On Butler $B(2)$-groups
decomposing over two base elements

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Abstract. A $B(2)$-group is a sum of a finite number of torsionfree Abelian groups of rank 1, subject to two independent linear relations. We complete here the study of direct decompositions over two base elements, determining the cases where the relations play an essential role.

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Introduction

In this paper by group we mean torsionfree Abelian group of finite rank.

A Butler $B(n)$-group $G$ is a finite sum of rank 1 groups, $G = \langle g_1 \rangle_* + \cdots + \langle g_m \rangle_*$, subject to $n$ ($\leq m$) independent relations. For basics we refer to [F II]; general results on $B(n)$-groups can be found in [A], [AV], [DVM 10], [DVM 11]; on $B(2)$, besides the decomposition result in [VWW], and the characterization of $B(2)$-groups that are a direct sum of two $B(1)$-groups in [DVM 8], the results from which we move are in [DVM 12], where we studied a particular kind of decomposition “over two base elements”, that mimics the general case for $B(1)$-groups. We gave there necessary and sufficient conditions in two out of three cases, and a counterexample in the third. The third case, which is addressed here, turns out to be the most intricate; a complete solution depends on one of the main open problems for a $B(2)$-group, the determination of its typeset (the set of types of its pure rank one subgroups).

Viewing $G$ as a quotient $X/K$, where $X = R_1x_1 \oplus \cdots \oplus R_mx_m$ and each $R_i \cong \langle g_i \rangle_*$ is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ of type $t_i$, we see that its structure is based on two features: a linear one, provided by the denominator $K$, purely generated in $X$ by the two relations between base elements; and an order-theoretical one, determined by the numerator $X$, i.e. the isomorphism types $t_i$ of the $R_i$, which form the type-base $(t_1, \ldots, t_m)$ of $G$.

We will show that the desired splitting depends on whether a certain type belongs to the typeset of $G$ (Theorem 2.4). This determination will require different procedures, depending on the tent of $G$, an order structure determined by the base types: in some cases, the order structure yields an answer; in other cases, the linear part comes into play (Theorems 1.10 B, 3.3).
The numerous examples will also enlighten the importance of the basic partition of $G$ (showing which base elements are not distinguished by the relations) and its interactions with the regularity of the representation (Examples 3.5 and 3.7).

Throughout, as is usual in this subject, we use as a basic equivalence quasi-isomorphism $[F \Pi]$ instead of isomorphism; we write “isomorphic, indecomposable, direct decomposition, . . . ” instead of “quasi-isomorphic, strongly indecomposable, quasi-direct decomposition, . . . ”.

1. Notation and previous results

Lower case greek letters, with the exception of $\sigma$ and $\tau$, denote rational numbers; $I = \{1, \ldots, m\}$. We will keep notation and tools introduced in our previous papers (in particular [DVM 4], [DVM 11], [DVM 12]); we quote here the essential ones. $T(\wedge, \vee)$ is the lattice of all types (= isomorphism classes of rank 1 groups), with the added maximum $\infty$ for the type of the 0 subgroup; if $g$ is an element of a group $G$, $t_G(g)$ denotes the type in $G$ of the pure subgroup $\langle g \rangle^*$; if $G$ is a Butler group, $\text{typeset}(G) = \{t_G(g) \mid g \in G\}$ is a finite sub-$\wedge$-semilattice of $T$, hence (having $\infty$ as a maximum) a lattice. $P(I)$ denotes the set of parts of $I$; $\mathcal{P}(I)(\vee, \wedge)$ is the lattice of partitions $A = \{A_1, \ldots, A_k\}$ of $I$ under the ordering “bigger = coarser”; blocks $A_i$ of partitions are nonempty by definition; their complements $I \setminus A_i$ are called coblocks.

If $E \subseteq I$ we set

$$E^{-1} = I \setminus E.$$ 

For a group $W = \langle w_1 \rangle^* + \cdots + \langle w_m \rangle^*$, and $E \subseteq I$, let

$$w_E = \sum \{w_i \mid i \in E\},$$

$$W_E = \langle w_i \mid i \in E \rangle^*.$$ 

The partition $C = \{C_1, \ldots, C_h\}$ associated to the element $w = \gamma_1 w_{C_1} + \cdots + \gamma_h w_{C_h}$ of $W$, where $\gamma_i \neq \gamma_j$ for $i \neq j$, is called a partition of $I$ into equal-coefficient blocks for $w$, or shortly a partition of $w$, w.r.t. the elements $w_1, \ldots, w_m$; when these elements are fixed, we set $C = \text{part}_W(w)$.

**Definition 1.1** ([DVM 11]). If $(t_1, \ldots, t_m)$ is a fixed $m$-tuple of types and $E \subseteq I$, set

$$\tau(E) = \bigwedge \{t_i \mid i \in E\},$$

in particular, $\tau(\emptyset) = \infty$; if $\mathcal{E}$ is a set of subsets of $I$ define

$$t(\mathcal{E}) = \bigvee \{\tau(E^{-1}) \mid E \in \mathcal{E}\};$$

the thus defined map $t : \mathcal{P}(\mathcal{P}(I)) \to T$ is called tent (details in [DVM 11]), and $(t_1, \ldots, t_m)$ is its base; we will often call tent the base itself. 

