**Extraresolvability of balleans**

**I.V. Protasov**

*Abstract.* A ballean is a set endowed with some family of balls in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. We introduce and study a new cardinal invariant of a ballean, the extraresolvability, which is an asymptotic reflection of the corresponding invariant of a topological space.

*Keywords:* ball structure, ballean, resolvability, extraresolvability

*Classification:* 54A25, 54A35, 05A18

1. Uniform spaces and balleans

A ball structure is a triple \(B = (X, P, B)\), where \(X, P\) are nonempty sets and, for any \(x \in X\) and \(\alpha \in P\), \(B(x, \alpha)\) is a subset of \(X\) which is called a ball of radius \(\alpha\) around \(x\). It is supposed that \(x \in B(x, \alpha)\) for all \(x \in X, \alpha \in P\). The set \(X\) is called the *support* of \(B\), \(P\) is called the *set of radii*. Given any \(x \in X, A \subseteq X, \alpha \in P\) we put

\[
B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},
\]

\[
B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha),
\]

\[
B^*(A, \alpha) = \bigcup_{a \in A} B^*(a, \alpha).
\]

A ball structure \(B = (X, P, B)\) is called

- *lower symmetric* if, for any \(\alpha, \beta \in P\), there exist \(\alpha', \beta' \in P\) such that, for every \(x \in X\),

\[
B^*(x, \alpha') \subseteq B(x, \alpha), \; B(x, \beta') \subseteq B^*(x, \beta);
\]

- *upper symmetric* if, for any \(\alpha, \beta \in P\), there exist \(\alpha', \beta' \) such that, for every \(x \in X\),

\[
B(x, \alpha) \subseteq B^*(x, \alpha'), \; B^*(x, \beta) \subseteq B(x, \beta');
\]

- *lower multiplicative* if, for any \(\alpha, \beta \in P\), there exists \(\gamma \in P\) such that, for every \(x \in X\),

\[
B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);
\]
• upper multiplicative if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ballean. Then the family

$$\left\{ \bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P \right\}$$

is a base of entourages for some (unique determined) uniformity on $X$. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on $X$, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, \mathcal{U}) = \{ y \in X : (x, y) \in \mathcal{U} \}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

A ballean is said to be a ballean (or a coarse structure) if it is upper symmetric and upper multiplicative. For motivation to study balleans as the asymptotic counterparts of the uniform topological spaces see [5], [9], [10], [14].

Now we define the mappings which play the part of uniformly continuous mappings on the ballean stage. Let $\mathcal{B}_1 = (X_1, P_1, B_1), \mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec$-mapping if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).$$

A bijection $f : X_1 \to X_2$ is called an asymorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$ if $f$ and $f^{-1}$ are $\prec$-mappings. If $X_1 = X_2$ and the identity mapping $\text{id} : X_1 \to X_2$ is an asymorphism, we identify $\mathcal{B}_1$ and $\mathcal{B}_2$, and write $\mathcal{B}_1 = \mathcal{B}_2$. For each ballean $\mathcal{B} = (X, P, B)$, replacing every ball $B(x, \alpha)$ with $B(x, \alpha) \cap B^*(x, \alpha)$, we get the same ballean. Therefore, in what follows, we assume that $B(x, \alpha) = B^*(x, \alpha)$ for all $x \in X, \alpha \in P$.

Let $\mathcal{B} = (X, P, B)$ be a ballean. We say that $\mathcal{B}$ is connected if, for any two points $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. In what follows, all balleans under consideration are supposed to be connected.

A subset $V \subseteq X$ is called bounded if there exist $x \in X$ and $\alpha \in P$ such that $V \subseteq B(x, \alpha)$. A ballean $\mathcal{B}$ is called bounded if its support is bounded.

For a ballean $\mathcal{B}$ we define a preordering $\leq$ on its set $P$ of radii by the rule: $\alpha \leq \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called cofinal if, for every $\alpha \in P$, there exists $\alpha' \in P$ such that $\alpha \leq \alpha'$. A cofinality $\text{cf}(\mathcal{B})$ of $\mathcal{B}$ is the minimal cardinality of cofinal subsets of $P$.

