Cardinal inequalities implying maximal resolvability

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Abstract. We compare several conditions sufficient for maximal resolvability of topological spaces. We prove that a space $X$ is maximally resolvable provided that for a dense set $X_0 \subset X$ and for each $x \in X_0$ the $\pi$-character of $X$ at $x$ is not greater than the dispersion character of $X$. On the other hand, we show that this implication is not reversible even in the class of card-homogeneous spaces.

Keywords: maximally resolvable space, base at a point, $\pi$-base, $\pi$-character

Classification: 54A10, 54A25

1. Preliminaries

The paper is a continuation of studies in [BT]. We will use the following notation (see e.g. [Ho], [J]). As usual, $|X|$ denotes the cardinality of $X$ and let $|\mathbb{R}| = c$. Suppose $(X, \mathcal{T})$ is a topological space. Then

- $w(X)$ denotes the weight of $X$: $w(X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a base of } X\}$,
- $\Delta(X)$ – the dispersion character of $X$: $\Delta(X) = \min\{|U|: U \in \mathcal{T} \setminus \{\emptyset\}\}$,
- $\chi(X, x)$ – the character of a space $X$ at a point $x$: $\chi(X, x) = \min\{|\mathcal{B}(x)|: \mathcal{B}(x) \text{ is a base of } X \text{ at } x\}$,
- $\chi(X)$ – the character of $X$: $\chi(X) = \sup\{\chi(X, x): x \in X\}$,
- $\pi w(X)$ – the $\pi$-weight of $X$: $\pi w(X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a } \pi\text{-base of } X\}$,
- $\pi\chi(X, x)$ – the $\pi$-character of a space $X$ at a point $x$: $\pi\chi(X, x) = \min\{|\mathcal{B}|: \mathcal{B} \subset \mathcal{T} \setminus \{\emptyset\} \land \forall U \in \mathcal{T}, x \in U \Rightarrow \exists B \in \mathcal{B} B \subset U\}$,
\bullet \pi \chi (X) - the \pi-character of \:
\pi \chi (X) = \sup \{ \pi \chi (X, x) : x \in X \}.

Let \kappa be a cardinal greater than 1. We say that \: X is \kappa-resolvable if it can
be decomposed into \kappa pairwise disjoint dense subsets; \: X is called maximally
resolvable (in short MR(X)) if it is \Delta(X)-resolvable (see [CGF], [B]); \: X is called
cardinality-homogeneous (card-homogeneous, shortly) if \Delta(X) = |X|.

All considered spaces are dense-in-itself. We study the following properties of
a space \:
\begin{align*}
P(X) & : \: w(X) \leq \Delta(X); \\
P'(X) & : \: \chi(X) \leq \Delta(X); \\
P''(X) & : \exists X_0 \subset X \left( \text{cl}(X_0) = X \land \forall x \in X_0 \left( \chi(X, x) \leq \Delta(X) \right) \right); \\
P_\pi(X) & : \: \pi w(X) \leq \Delta(X); \\
P'_\pi(X) & : \: \pi \chi(X) \leq \Delta(X); \\
P''_\pi(X) & : \exists X_0 \subset X \left( \text{cl}(X_0) = X \land \forall x \in X_0 \left( \pi \chi(X, x) \leq \Delta(X) \right) \right).
\end{align*}

Some of those conditions were considered in connection with resolvability of \: X. For example, the following facts were proved:

**Fact 1** ([CGF]). \: If a topological space \: X is card-homogeneous then \: P(X) implies MR(X).

**Fact 2** ([CGF], [B]). \: If \: X is card-homogeneous then \: P_\pi(X) implies MR(X).

**Fact 3** ([BT]). \: If \: X is card-homogeneous then \: P''_\pi(X) implies MR(X).

It is clear that the statement \: P''_\pi(X) is the most general among considered
conditions. The aim of this note is to show that \: P''_\pi(X) implies MR(X), and
that MR(X) does not imply \: P_\pi(X) even for card-homogeneous spaces. These
theorems will be proved in the final sections of the paper. We start with some
construction and next we compare the introduced properties.