In the following, our $B(2)$-group

$$G = \langle g_1 \rangle^* + \cdots + \langle g_m \rangle^*$$
will be regular, i.e. the base elements \( g_1, \ldots, g_m \) will satisfy (in order to constitute a regular base) the two basic relations:

\[
g_I = g_1 + \cdots + g_m = 0 \quad \text{(the diagonal relation), and} \quad g_0 = \alpha_1 g_{A_1} + \cdots + \alpha_k g_{A_k} = 0 \quad \text{(the second relation)};
\]

here \( k \geq 3 \) (otherwise \( G \) is a direct sum of two \( B(1) \)-groups \([DVM \; 8]\)), and for \( j', j'' \in J = \{1, \ldots, k\} \) we have \( \alpha_{j'} \neq \alpha_{j''} \) iff \( j' \neq j'' \). The partition

\[
A = \{A_1, \ldots, A_k\}
\]

of \( I \) is called the basic partition of \( G \), and its blocks \( A_j \) are called sections. Base elements indexed in the same section are called adjacent. Since in the following we will privilege \( A_1 \), we set

\[
A = A_2 \cup \cdots \cup A_k.
\]

An element \( g \in G \) can be written in many ways as a linear combination of base elements; each such linear combination is a representative of \( g \). If \( G = X/K \), with \( X = R_1x_1 \oplus \cdots \oplus R_mx_m \) and \( K = \langle x_I, x_0 \rangle \), purely generated by the two relations, the representatives of \( g = x + K \) are the elements \( x + y \) of \( X \) with \( y \in K \); when we look for types, w.l.o.g. \( y = \lambda x_I + \mu x_0 \). For representatives of 0 we have

**Lemma 1.2.** If \( 0 = \gamma_1g_{C_1} + \cdots + \gamma_hg_{C_h} \) with \( \gamma_i \neq \gamma_j \) if \( i \neq j \) then either \( h = 1 \) or \( C = A \). \( \square \)

For all \( i \in I \), \( t_i = t_G(g_i) \) is a base type of \( G \), and \( (t_1, \ldots, t_m) \) the type-base of \( G \).

The type in \( G \) of an element \( g = x + K \) is the supremum of the types in \( X \) of its representatives \( x + y \); \( y \in K \). The type in \( X \) of an element \( x = \gamma_1x_1 + \cdots + \gamma_mx_m \) with \( \gamma_i \neq \gamma_j \) if \( i \neq j \) is the infimum of the types of the base elements of \( X \) effectively occurring in \( x \). Setting

\[
supp(x) = \{i \in I \mid \gamma_i \neq 0\} \quad \text{and} \quad Z(x) = \{i \in I \mid \gamma_i = 0\} \quad \text{(the zero-block of } x) \}; \text{we then have} \quad t_X(x) = \tau(\text{supp}(x)) = \tau((Z(x))^{-1}), \text{ hence} \quad t_G(g) = \tau(\{t_X(x + y) \mid y \in K\}) = \tau((Z(x + y))^{-1}) \mid y \in K\}.
\]

We call zero-blocks of \( g \) the zero-blocks of its representatives \( x + y \); \( \text{fam}_G(g) = \{Z(x) \mid x + K = g\} \), the set of zero-blocks of \( g \); \( \text{maxfam}_G(g) \) the set of maximal elements of \( \text{fam}_G(g) \); \( \text{Maxfam}(G) = \{\text{maxfam}_G(g) \mid g \in G\} \).

**Lemma 1.3.** \( t_G(g) = t(\text{fam}_G(g)) = t(\text{maxfam}_G(g)) \);

\( \text{typeset}(G) = t(\text{Maxfam}(G)) \);

thus the elements of \( \text{typeset}(G) \) are suprema of infima of base types. \( \square \)

A useful special case is the following:
Lemma 1.4. If $\emptyset \neq E \subseteq A_1$ we have [DVM 12]:

\[
t_G(g_E) = \tau(E) \vee \tau(E^{-1} \cap A_2^{-1}) \vee \cdots \vee \tau(E^{-1} \cap A_k^{-1}) = \tau(E) \vee [\tau(A_1 \setminus E) \wedge (\vee\{\tau(A_{j'}) \mid j' \neq 1, j \} \mid j = 2, \ldots, k)]
\]

Following [DVM 4] and [DVM 11], without loss of generality we will take the base types to consist of all zeros but a finite number of infinities; they form a finite table (called tent as well) where sections are marked; its columns are also called primes, and we consider each type as a product of its primes: e.g., writing primes instead of infinities and dots instead of zeros:

Example 1.5.

\[
| A_1 & t_1 = p_1 & p_2 & \cdots & \cdots & \cdots \\
& t_2 = & \cdots & p_3 & p_4 & \cdots & \cdots \\
| A_2 & t_3 = & \cdots & p_2 & p_4 & s_3 & \cdots & \cdots \\
| A_3 & t_4 = & \cdots & p_2 & p_3 & \cdots & s_4 & \cdots \\
| A_4 & t_5 = & p_1 & \cdots & p_4 & \cdots & s_5 & \cdots \\
| A_5 & t_6 = & p_1 & \cdots & p_3 & \cdots & \cdots & s_6 \\
\]

(This is in fact the tent of the main counterexample 4.8 in [DVM 12]). The partition $\{A_1, \ldots, A_5\} = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ adds a linear information, i.e. it tells that the second relation is of the form

\[
\alpha_1 g_{\{1, 2\}} + \alpha_2 g_3 + \alpha_3 g_4 + \alpha_4 g_5 + \alpha_5 g_6 = 0
\]

with $\alpha_i \neq \alpha_j$ for $i \neq j$.

If the type $\sigma$ is a product of primes among which there is $p$, we say $p$ divides $\sigma$ ($p \mid \sigma$), or $\sigma$ has the prime $p$; $p$ is a prime of $g$, or divides $g$, if $p$ divides $t_G(g)$. Each prime $p$ has a zero-block $Z(p)$, with $\text{supp}(p) = Z(p)^{-1}$: e.g., $Z(p_1) = \{2, 3, 4\}$, $\text{supp}(s_3) = \{3\}$. For a type $\sigma$, the set of zero-blocks of primes dividing $\sigma$ is

\[
ZB_t(\sigma) = \{Z(p) \mid p \text{ divides } \sigma\}.
\]

A prime $p$ with $Z(p) = I \setminus \{i\}$ (like the primes $s_i$ above) is called a locking prime; by Lemma 1.3 the base type $t_i$ where $p$ occurs (a locked type, e.g. type $t_3$ above) is the only such type, and is bound to belong to every type-base of $G$. Here we will not consider the empty prime (one that does not divide any base type).