Every metric space $(X, d)$ determines the metric ballean $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$, where $B_d(x, r) = \{ y \in X : d(x, y) \leq r \}$. A ballean $\mathcal{B}$ is called metrizable if $\mathcal{B}$ is asymorphic to $\mathcal{B}(X, d)$ for some metric space $(X, d)$. By [9, Theorem 9.1], a ballean $\mathcal{B}$ is metrizable if and only if $\text{cf}(\mathcal{B}) \leq \aleph_0$. 
2. Types of subsets of a ballean

Let \( B = (X, P, B) \) be a ballean. We say that a subset \( A \) of \( X \) is

- **large** if there exists \( \alpha \in P \) such that \( X = B(A, \alpha) \);
- **small** if \( X \setminus B(A, \alpha) \) is large for every \( \alpha \in P \);
- **thick** if \( \text{int}(A, \alpha) \neq \emptyset \) for every \( \alpha \in P \), where \( \text{int}(A, \alpha) = \{ x \in X : B(x, \alpha) \subseteq A \} \);
- **extralarge** if \( \text{int}(A, \alpha) \) is large for every \( \alpha \in P \);
- **piecewise large** if there exists \( \beta \in P \) such that \( \text{int}(B(A, \beta), \alpha) \neq \emptyset \) for every \( \alpha \in P \);
- **pseudodiscrete** if, for every \( \alpha \in P \), there exists a bounded subset \( V \) of \( X \) such that \( B(a, \alpha) \cap A = \{ a \} \) for every \( a \in A \setminus V \).

We shall use the following elementary observations from [9, Chapter 12].

1. For a subset \( S \) of \( X \), the following properties are equivalent: \( S \) is small, \( S \) is not piecewise large, \( X \setminus S \) is extralarge, \( (X \setminus S) \cap L \) is large for every large subset \( L \) of \( X \).
2. A subset \( A \) of \( X \) is thick if and only if \( A \cap L \neq \emptyset \) for every large subset \( L \) of \( X \).
3. If the subsets \( X_1, X_2, \ldots, X_n \) of \( X \) are extralarge, then \( X_1 \cap \cdots \cap X_n \) is extralarge. If the subsets \( S_1, \ldots, S_n \) of \( X \) are small, then \( S_1 \cup \cdots \cup S_n \) is small. If a piecewise large subset \( A \) of \( X \) is finitely partitioned \( A = A_1 \cup \cdots \cup A_n \), then at least one cell \( A_i \) of the partition is piecewise large.

These observations give a foundation for the following uniform spaces-balleans vocabulary:

- dense subset
- nowhere dense subset
- subset with nonempty interior
- subset with dense interior
- somewhere dense subset
- discrete subset
- large subset
- small subset
- thick subset
- extralarge subset
- piecewise large subset
- pseudodiscrete subset

Using this vocabulary, we get the following cardinal invariants of a ballean:

\[
\text{density } (B) = \min \{|L| : L \text{ is a large subset of } X\},
\]

\[
\text{cellularity } (B) = \sup \{|F| : F \text{ is a disjoint family of thick subsets of } X\},
\]

\[
\text{spread } (B) = \sup \{|Y|_B : Y \text{ is a pseudodiscrete subset of } X\}, \text{ where } |Y|_B = \min \{|Y \setminus V| : V \text{ is a bounded subset of } X\}.
\]

For interrelations between these invariants see [6], [13].

3. Resolvability

A topological space \( X \) is called resolvable [8] if \( X \) has two disjoint dense subsets. For a cardinal \( \kappa \), a topological space \( X \) is called \( \kappa \)-resolvable [2] if \( X \) contains
\( \kappa \) pairwise disjoint dense subsets. For resolvability of topological spaces and topological groups see the surveys [3], [4], [11].

Given a cardinal \( \kappa \), we say that a ballean \( B = (X, P, B) \) is \( \kappa \)-resolvable if \( X \) can be partitioned to \( \kappa \) large subsets. The resolvability of \( B \) is the cardinal

\[
\text{res}(B) = \sup\{ \kappa : B \text{ is } \kappa\text{-resolvable} \}.
\]

Clearly, \( \text{res}(B) \leq \Delta(B) \), where \( \Delta(B) = \min \{|Y| : Y \text{ is a thick subset of } X\} \).