2. Small ideals with big cofinality

Let \: \kappa be an infinite cardinal. For \: E \subset \kappa define \: 1E = E and \: (-1)E = \kappa \setminus E.
A family \: \mathcal{A} \subset \mathcal{P}(\kappa) is called strongly independent if \: | \bigcap_{i=0}^{m} E_i | = \kappa
for any sequence \: E_0, \ldots, E_m of distinct elements of \: \mathcal{A} and any sequence \: \varepsilon_0, \ldots, \varepsilon_m
of numbers from \: \{-1, 1\}. A theorem by Fichtenholz, Kantorovitch and Hausdorff
(see [M]) states that there exists a strongly independent family \: \mathcal{A} \subset \mathcal{P}(\kappa)
of cardinality \: 2^\kappa. A family \: \mathcal{F} \subset \mathcal{P}(\kappa) is called a base of an ideal \: \mathcal{I} \subset \mathcal{P}(\kappa)
if \: \mathcal{F} \subset \mathcal{I} and each set \: A \in \mathcal{I} is contained in a set \: B \in \mathcal{F}. The cardinal \: \text{cf}(\mathcal{I})
stands for the minimal cardinality of a base of \: \mathcal{I}. 
Theorem 4. For each infinite cardinal \( \kappa \) there is an ideal \( I \subset P(\kappa) \) such that \( \bigcup I = \kappa \) and \( \text{cf}(I) = 2^\kappa \).

Proof: Consider a strongly independent family \( A \subset P(\kappa) \) of cardinality \( 2^\kappa \) and let \( I \subset P(\kappa) \) stand for the ideal generated by \( A \). (Thus \( I = \{ F \subset \bigcup B : B \in [A]^{<\omega} \} \), where \([A]^{<\omega}\) denotes the family of all finite subsets of \( A \).) We may assume that \( \bigcup A = \kappa \) (adding \( \kappa \setminus \bigcup A \) to one of the sets from \( A \)). Thus \( \bigcup I = \kappa \).

Suppose that \( F \) is a base of \( I \) such that \( |F| = \lambda \) and \( \omega \leq \lambda < 2^\kappa \). For each \( F \in F \) pick a family \( A_F \in [A]^{<\omega} \). Thus \( |\bigcup F \in F A_F| \leq \lambda \) and since \( |A| = 2^\kappa > \lambda \), we can find an \( A_* \in A \setminus \bigcup F \in F A_F \). Pick an \( F_* \in F \) such that \( A_* \subset F_* \). Hence \( A_* \subset F_* \subset \bigcup A_{F_*} \). On the other hand, by the strong independence of \( A \), we have

\[
|A_* \setminus \bigcup A_{F_*}| = |A_* \cap \bigcap_{A \in A_{F_*}} (-1)A| = \kappa,
\]

a contradiction. \( \square \)

For an ideal \( I \subset P(X) \) and \( Y \subset X \) denote \( I \mid Y = \{ A \cap Y : A \in I \} \).

Corollary 5. There is an ideal \( I \subset P(\mathbb{R}) \) such that \( \bigcup I = \mathbb{R} \), \( I \) consists of nowhere dense subsets of \( \mathbb{R} \) and \( \text{cf}(I \mid C) = 2^c \) for each perfect set \( C \subset \mathbb{R} \).