Lemma 1.6. (a) (Regularity, [DVM 12]) In the tent of a regular $B(2)$-group there are no primes with only one zero in a section, and all other zeros in another section. If such a situation should occur, the single hole will be filled by the prime (regularization).

(b) The subset $E$ of $I$ contains the zero-block $Z(p)$ if and only if $p$ divides $\tau(E^{-1})$; $p$ is a prime of $g$ (equivalently: $p$ divides $g$, or $g$ covers $p$) if and only if some zero-block $E$ of $g$ contains $Z(p)$; a type $\sigma$ divides $g$ if and only if every set in $ZB_t(\sigma)$ is contained in a set of maxfam$_G(g)$.
We quote now from [DVM 12, Proposition 1.12] a useful description of the pure subgroups \( G_E \) of a \((2)-group \( G \) viewed as \( B(n) \)-groups), when \( E \) or \( E^{-1} \) is contained in a section.

**Proposition 1.7 ([DVM 12]).** Let \( G \) be a \((2)-group, E \subseteq A_j \) for some \( j \in J \). Then

(i) \( G_E = \sum \{ (g_i)_* \mid i \in E \} + (g_{E^{-1}})_* \) is a \((1)-group of rank \( |E| \), with the diagonal relation, and

(ii) \( G_{E^{-1}} = \sum \{ (g_i)_* \mid i \in E^{-1} \} + (g_E)_* \) is a \((2)-group of rank \( |E^{-1}| - 1 \), with the same basic relations as \( G \).

We get now to \((2)-groups decomposing over two base elements.

**Definition 1.8.** Let \( G = G' \oplus G'' \) be a \((2)-group. We say \( G \) splits over \( d \) base elements if exactly \( d \) of its base elements do not belong to the set \( G' \cup G'' \). \( \Box \)

If \( d = 2 \), the following hold:

**Lemma 1.9 ([DVM 12]).** If a \((2)-group splits over 2 base elements then it splits into the direct sum of two \((2)-groups. \( \Box \)

**Theorem 1.10 ([DVM 12]).** Let the \((2)-group \( G \) split over the two base elements \( g_i \) and \( g_j \) \( (i \neq j) \). Then

(1) if \( g_i \) and \( g_j \) are not adjacent, there is a partition \( \{ \{i, j\}, E, F \} \) of \( I \), such that \( G = G_E \oplus G_F \), and we have necessary and sufficient conditions on the tent of \( G \) for this to happen [DVM 12, 3.5];

(2) if \( g_i \) and \( g_j \) are adjacent, let w.l.o.g. \( i = 1, j = 2, \) with \( 1, 2 \in A_1 \). Let \( G = G' \oplus G'', \{\{1, 2\}, E, F\} \) the partition of \( I \) such that \( G' \geq G_E, G'' \geq G_F \). Then \( E \subseteq A_1 \), and one of two situations occurs:

(A) \( G' = G_E \) and \( G'' > G_F \). Here again we have necessary and sufficient conditions on the tent of \( G \) for this splitting [DVM 12, 4.6].

(B) \( G' > G_E \) and \( G'' = G_F \). The tent in Example 1.5 yields two \((2)-groups, one with second relation \( 3g_3 - 3g_4 + 6g_5 - 2g_6 = 0 \), where \( G \) splits over \( g_1 \) and \( g_3 \); another with second relation \( g_3 - g_4 + 2g_5 - 2g_6 = 0 \), where \( G \) does not split. \( \Box \)

Our goal in this paper is to examine Case (B), determining when conditions on the tent suffice to cause the splitting, and when instead the second relation comes into play.

2. The setting

Given \( G = G' \oplus G'' \), decompose accordingly each base element: \( g_i = g_{i}^l + g_i'' \). Set \( E = \{ i \in I \mid g_i^l = 0 \}, F = \{ i \in I \mid g_i'' = 0 \}; \) if \( G \) splits over two base elements, we have \( m = |E| + |F| + 2, \) \( \text{rk}(G) = |E| + |F| \).

Since \( g_i = g_i^l + g_i'' \) implies \( t_i = t_G(g_i^l) \land t_G(g_i'') \), we have \( (g_i)_* = (g_i^l)_* + (g_i'')_* \); then \( G = \sum \{ (g_i^l)_* + (g_i'')_* \mid i \in I \} \), hence \( G' \) is the sum of \( |E| + 2 \) rank 1 groups, \( G'' \) is the sum of \( |F| + 2 \) rank 1 groups.
For case (B), let $G' > G_E$ and $G'' = G_F$. Then $G''$ – in its form as $G_F$ – is by Proposition 1.7 a $B(2)$-group of rank $|F| - 1$ (and since $E \subseteq A_1$, hence $A \subseteq F$, the second relation of $G$ holds in $G_F$). Then $\text{rk}(G') = |E| + 1$ (thus $\text{rk}(G'/G_E) = 1$); since $G'$ is the sum of $|E| + 2$ rank 1 groups, $G'$ is a $B(1)$-group. Again from Proposition 1.7, it is easy to see that

**Proposition 2.1.** In the above setting, $G$ splits over $g_1$ and $g_2$ (with $1, 2 \in A_1$) into $G' \oplus G_F$ if and only if $\{\{1, 2\}, E, F\}$ is a partition of $I$ with $E \subseteq A_1$ such that

1. $G/G_E$ splits over $g_1 + G_E$ and $g_2 + G_E$ into $(G'/G_E) \oplus (G_{E\cup F}/G_E)$;
2. $G_{E\cup F}$ splits over its base element $g_{\{1, 2\}}$ into $G_E \oplus G_F$. □