We say that a subset \( Y \) of \( X \) is \( \kappa \)-crowded if there exists \( \alpha \in P \) such that \( |B(y, \alpha) \cap Y| \geq \kappa \) for every \( y \in Y \). A ballean \( B \) is called \( \kappa \)-crowded if its support \( X \) is \( \kappa \)-crowded. The crowdedness of \( B \) is the cardinal

\[
\text{cr}(B) = \sup\{ \kappa : X \text{ is } \kappa\text{-crowded} \}.
\]

The following two theorems are from [12].

**Theorem 1.** For every ballean \( B = (X, P, B) \), the following statements hold:

(i) if \( B \) is \( \kappa \)-crowded, then \( B \) is \( \kappa \)-resolvable;
(ii) \( \text{cr}(B) \leq \text{res}(B) \leq \text{cr}(B) \cdot \text{cf}(B) \);
(iii) if \( \kappa \) is a finite cardinal and \( B \) is \( \kappa \)-resolvable, then \( B \) is \( \kappa \)-crowded.

**Theorem 2.** Let \( (X, d) \) be a metric space, \( B = B(X, d) \). Then \( \text{res}(B) = \text{cr}(B) \) and \( X \) can be partitioned to \( \text{cr}(B) \) large subsets.

### 4. Extraresolvability

A topological space \( X \) is called extraresolvable [7], [1], if there exists a family \( F \) of dense subsets of \( X \) such that \( |F| > \Delta(X) \), where \( \Delta(X) = \min\{|U| : U \text{ is a nonempty open subset of } X\} \), and \( F \cap F' \) is nowhere dense whenever \( F, F' \in F \) and \( F \neq F' \).

Given a cardinal \( \kappa \), we say that a ballean \( B = (X, P, B) \) is \( \kappa \)-extraresolvable if there exists a family \( F \) of large subsets of \( X \) such that \( |F| = \kappa \) and \( F \cap F' \) is small whenever \( F, F' \in F \) are distinct elements of \( F \). The extraresolvability of \( B \) is the cardinal

\[
\text{exres}(B) = \sup\{ \kappa : B \text{ is } \kappa\text{-extraresolvable} \}.
\]

Clearly, \( \text{res}(B) \leq \text{exres}(B) \). We note also that \( \text{res}(B) = \text{exres}(B) = |X| \) for every bounded ballean \( B \).

**Lemma 1.** Let \( B = (X, P, B) \) be a ballean, \( \alpha \in P, n \in \mathbb{N} \). Assume that there exists a piecewise large subset \( Y \) of \( X \) such that the family \( \{B(y, \alpha) : y \in Y\} \) is disjoint and \( |B(y, \alpha)| \leq n \) for every \( y \in Y \). If \( F_1, \ldots, F_{n+1} \) are subsets of \( X \) such that \( B(F_i, \alpha) = X, i \in \{1, \ldots, n+1\} \), then there exist distinct \( i, j \in \{1, \ldots, n+1\} \) such that \( F_i \cap F_j \) is piecewise large.
Proof: We assume on the contrary that each subset $F_i \cap F_j$ is small and put

$$Z = \bigcup_{i \neq j} (F_i \cap F_j).$$

Then $Z$ is small and $(F_i \setminus Z) \cap (F_j \setminus Z) = \emptyset$, $i \neq j$. Since $Y$ is piecewise large and $Z$ is small, there exists $y \in Y$ such that $B(y, \alpha) \cap Z = \emptyset$. Since $B(F_i, \alpha) = X$, we have $F_i \cap B(y, \alpha) \neq \emptyset$. Since $|B(y, \alpha)| \leq n$, there are distinct $i, j \in \{1, \ldots, n+1\}$ such that $F_i \cap F_j \cap B(y, \alpha) \neq \emptyset$. We take any $y' \in F_i \cap F_j \cap B(y, \alpha)$. Then $y' \in Z$ contradicting to the choice of $y$.

