_0} \|a(u,v) \cdot m_\lambda\|$$

$$+ \sum_{(u,v) \in J_0} \|m_\lambda \cdot a(u,v)\|$$

$$< \epsilon + \epsilon M + \epsilon M,$$

and analogously,

$$\sum_{v \in F, u \in I - F} \|a(u,v) \cdot m_\lambda\| < \epsilon M$$

and

$$\sum_{u \in F, v \in I - F} \|m_\lambda \cdot a(u,v)\| < \epsilon M.$$

Thus by (6), we have $\|a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a\| < \epsilon + 4\epsilon M$.

Therefore, we proved that $a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a \longrightarrow 0$, for every $a \in \ell^1(T)$.

Now, we prove that $\pi(W_{F,\lambda})a \longrightarrow a$ for any $a \in \ell^1(T)$.

By (5), we have

$$\pi(W_{F,\lambda}) = \frac{1}{\#F} \sum_{i,j \in F} H_{(i,j)}^{\pi(m_\lambda)} = \sum_{i \in F} H_{(i,i)}^{\pi(m_\lambda)}.$$

Thus, (4) implies that

$$\pi(W_{F,\lambda})a = \sum_{i \in F, v \in I} H_{(i,v)}^{\pi(m_\lambda)} a_{(i,v)},$$
since \( a = \sum_{u,v \in I} H_{(u,v)}^{a_{(u,v)}} \). Then we have,

\[
\| \pi(W_{F,\lambda})a - a \| \leq \sum_{i \in F, v \in I} \| H_{(i,v)}^{\pi(m_{\lambda})a_{(i,v)} - a_{(i,v)}} \| + \sum_{v \in I, u \in I - F} \| H_{(u,v)}^{a_{(u,v)}} \|.
\]

(7)

Let \( \epsilon > 0 \) be arbitrary, and let \( F_0 \) and \( J_0 \) be as above. Choose a \( \lambda_1 \in \Lambda \) such that for every \( \lambda \geq \lambda_1 \),

\[
\sum_{i,j \in F_0} \| \pi(m_{\lambda})a_{(i,j)} - a_{(i,j)} \| < \epsilon.
\]

Now, if \( (F, \lambda) \in \Gamma \times \Lambda \) is such that \( F_0 \subseteq F \), \( \lambda \geq \lambda_1 \), then by (1) we have,

\[
\sum_{i \in F, v \in I} \| H_{(i,v)}^{\pi(m_{\lambda})a_{(i,v)} - a_{(i,v)}} \| \leq \sum_{i,j \in F_0} \| \pi(m_{\lambda})a_{(i,j)} - a_{(i,j)} \|
\]

\[
+ \sum_{(u,v) \in J_0} \| \pi(m_{\lambda})a_{(u,v)} \| + \sum_{(u,v) \in J_0} \| a_{(u,v)} \|
\]

\[
< \epsilon + \epsilon M + \epsilon,
\]

and

\[
\sum_{v \in I, u \in I - F} \| H_{(u,v)}^{a_{(u,v)}} \| = \sum_{v \in I, u \in I - F} \| a_{(u,v)} \| < \epsilon.
\]

Thus, by (7) we have

\[
\| \pi(W_{F,\lambda})a - a \| < 3\epsilon + \epsilon M.
\]

This completes the proof. \( \square \)

We end with a natural question:

**Question 4.** Does pseudo-amenability of \( \ell^1(B(I,G)) \) imply amenability of \( G \)?

**References**


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