Conditions (1a) and (1b) are independent: to see that (1b) does not imply (1a), let $G = \langle g_1 \rangle + \cdots + \langle g_5 \rangle$ with second relation $\alpha_4 g_4 + \alpha_5 g_5 = 0$, $A = \{\{1, 2, 3\}, \{4\}, \{5\}\}$, and tent

\[
\begin{align*}
A_1 & \quad t_1 = p \quad . \\
& \quad t_2 = . \quad q \\
& \quad t_3 = . \\
A_2 & \quad t_4 = . \\
A_3 & \quad t_5 = . 
\end{align*}
\]

(hence $G$ is also $B(1)$ with base $(h_1 = g_1, h_2 = g_2, h_3 = g_3, h_4 = \frac{\alpha_5 - \alpha_4}{\alpha_5} g_4)$). For $E = \{3\}$, $F = \{4, 5\} = A$, (1b) holds, since $G_{E\cup F} = \langle h_3, h_4 \rangle* = \langle h_3 \rangle* \oplus \langle h_4 \rangle*$; but $G/G_E$ cannot split over $g_1 + G_E, g_2 + G_E$, whose types are locked. Note also that $G$ has no element of type $t_1 \lor t_2 = p q$.

To see that (1a) does not imply (1b), let $G, E, F$ be as above with tent

\[
\begin{align*}
A_1 & \quad t_1 = p \quad . \quad q \\
& \quad t_2 = p \quad . \\
& \quad t_3 = . \quad r \\
A_2 & \quad t_4 = . \quad q \\
A_3 & \quad t_5 = . \quad q 
\end{align*}
\]

The tent of $G_{E\cup F}$ is

\[
\begin{align*}
A_1 & \quad t_{1,2} = p \quad . \\
& \quad t_3 = . \quad r \\
A_2 & \quad t_4 = . \quad q \\
A_3 & \quad t_5 = . \quad q 
\end{align*}
\]

Here $G_{E\cup F}$ cannot split into $G_E \oplus G_F$, because $t_{1,2}$ is locked; while $G/G_E$ splits over $g_1 + G_E, g_2 + G_E$, since it becomes homogeneous by regularization (see below).

The next corollary makes condition (1b) easy to check first:
**Corollary 2.2.** Condition (1b) is equivalent to

(\*) \quad t_G(g_{\{1,2\}}) \leq t_G(g_E).

Moreover, there is a maximum \( E \subseteq A_1 \setminus \{1, 2\} \) satisfying (\*).

**Proof:** The first assertion is in [DVM 10, 2.4]. Let then \( t' \) be the tent of the \( B(1) \)-group \( G' = G_{A_1} = \sum \{g_i \mid i \in A_1\} + \langle g_{A_1} \rangle \) (Proposition 1.7), with \( t'_i = t_i \) for \( i \in A_1 \), and

\[
t'_0 = t_G(g_{A_1}) = \tau(A_1) \vee \tau(A \setminus A_2) \vee \cdots \vee \tau(A \setminus A_k) \quad \text{(Lemma 1.4)}.
\]

Then a prime \( p \) divides \( t'_0 \) if and only if it either divides \( \tau(A_1) \) (hence its zero-block \( Z'(p) \) is contained in \( A \)), or it divides all but one of the \( \tau(A_j) \) with \( j' \neq 1 \) (that is \( Z'(p) \subseteq A_1 \cup A_j \) for some \( j = 2, \ldots, k \)).

Let \( \text{part}_{\tau'}(t_1 \wedge t_2) = C = \{\{1\}, \{2\}, C_0, \ldots, C_s\} \), where \( s \geq 0 \) and \( C_0 = \{\} \) is the block containing 0. Set

\[ E = \bigcup \{C_i \mid i = 1, \ldots, s\}. \]

We show that \( E \) is maximum satisfying (\*).

Computing types in the \( B(1) \)-group \( G' \) (pure in \( G \)) we have

\[
t_{G'}(g_{\{1,2\}}) = (t_1 \wedge t_2) \vee \tau(C_0 \cup E) = (t_1 \wedge t_2) \vee (\tau(C_0) \wedge \tau(E)),
\]

\[
t_{G'}(g_E) = (t_1 \wedge t_2 \wedge \tau(C_0)) \vee \tau(E).
\]

A prime \( p \) dividing \( g_{\{1,2\}} \) either divides \( \tau(C_0 \cup E) \), hence \( \tau(E) \), hence \( g_E \); or it divides \( t_1 \wedge t_2 \); if \( p \) does not divide \( \tau(E) \), it has a hole – say – in \( C_1 \). But then all of its holes are in \( C_1 \), since \( C = \text{part}_{\tau'}(t_1 \wedge t_2) \); thus it divides \( \tau(C_0) \), hence \( g_E \). Therefore \( t_{G'}(g_{\{1,2\}}) \leq t_{G'}(g_E) \).

To show maximality, let \( E' \subseteq A_1 \setminus \{1, 2\} \) with \( C_0 \cap E' \neq \emptyset \); note that \( 0 \notin E' \), thus \( E' \neq C_0 \). Since \( C_0 \) is a block of \( C \), there must be a prime dividing \( t_1 \wedge t_2 \) (hence \( g_{\{1,2\}} \)) that has a hole in \( E' \) and a hole in \( C_0 \setminus E' \). But then \( p \) does not divide \( \tau(E') \) nor \( \tau(C_0) \), hence does not divide \( t_{G'}(g_E) \); thus (\*) does not hold. \( \square \)

Our check on \( G \) then starts with (1b): when it holds, we can operate a first reduction, modding out \( G_E \): a simple operation, reducing \( I \) to \( \{1, 2\} \cup C_0 \cup A = \{1, 2\} \cup F \).