**Theorem 3.** Let $\mathcal{B} = (X, P, B)$ be a ballean. If $\text{cr}(\mathcal{B})$ is finite then

$$\text{cr}(\mathcal{B}) = \text{res}(\mathcal{B}) = \text{exres}(\mathcal{B}).$$

Proof: Let $\text{cr}(\mathcal{B}) = n$. By Theorem 1(i), $n \leq \text{res}(\mathcal{B})$. Let $F_1, \ldots, F_{n+1}$ be distinct large subsets of $X$. We show that $F_i \cap F_j$ is piecewise large for some distinct $i, j \in \{1, \ldots, n+1\}$ so $\text{exres}(\mathcal{B}) \leq n$. For every $\alpha \in P$, we put

$$X_\alpha = \{x \in X : |B(x, \alpha)| \leq n\}.$$

We assume that $X_\alpha$ is small for some $\alpha \in P$. Then $X \setminus B(X_\alpha, \alpha)$ is large. We pick $\beta \in P$ such that

$$\bigcup\{B(x, \beta) : x \in X \setminus B(X_\alpha, \alpha)\} = X.$$

Then we choose $\gamma \in P$ such that $B(B(x, \beta), \beta) \subseteq B(x, \gamma)$ for each $x \in X$. We take an arbitrary $y \in X$ and choose $x \in X \setminus B(X_\alpha, \alpha)$ such that $y \in B(x, \beta)$. Then $B(x, \beta) \subseteq B(y, \gamma)$. It follows that $|B(y, \gamma)| > n$ for every $y \in X$ contradicting $\text{cr}(\mathcal{B}) = n$. Hence, $X_\alpha$ is piecewise large for every $\alpha \in P$.

We choose $\alpha \in P$ such that $B(F_i, \alpha) = X$ for each $i \in \{1, \ldots, n+1\}$. Then we take a subset $Y \subseteq X_\alpha$ such that the family $\{B(y, \alpha) : y \in Y\}$ is maximal disjoint. By the above paragraph, $X_\alpha$ is piecewise large so $Y$ is piecewise large. Since $|B(y, \alpha)| \leq n$ for each $y \in Y$, we can apply Lemma 1 to conclude that $F_i \cap F_j$ is not small for some distinct $i, j \in \{1, \ldots, n+1\}$.

Let $\mathcal{B} = (X, P, B)$ be a ballean. Given any $\alpha \in P$ and a cardinal $\kappa$, we put

$$X(\alpha, \kappa) = \{x \in X : |B(x, \alpha)| \leq \kappa\}.$$
**Lemma 2.** Let $\mathcal{B} = (X, P, B)$ be an unbounded ballean. Assume that, for every $\alpha \in P$, there exists a natural number $n$ such that $X(\alpha, n)$ is piecewise large. Let $\mathcal{F}$ be a family of large subsets of $X$ such that $|\mathcal{F}| > \text{cf}(\mathcal{B})$. Then there exists an infinite subfamily $\mathcal{F}'$ of $\mathcal{F}$ such that $F \cap F'$ is piecewise large for all $F, F' \in \mathcal{F}'$. In particular, $\text{exres}(\mathcal{B}) \leq \text{cf}(\mathcal{B})$.

**Proof:** Let $P'$ be a cofinal subset of $P$ such that $|P'| = \text{cf}(\mathcal{B})$. For every $\alpha \in P$, we put

$$\mathcal{F}_\alpha = \{ F \in \mathcal{F} : B(F, \alpha) = X \}.$$  

Then $\mathcal{F} = \bigcup_{\alpha \in P'} \mathcal{F}_\alpha$. If every family $\mathcal{F}_\alpha$, $\alpha \in P'$, is finite then $|\mathcal{F}| \leq \text{cf}(\mathcal{B})$ contradicting the assumption. Hence, there exists $\alpha \in P'$ such that $\mathcal{F}_\alpha$ is infinite. We choose a natural number $n$ such that $X(\alpha, n)$ is piecewise large. Let $Y$ be a subset of $X(\alpha, n)$ such that the family $\{ B(y, \alpha) : y \in Y \}$ is maximal disjoint. Since $X(\alpha, n)$ is piecewise large, $Y$ is piecewise large. Let $F_1, \ldots, F_n + 1 \in \mathcal{F}_\alpha$. By Lemma 1, there are distinct $i, j \in \{1, \ldots, n + 1\}$ such that $F_i \cap F_j$ is piecewise large.