**Lemma 2.3.** The tent of \( G/G_E \) is obtained from the tent of \( G \) by eliminating the base types indexed in \( E \) and then regularizing.

**Proof:** \( G/G_E = \sum \{\langle g_i \rangle \mid i \in \{1, 2\} \cup F\} \), with the relations of \( G \) inherited by the cosets, hence in particular with the same second relation. A surviving base type \( t_i \) (\( i \notin E \)) might change only if a prime \( p \) that did not divide a base element \( g_i \) will divide \( g_i \): this means that

- no representative \( g_i + \lambda g_{l} + \mu g_0 \) of \( g_i \) covers \( p \), while
- there is a representative \( g_i \), \( g_E \): \( g = g_i + \sum \{\beta_r g_r \mid r \in E\} + \lambda g_{l} + \mu g_0 \), covering \( p \); that is, \( Z(p) \setminus E \subseteq \{i\} \cup A_j \) for some \( j \in J \), with \( i \notin A_j \).
But this is the situation described in Lemma 1.6, where the hole of \( p \) in \( \{ i \} \) will be filled by regularization, hence \( t_i \) would have the prime \( p \), against our hypothesis.

After modding out \( G_E \), the remaining condition (1a) is reduced to the solution of the following

**Theorem 2.4.** Given the \( B(2) \)-group \( G \), let \( I = \{ 1, 2 \} \cup F \) with \( A_1 \supseteq \{ 1, 2 \} \).
Then \( G \) splits over \( g_1 \) and \( g_2 \) into \( \langle g' \rangle \oplus G_F \) if and only if there is an element \( g' \) such that

(a) \( g' = \sum \{ \beta_i g_i \mid i \in I \} \) with \( \beta_1 \neq \beta_2 \);

(b) \( t_G(g') = t_1 \lor t_2 \).

If in particular \( t_1 \leq t_2 \), \( G \) splits over \( g_1 \).

**Proof:** To recover \( g_1 \) linearly inside \( \langle g' \rangle \oplus G_F \) we must have in \( G \) a relation \( \gamma g_1 = \gamma' g' - g'' \), with \( g'' \in G_F \), that is \( \gamma' g' = \gamma g_1 + g'' \), where \( \gamma \neq 0 \): here the coefficient of \( g_2 \) is 0. As a consequence, no representative of \( g' \) has both 1 and 2 in the same zero-block; but since a prime \( p \) not dividing \( t_1 \lor t_2 \) (i.e. not dividing either \( t_1 \) or \( t_2 \)) has \( \{ 1, 2 \} \subseteq Z(p) \), \( p \) cannot be covered by \( g' \); therefore \( t_G(g') \leq t_1 \lor t_2 \).
Finally, to recover the type of \( g_1 \) from \( t_G(g_1) = t_G(\gamma' g' ) \land t_G(g'') \), we must have \( t_G(g') \geq t_1 \); same for \( g_2 \); hence we get \( t_G(g') \geq t_1 \lor t_2 \). \( \square \)

Here is an example showing that condition (a) does not depend from (b):

**Example 2.5.** Let

\[
\begin{array}{ccc}
A_1 & t_1 & = & q \\
 & t_2 & = & p \\
A_2 & t_3 & = & p & q \\
A_3 & t_4 & = & . & . \\
A_4 & t_5 & = & . & . \\
\end{array}
\]

Here the only element of maximum type is \( g_3 \), and all of its representatives have \( \beta_1 = \beta_2 \). \( \square \)

3. **Elements of given type**

Condition (b) asks for the existence in \( G \) of an element of a given type. This is a version of the open question of the determination of typeset(\( G \)). The typeset of a \( B(1) \)-group is the image of the restriction of the map \( t : \mathcal{P}(P(I)) \to \mathbb{T} \) to \( \mathbb{P}(I) \); it is well understood and easy to work on. The typeset of a \( B(2) \)-group is the image of the restriction of the same map \( t : \mathcal{P}(P(I)) \to \mathbb{T} \) to Maxfam(\( G \)), which depends on the linear part of \( G \); the typeset shows no clear structure at this point of research.

What we are required by our problem is, **given a type \( \sigma \) as a product of primes of the tent, whether or not \( \sigma \in \text{typeset}(G) \).** The answer – which always depends on the tent – may in some cases also depend on the linear structure, that is, on the second relation of \( G \).
Starting with our $B(2)$-group $G$, and with a product of primes of $G : \sigma = p_1p_2\ldots p_n$, we look for an element $pg'$ (defined up to a scalar multiple $\rho$, with $g' = \sum \{\beta_i g_i \mid i \in I\}$) of type $\sigma$; that is, such that, for every prime $p_r$ of $\sigma$, there is a representative $pg' + \lambda g_I + \mu g_0$ of $pg'$ whose zero-block contains $Z(p_r)$. We are thus looking for an element

$$
\rho g' + \lambda g_I + \mu g_0 = \\
= \sum \{\rho \beta_i g_i \mid i \in I\} + \lambda \sum \{g_i \mid i \in I\} + \mu \sum \{\alpha_j g_{A_j} \mid j \in J\} = \\
= \sum \{(\rho \beta_i + \lambda + \mu \alpha_{j(i)}) g_i \mid i \in I \cap A_{j(i)}\}
$$

(where $j(i)$ is the index of the section containing $i$) that covers $p_r$ for $r = 1, \ldots, n$; that is,

**Proposition 3.1.** To find an element of type $\sigma$ we must solve the following system of equations (0) in the unknowns $\rho, \lambda_r, \mu_r, \beta_i$:

$$
(0) \quad \rho \beta_i - \lambda_r - \mu_r \alpha_{j(i)} = 0,
$$

for all $i \in Z(p_r)$ and all $r = 1, \ldots, n$, looking for solutions different from the trivial ones ($\rho, \lambda_r = \lambda, \mu_r = \mu, \beta_i \mid r = 1, \ldots, n, \ i = 1, \ldots, m$) which yield the zero element.