We consider a complete graph $\Gamma$ with the set of vertices $\mathcal{F}_\alpha$. We color an edge $\{F, F'\}$ of $\Gamma$ in yellow if $F \cap F'$ is piecewise large, otherwise we color this edge in blue. By Ramsey theorem, there exists an infinite subfamily $\mathcal{F}'$ of $\mathcal{F}_\alpha$ such that the complete subgraph $\Gamma'$ determined by $\mathcal{F}'$ is monochrome. By above paragraph, $\Gamma'$ must be yellow. Hence, $F \cap F'$ is piecewise large for all $F, F' \in \mathcal{F}'$. \hfill $\square$

**Theorem 4.** Let $(X, d)$ be an unbounded metric space, $\mathcal{B} = \mathcal{B}(X, d)$. Assume that $\text{cr}(\mathcal{B}) = \aleph_0$ and, for every natural number $m$, there exists a natural number $n$ such that $X(m, n)$ is piecewise large. Then

$$\text{res}(\mathcal{B}) = \text{exres}(\mathcal{B}) = \aleph_0.$$  

**Proof:** By Lemma 2, $\text{exres}(\mathcal{B}) \leq \aleph_0$. By Theorem 2, $\text{res}(\mathcal{B}) = \aleph_0$. \hfill $\square$

**Corollary.** Let $(X, d)$ be an unbounded metric space, $\mathcal{B} = \mathcal{B}(X, d)$. Assume that $\text{cr}(\mathcal{B}) = \aleph_0$ and, for every natural number $m$, there exists a natural number $n$ such that $|B(x, m)| \leq n$ for every $x \in X$. Then

$$\text{res}(\mathcal{B}) = \text{exres}(\mathcal{B}) = \aleph_0.$$  

**Lemma 3.** Let $\mathcal{B} = (X, P, B)$ be a ballean, $Y$ be a countable large subset of $X$, $\kappa$ be an infinite cardinal. Assume that there exists $\alpha \in P$ such that the family $\{ B(y, \alpha) : y \in Y \}$ is disjoint and $|B(y, \alpha)| \geq \kappa$ for each $y \in Y$. Then there exists a disjoint family $\mathcal{F}$ of countable subsets of $X$ such that $|\mathcal{F}| = \kappa^{\aleph_0}$ and $F \cap F'$ is finite for all distinct $F, F' \in \mathcal{F}$. In particular, $\text{exres}(\mathcal{B}) \geq \kappa^{\aleph_0}$.

**Proof:** Let $Y = \{ y_n : n \in \omega \}$. For each $n \in \omega$, we choose some subset $Z_n = \{ z(n, \gamma_0, \gamma_1, \ldots, \gamma_n) : \gamma_i \in \kappa \}$ of distinct elements from $B(y_n, \alpha)$. Then we fix
some symbol \( v \not \in X \) and construct a homogeneous tree \( T \) of local degree \( \kappa \) with the root \( v \) and the set of vertices \( \{ v \} \cup \bigcup_{n \in \omega} Z_n \). At the first step we connect \( v \) with all the vertices \( \{ z(0, \gamma_0) : \gamma_0 \in \kappa \} \). At the second step we connect each vertex \( z(0, \gamma_0) \) with all the vertices \( \{ z(1, \gamma_0, \gamma_1) : \gamma_1 \in \kappa \} \). At the third step we connect each vertex \( z(1, \gamma_0, \gamma_1) \) with all the vertices \( \{ z(2, \gamma_0, \gamma_1, \gamma_2) : \gamma_2 \in \kappa \} \), and so on. After \( \omega \) steps we get the tree \( T \). Each ray in \( T \) starting at \( v \) determines a subset \( F \) of \( X \) consisting of all vertices on this ray except \( v \). We denote by \( \mathcal{F} \) the family of all obtained subsets. By the construction of \( T \), we have \( F \cap B(y, \alpha) \neq \emptyset \) for all \( y \in Y, F \in \mathcal{F} \). Since \( Y \) is large, each member of \( \mathcal{F} \) is large. Clearly, \( |\mathcal{F}| = \kappa^{\aleph_0} \) and \( F \cap F' \) is finite for all distinct \( F, F' \in \mathcal{F} \).