This search does not involve primes of the tent not dividing $\sigma$, hence

**Main reduction.** Cancel from the tent all primes not dividing $\sigma$.

Our problem now amounts to asking whether the tent of $G$ – thus reduced – has a proper ($\neq \infty$) maximum $\sigma$. Recalling the definition of the classical fully invariant subgroup $G(\sigma) = \{g \in G \mid t_G(g) \geq \sigma\}$, the question can also be refined by looking for the rank of $G(\sigma)$.

An immediate observation is the following:

**Lemma 3.2.** If a tent has a locked type, either this type is maximum, or there is no maximum.

System (0) can be visualized from the tent, as in the following example:

**Example 1.5 (continued).** Consider the tent of Example 1.5; say we look for $\sigma = p_1p_2p_3p_4$; then with the main reduction we may simplify the tent into:

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$t_4$</th>
<th>$t_5$</th>
<th>$t_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$p_1$</td>
<td>$p_2$</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>$A_2$</td>
<td>.</td>
<td>.</td>
<td>$p_2$</td>
<td>$p_3$</td>
<td>$p_4$</td>
<td>.</td>
</tr>
<tr>
<td>$A_3$</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>$p_2$</td>
<td>$p_3$</td>
<td>.</td>
</tr>
<tr>
<td>$A_4$</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>$p_3$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>
Attach to the tent a \textit{grid} (1) where the holes are represented by dots:

\begin{center}
\begin{tabular}{c|c|c|c|c|c}
 & $P_1$ & $P_2$ & $P_3$ & $P_4$ \\
\hline
$A_1$ & $t_1$ & $\cdot$ & $\cdot$ & $\cdot$ \\
$A_2$ & $t_3$ & $\cdot$ & $\cdot$ & $\cdot$ \\
$A_3$ & $t_4$ & $\cdot$ & $\cdot$ & $\cdot$ \\
$A_4$ & $t_5$ & $\cdot$ & $\cdot$ & $\cdot$ \\
$A_5$ & $t_6$ & $\cdot$ & $\cdot$ & $\cdot$ \\
\end{tabular}
\end{center}

Then, viewing the linear system (0) in the unknowns $\lambda_1, \mu_1, \ldots, \lambda_4, \mu_4$ (although the $\rho\beta$’s should be considered unknowns as well), its matrix is shaped by the above grid in the following way:

\begin{center}
\begin{bmatrix}
\lambda_1 & \mu_1 & \lambda_2 & \mu_2 & \lambda_3 & \mu_3 & \lambda_4 & \mu_4 \\
\rho\beta_1 & 0 & 0 & 0 & 1 & \alpha_1 & 0 & 0 \\
\rho\beta_2 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_1 \\
\rho\beta_2 & 1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\
\rho\beta_3 & 0 & 0 & 1 & \alpha_1 & 0 & 0 & 0 \\
\rho\beta_3 & 1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\
\rho\beta_4 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_2 \\
\rho\beta_4 & 1 & \alpha_3 & 0 & 0 & 0 & 0 & 0 \\
\rho\beta_5 & 0 & 0 & 1 & \alpha_3 & 0 & 0 & 0 \\
\rho\beta_5 & 0 & 0 & 0 & 1 & \alpha_4 & 0 & 0 \\
\rho\beta_6 & 0 & 0 & 1 & \alpha_5 & 0 & 0 & 0 \\
\rho\beta_6 & 0 & 0 & 0 & 0 & 1 & \alpha_5 \\
\end{bmatrix}
\end{center}

E.g., the first dot in the third row refers to the equation $\rho\beta_2 + \lambda_1 + \mu_1\alpha_1 = 0$, which looks for a representative $\rho g' + \lambda_1 g_I + \mu_1 g_0$ that is 0 on the hole that $p_1$ has on $t_2$. We see how the tent (an order structure) determines the system (a linear structure).

We start by looking for conditions independently from the partition, hence we \textit{renote the second relation} as

\[ \alpha_1 g_1 + \cdots + \alpha_m g_m = 0; \]

the section equalities $\alpha_i = \alpha_j$ will then represent additional linear conditions to be satisfied. We have a simple sufficient condition:

**Theorem 3.3.** Let $t$ be a tent where no prime has two holes in the same section, $Z(t)$ the number of holes of $t$. Then a sufficient condition for the existence of an element of maximum type is

\begin{equation}
Z(t) - 2n \leq m - 3.
\end{equation}
Proof: Let the prime $p_r$ have its $z_r = |Z(p_r)|$ holes on – say – $t_1, t_2, \ldots, t_{z_r}$ ($z_r \leq m$); note that, in our hypothesis, $\alpha_1, \alpha_2, \ldots, \alpha_{z_r}$ are pairwise distinct. The part of system (0) determined by $p_r$ is

\[ \begin{align*}
p\beta_1 + \lambda_r + \mu_r \alpha_1 &= 0 \\
p\beta_2 + \lambda_r + \mu_r \alpha_2 &= 0 \\
\vdots & \\
p\beta_{z_r} + \lambda_r + \mu_r \alpha_{z_r} &= 0,
\end{align*} \]

a homogeneous system in the unknowns $p, \lambda_r, \mu_r$ which in our hypothesis has a nonzero solution if and only if the determinants

\[ \left| \begin{array}{cc}
1 & \alpha_1 \\
1 & \alpha_2 \\
1 & \alpha_i
\end{array} \right| = 0 \quad \forall i = 3, \ldots, z_r. \]

We have thus a system of $|Z(p_r)| - 2$ linear equations for the prime $p_r$:

\[ (\alpha_2 - \alpha_i)\beta_1 + (\alpha_i - \alpha_1)\beta_2 + (\alpha_1 - \alpha_2)\beta_i = 0, \quad i = 3, \ldots, z_r \]

homogeneous in the unknowns $\beta$. On the whole, we have $\sum\{|Z(p_r)| - 2 \mid r = 1, \ldots, n\} = |Z(t)| - 2n$ homogeneous linear equations in the $m$ unknowns $\beta_1, \ldots, \beta_m$.