**Lemma 4.** Let \((X, d)\) be a metric space, \( \mathcal{B} = \mathcal{B}(X, d) \). Assume that there exists \( n \in \mathbb{N} \) such that \( X(n, k) \) is small for each \( k \in \mathbb{N} \), and \( \bigcup_{k \in \mathbb{N}} X(n, k) \) is large. Then \( \text{exres}(\mathcal{B}) \geq 2^{\aleph_0} \).

**Proof:** We put \( Y = \bigcup_{k \in \mathbb{N}} X(n, k) \) and choose a subset \( Z \) of \( Y \) such that the family \( \{ B(z, n) : z \in Z \} \) is maximal disjoint. Clearly, \( Z \) is large. For every \( k \in \mathbb{N} \), we put \( Z_k = \{ z \in Z : |B(z, n)| = k \} \).

Since each subset \( X(n, k), k \in \mathbb{N} \) is small and \( Z \) is large, for every \( m \in \omega \), there exists \( k \in \mathbb{N} \) such that \( Z_k \neq \emptyset \) and \( k > m \). Hence, we can choose an increasing sequence \( (k_m)_{m \in \omega} \) of natural numbers such that \( Z_{k_m} \neq \emptyset \) and \( k_m \geq 2^m \).

For each \( m \in \omega \), we choose the pairwise disjoint subsets \( Z(m, 1), Z(m, 2), \ldots, Z(m, 2^m) \) of \( X \) such that for each \( z \in Z_{k_m} \cup \cdots \cup Z_{k_{m+1}-1}, \) we have

\[
|B(z, m) \cap Z(m, i)| = 1, \quad Z(m, i) \subseteq B(Z_{k_m} \cup \cdots \cup Z_{k_{m+1}-1}, n), \quad i \in \{1, \ldots, 2^m\}.
\]

Then we construct a binary tree \( T \) with the root \( Z(0, 1) \) and the set of vertices \( \{ Z(m, i) : m \in \omega, i \in \{1, \ldots, 2^m\} \} \). We define the edges of \( T \) as follows. At the first step we define the edges \( \{ Z(0, 1), Z(1, 1) \}, \{ Z(0, 1), Z(1, 12) \}. \) At the second step we define the edges

\[
\{ Z(1, 1), Z(2, 1) \}, \{ Z(1, 1), Z(2, 2) \}
\]

\[
\{ Z(1, 2), Z(2, 3) \}, \{ Z(1, 2), Z(2, 4) \},
\]

and so on.

Every ray in \( T \) starting at the root \( Z(0, 1) \) determines a subset \( S \) of \( X \) which is a union of all vertices of \( T \) (as the subsets of \( X \)) on this ray. By the construction of \( T \), \( S \) is large and the intersection of any two distinct subsets \( S, S' \) of this form is small. Since there are \( 2^{\aleph_0} \) distinct rays in \( T \), we conclude \( \text{exres}(\mathcal{B}) \geq 2^{\aleph_0} \).  

For a ballean \( \mathcal{B} = (X, P, B) \) and a subset \( Y \) of \( X \), we put \( \mathcal{B}_Y = (Y, P, B_Y) \), where \( B_Y(y, \alpha) = B(y, \alpha) \cap Y \).
Theorem 5. Let \((X, d)\) be an unbounded countable metric space, \(\mathcal{B} = \mathcal{B}(X, d)\). Assume that there exists \(n \in \mathbb{N}\) such that \(X(n, k)\) is small for each \(k \in \mathbb{N}\). Then \(\text{exres}(\mathcal{B}) = 2^{\aleph_0}\).

Proof: We put \(Y = \bigcup_{k \in \mathbb{N}} X(n, k)\), \(Z = X \setminus Y\) and construct a family \(\mathcal{F}'\) of large subsets of \(X\) such that \(|\mathcal{F}'| = 2^{\aleph_0}\) and \(F \cap F'\) is small for all distinct \(F, F' \in \mathcal{F}'\). We consider three cases.

Case \(\mathcal{B}_Y\) is bounded, \(\mathcal{B}_Z\) is unbounded. By Lemma 3, there exists a family \(\mathcal{F}, |\mathcal{F}| = 2^{\aleph_0}\), of large subsets of \(\mathcal{B}_Z\) such that \(F \cap F'\) is finite for all distinct \(F, F' \in \mathcal{F}\). We put \(\mathcal{F}' = \{Y \cup F : F \in \mathcal{F}\}\).