A nonzero solution of this system determines a single representative of our $\rho g^\prime$. In order to have all representatives we need three independent solutions, and this happens whenever $Z(t) - 2n \leq m - 3$. \hfill $\square$

Observation 3.4. In general, system (3) consists of equations in the differences $\alpha_i - \alpha_j$; therefore such will also be the compatibility conditions for the system. There will thus always be solutions, at least those of the form $\alpha_i = \alpha_j$ for suitable $i \neq j$; but this does not mean that, given an $m$-tuple of base types, there will always be second conditions guaranteeing, for the group they define, the existence of a maximum type. The fact is, the sections defined by these conditions may collide with regularity, as shown both in the next and especially in the last example. \hfill $\square$

What may happen if a prime has two holes in the same section is shown in

Example 3.5. Consider the following tent:

\[
\begin{array}{c|c}
A_1 & t_1 \\
\hline
& \cdot \quad q \\
& \cdot \\
A_2 & t_2 \\
A_3 & t_4 \\
& \cdot \quad q \\
& \cdot \\
t_5 & \cdot
\end{array}
\]

Here $Z(t) = 6, n = 2, m = 5$, hence (2) is satisfied (but $p$ has two holes in the same section); $t_4$ is locked, thus by Lemma 3.2 there is no maximum. Observe that, if we replaced the partition $\{\{1, 2\}, \{3\}, \{4, 5\}\}$ of the above tent with the partition $\{\{1\}, \{2, 3\}, \{4\}, \{5\}\}$, removing the obstacle to Theorem 3.3, the numbers in (2) would not change, so the conditions would seemingly be satisfied, provoking an apparent contradiction with Lemma 3.2. What really happens is that the new tent
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\[
\begin{align*}
A_1 & \quad t_1 = . q \\
A_2 & \quad t_2 = . q \\
A_3 & \quad t_3 = . q \\
A_4 & \quad t_4 = p . \\
A_5 & \quad t_5 = . \\
\end{align*}
\]

is not regular any more; by regularity, \(q\) divides both \(t_4\) and \(t_5\), hence \(t_4\) is maximum, as predicted by Lemma 3.2.

A practical rule in the case of a prime with \(\geq 2\) holes in the same section is found in [DVM 14]:

**Proposition 3.6** ([DVM 14]). If, for a prime \(p\), \(E = Z(p) \cap A_j\) for some \(j = 1, \ldots, k\), the maximum of \(G\) can be found in the tent obtained by replacing the types indexed in \(E\) with the type \(\tau(E)\) and regularizing.

We can now conclude the analysis of our initial example:

**Example 1.5 (completed).** Without the partition, the grid (1) becomes

\[
\begin{align*}
A_1 & \quad t_1 & & & \\
A_2 & \quad t_2 & & & \bullet \quad \bullet \\
A_3 & \quad t_3 & & & \bullet \quad \bullet \\
A_4 & \quad t_4 & & & \bullet \quad \bullet \\
A_5 & \quad t_5 & & & \bullet \quad \bullet \\
A_6 & \quad t_6 & & & \bullet \quad \bullet \\
\end{align*}
\]

yielding (since every prime has three holes in different sections) the transposed matrix of system (3):

\[
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5 \\
\beta_6
\end{pmatrix} =
\begin{bmatrix}
0 & 0 & \alpha_3 - \alpha_5 & \alpha_4 - \alpha_6 \\
\alpha_3 - \alpha_4 & \alpha_5 - \alpha_6 & 0 & 0 \\
\alpha_4 - \alpha_2 & 0 & \alpha_5 - \alpha_1 & 0 \\
\alpha_2 - \alpha_3 & 0 & 0 & \alpha_6 - \alpha_1 \\
0 & \alpha_6 - \alpha_2 & \alpha_1 - \alpha_3 & 0 \\
0 & \alpha_2 - \alpha_5 & 0 & \alpha_1 - \alpha_4
\end{bmatrix}
\]

Here \(Z(t) = 12\), \(2n = 8\), \(m = 6\); \(Z(t) - 2n = 4\) is not \(\leq m - 3 = 3\), thus the sufficient condition does not hold.

We need three independent solutions of the system; the rank of the matrix must be \(\leq 3\), which sets a condition on the \(\alpha\)'s. In order to simplify the problem, w.l.o.g. place \(P_1\) in the origin, setting \(\lambda_1 = \mu_1 = 0\), hence \(\beta_2 = \beta_3 = \beta_4 = 0\); we are left with three equations:
\[(\alpha_6 - \alpha_2)\beta_5 + (\alpha_2 - \alpha_5)\beta_6 = 0,\]
\[(\alpha_3 - \alpha_5)\beta_1 + (\alpha_1 - \alpha_3)\beta_5 = 0,\]
\[(\alpha_4 - \alpha_6)\beta_1 + (\alpha_1 - \alpha_4)\beta_6 = 0,\]

whose determinant needs to be 0:

\[
\begin{vmatrix}
0 & \alpha_6 - \alpha_2 & \alpha_2 - \alpha_5 \\
\alpha_3 - \alpha_5 & \alpha_1 - \alpha_3 & 0 \\
\alpha_4 - \alpha_6 & 0 & \alpha_1 - \alpha_4
\end{vmatrix}
= (\alpha_1 - \alpha_3)(\alpha_4 - \alpha_6)(\alpha_2 - \alpha_5) + (\alpha_1 - \alpha_4)(\alpha_3 - \alpha_5)(\alpha_6 - \alpha_2) = 0.
\]

The solution then exists when the 6-tuple \((\alpha_1, \ldots, \alpha_6)\) lies on the above hypersurface of \(PS(5)\) and \(\alpha_i \neq \alpha_j\) whenever \(i \neq j\) are in some \(Z(p_r)\).