Case \(\mathcal{B}_Z\) is bounded, \(\mathcal{B}_Y\) is unbounded. By Lemma 4, there exists a family \(\mathcal{F}, |\mathcal{F}| = 2^{\aleph_0}\), of large subsets of \(\mathcal{B}_Y\) such that \(F \cap F'\) is small for all distinct \(F, F' \in \mathcal{F}\). We put \(\mathcal{F}' = \{Z \cup F : F \in \mathcal{F}\}\).

Case \(\mathcal{B}_Y, \mathcal{B}_Z\) are unbounded. Applying Lemmas 3 and 4 to \(\mathcal{B}_Z\) and \(\mathcal{B}_Y\), we get two corresponding families \(\mathcal{F}_1, \mathcal{F}_2, |\mathcal{F}_1| = |\mathcal{F}_2| = 2^{\aleph_0}\) of large subsets of \(\mathcal{B}_Z\) and \(\mathcal{B}_Y\). Let \(\mathcal{F}_1 = \{F_\lambda : \lambda \in 2^{\aleph_0}\}, \mathcal{F}_2 = \{F'_\lambda : \lambda \in 2^{\aleph_0}\}\). We put \(\mathcal{F}' = \{F_\lambda \cup F'_\lambda : \lambda \in 2^{\aleph_0}\}\).

It follows from Theorems 3, 4, 5 that, for a countable metric space \((X, d)\), \(\text{exres}(\mathcal{B}(X, d))\) could be either a natural number, or \(\aleph_0\), or \(2^{\aleph_0}\). It is easy to construct an example for each case.

Theorem 6. Let \(\kappa\) be an infinite regular cardinal such that \(2^\gamma < \kappa\) for each cardinal \(\gamma < \kappa\). Then there exists a balleans \(\mathcal{B} = (X, P, B)\) such that \(|X| = \kappa, \text{res}(\mathcal{B}) = \kappa\) and \(\text{exres}(\mathcal{B}) = 2^\kappa\).

Proof: We denote by \(S\) the family of all subsets of \(X\) of cardinality \(< \kappa\). Let \(P\) be the set of all mappings \(f : X \to S\) such that, for every \(x \in X\), we have \(x \in f(x)\) and

\(|\{y \in X : x \in f(y)\}| < \kappa.\)

Given any \(x \in X\) and \(\alpha \in P\), we put \(B(x, f) = f(x)\) and consider the ball structure \(\mathcal{B} = (X, P, B)\). Since \(B^*(x, f) = \{y \in X : x \in f(y)\}\), \(\mathcal{B}\) is upper symmetric. Since \(\kappa\) is regular, \(|B(B(x, f), g)| < \kappa\) so \(\mathcal{B}\) is upper multiplicative. Hence, \(\mathcal{B}\) is a balleans.

Let \(Y\) be a subset of \(X\). If \(|Y| < \kappa\), by regularity of \(\kappa\), we conclude that \(Y\) is bounded so \(Y\) is small. We assume that \(|Y| = \kappa\) and show that \(Y\) is large. We choose a subset \(Z\) of \(Y\) such that \(|Z| = |X \setminus Y|\) and fix some bijection \(g : Z \to X \setminus Y\). Then we define \(f \in P\) by the rule: \(f(x) = \{x, g(x)\}\) for each \(x \in Z\), and \(f(x) = \{x\}\) for each \(x \in X \setminus Z\). Then \(B(Y, f) = X\).

To conclude the proof, it suffices to point out a family \(\mathcal{F}\) of subsets of \(X\) such that \(|\mathcal{F}| = 2^\kappa\) and \(|F \cap F'| < \kappa\) for all distinct \(F, F' \in \mathcal{F}\). To this end we use the standard construction. By the assumption \(2^\gamma < \kappa, \gamma < \kappa\), we identify \(X\) with the set of vertices of the binary tree \(T\) of height \(\kappa\), and denote by \(\mathcal{F}\) the family of all rays starting at the root of \(T\).
References


**Department of Cybernetics, Kiev University, Volodimirskaya 64, Kiev 01033, Ukraine**

*E-mail*: protasov@unicyb.kiev.ua

(Received May 14, 2006, revised August 21, 2006)