Bringing back our initial section \(A_1\) by requiring \(\alpha_2 = \alpha_1\) does not eliminate the condition, which thus remains in place, as we knew from the start. \(\square\)

A last example clarifies Observation 3.4:

**Example 3.7.** Consider a group \(G\) with the following tent:

| \(A_1\) | \(t_1\) | \(\cdot\) | \(\cdot\) | \(p\) | \(q\) |
| \(t_2\) | \(p_1\) | \(p_2\) | \(p_3\) | \(\cdot\) |
| \(A_2\) | \(t_3\) | \(\cdot\) | \(p_2\) | \(p_3\) | \(q\) |
| \(t_4\) | \(p_1\) | \(p_3\) | \(p\) |
| \(A_3\) | \(t_5\) | \(\cdot\) | \(p_2\) | \(p_3\) | \(\cdot\) | \(q\) |
| \(t_6\) | \(p_1\) | \(p_2\) | \(\cdot\) | \(\cdot\) | \(q\) |
| \(A_4\) | \(t_7\) | \(p_1\) | \(p_3\) | \(\cdot\) | \(q\) |
| \(t_8\) | \(p_1\) | \(p_2\) | \(\cdot\) | \(p\) |

(by which we mean that \(\alpha_1 = \alpha_2\), \(\alpha_3 = \alpha_4\), \(\alpha_5 = \alpha_6\), \(\alpha_7 = \alpha_8\) and \(\alpha_1, \alpha_3, \alpha_5, \alpha_7\) are pairwise distinct). We look for an element \(g = \sum \{\beta_i g_i \mid i = 1, \ldots, 8\}\) of maximum type, hence divisible by all the 5 primes above. Starting with \(Z(p)\), we set \(\beta_2 = \beta_3 = \beta_6 = \beta_7 = 0\). Then divisibility by \(p_1\) requires \(\det \begin{vmatrix} 1 \alpha_1 & \beta_1 \\ 1 \alpha_3 & 0 \\ 1 \alpha_5 & \beta_5 \end{vmatrix} = 0\); proceeding analogously for the other primes we obtain the system

\[
\beta_1(\alpha_5 - \alpha_3) + \beta_5(\alpha_3 - \alpha_1) = 0 \quad \text{(for } p_1\text{)}
\]
\[
\beta_1(\alpha_7 - \alpha_3) + \beta_4(\alpha_1 - \alpha_7) = 0 \quad \text{(for } p_2\text{)}
\]
\[
\beta_1(\alpha_7 - \alpha_5) + \beta_8(\alpha_5 - \alpha_1) = 0 \quad \text{(for } p_3\text{)}
\]
\[
\beta_4(\alpha_1 - \alpha_5) + \beta_5(\alpha_3 - \alpha_1) = 0 \quad \text{(for } q\text{)}
\]
\[
\beta_4(\alpha_1 - \alpha_7) + \beta_8(\alpha_3 - \alpha_1) = 0 \quad \text{(for } q\text{)}
\]

which is solvable (in our hypotheses on the \(\alpha\)'s) if and only if

\[(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_5)(\alpha_1 - \alpha_5) = 0,\]

whose solutions are forbidden by our hypotheses: \(G\) has no element of maximum type.
If on the other hand we consider e.g. $\alpha_3 = \alpha_1$, not only the partition but also the base-types change (by regularity) into

$$
\begin{align*}
A_1 & \quad t_1 = \cdot \cdot \cdot p \; q \\
 & \quad t_2 = p_1 \; p_2 \; p_3 \; \cdot \\
 & \quad t_3 = \cdot \; p_2 \; p_3 \; \cdot \; q \\
 & \quad t_4 = p_1 \; \cdot \; p_3 \; p \\
A_3 & \quad t_5 = \mathbf{p_1} \; p_2 \; p_3 \; p \; \cdot \\
 & \quad t_6 = p_1 \; p_2 \; \cdot \; \cdot \; q \\
A_4 & \quad t_7 = p_1 \; \mathbf{p_2} \; p_3 \; \cdot \; q \\
 & \quad t_8 = p_1 \; p_2 \; \cdot \; p \\
\end{align*}
$$

An element of maximum type for such a new group $G'$ would have $\beta_1 = \beta_3 = \beta_4 (= 0)$; divisibility by $p_3$ would require $\beta_8 (\alpha_5 - \alpha_1) = 0$, thus either $\beta_8 = 0$ (no nonzero solution) or again a change of group. In fact, setting $\alpha_5 = \alpha_1$ we get a $G''$ with tent

$$
\begin{align*}
A_1 & \quad t_1 = \cdot \cdot \cdot p \; q \\
 & \quad t_2 = p_1 \; p_2 \; p_3 \; \cdot \\
 & \quad t_3 = \cdot \; p_2 \; p_3 \; \cdot \; q \\
 & \quad t_4 = p_1 \; \cdot \; p_3 \; p \\
 & \quad t_5 = \mathbf{p_1} \; p_2 \; p_3 \; p \; \cdot \\
 & \quad t_6 = p_1 \; p_2 \; \cdot \; \cdot \; q \\
A_4 & \quad t_7 = p_1 \; \mathbf{p_2} \; p_3 \; \cdot \; q \\
 & \quad t_8 = p_1 \; p_2 \; \mathbf{p_3} \; p \; q \\
\end{align*}
$$

which is the tent of a degenerate $B(2)$-group: the direct sum of a rank 1 group (finally, of maximum type) and a $B(1)$-group of rank 5 without maximum.  

\[\Box\]

References


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