Proof: Let \( C_\alpha, \alpha < c \), be an enumeration of all nowhere dense perfect subsets of \( \mathbb{R} \). By a Bernstein-type construction we find a family \( \{ B_\alpha : \alpha < c \} \) of pairwise disjoint sets such that \( \bigcup_{\alpha < c} B_\alpha = \mathbb{R} \) and \( B_\alpha \subset C_\alpha \), \( |B_\alpha| = \mathfrak{c} \) for each \( \alpha < c \). By Theorem 4, for each \( \alpha < c \) pick an ideal \( I_\alpha \subset P(B_\alpha) \) with \( \text{cf}(I_\alpha) = 2^c \). Let \( I \) consist of all sets \( A \subset \mathbb{R} \) such that \( A \cap B_\alpha \in I_\alpha \) for each \( \alpha < c \). So \( I \mid B_\alpha = I_\alpha \) and thus \( \text{cf}(I \mid B_\alpha) = 2^c \) (hence \( \text{cf}(I \mid C_\alpha) = 2^c \)) for all \( \alpha < c \). \( \square \)

3. Relationships between considered properties

Theorem 6. For any dense-in-itself topological space \( X \) the following implications hold

\[
\begin{array}{cccc}
P(X) & \longrightarrow & P'(X) & \longrightarrow \ P''(X) \\
\downarrow & & \downarrow & \downarrow \\
P_\pi(X) & \longrightarrow & P'_\pi(X) & \longrightarrow \ P''_\pi(X)
\end{array}
\]

Moreover, all considered implications are not reversible.

Proof: All implications considered in Theorem 6 are obvious. The following examples show that those implications do not reverse. \( \square \)
Example 7 (see [BT]). Let $D(c)$ be the discrete space of size $c$ and let $\mathbb{Q}$ be the space of all rationals with the Euclidean topology. Put $X_1 = D(c) \times \mathbb{Q}$ with the product topology. Then $w(X_1) = \pi w(X_1) = c$, $\Delta(X_1) = \omega$, $\chi(X_1) = \pi \chi(X_1) = \omega$. Hence $P'(X) \Rightarrow P_\pi(X)$ (and consequently, $P''(X) \Rightarrow P_\pi(X)$) and $P'(X) \Rightarrow P(X)$.

Example 8. Let $\approx$ be the equivalence relation on $\mathbb{R} \times \mathbb{Q}$ defined by the formula $(x, y) \approx (x', y')$ iff $(x, y) = (x', y')$ or $y = y' = 0$. Let $X_2$ be the space $(\mathbb{R} \times \mathbb{Q})/\approx$ with the topology introduced by a complete system of neighbourhoods (a hedgehog-type space). If $y \neq 0$ then define neighbourhoods of $(x, y) \approx$ as $U_n((x, y) \approx) = \{x\} \times (y - \frac{|y|}{n}, y + \frac{|y|}{n})$, $n \in \mathbb{N}$. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{R})$ be the ideal of countable sets. Neighbourhoods of the point $(0, 0) \approx$ are the sets of the form $U_I((0, 0) \approx) = (\mathbb{R} \setminus I) \times \mathbb{Q} / \approx \cup \{(0, 0) \approx\}$ where $I \in \mathcal{I}$. Then $X_2 \setminus \{(0, 0) \approx\}$ is dense in $X_2$ and $\Delta(X_2) = \omega$. For all $(x, y) \neq (0, 0)$ we have $\chi(X_2, (x, y) \approx) = \pi \chi(X_2, (x, y) \approx) = \omega$, $\chi(X_2, (0, 0) \approx) = c$, $\pi \chi(X_2, (0, 0) \approx) = \omega_1 > \omega$. Hence $P''(X) \Rightarrow P'_\pi(X)$ (so $P''_\pi(X) \Rightarrow P'_\pi(X)$).

Example 9. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{R})$ be an ideal of nowhere dense sets with $\text{cf}(\mathcal{I}) = 2^c$ (as in Corollary 5), $T^*$ be the Hashimoto topology on $\mathbb{R}$ with respect to $\mathcal{I}$ (see [Ha]), i.e. the family of all sets of the form $U \setminus I$ where $U$ is open in the Euclidean topology and $I \in \mathcal{I}$. Let $X_3 = (\mathbb{R}, T^*)$. Then $X_3$ is card-homogeneous, $\Delta(X_3) = c$, $w(X_3) = 2^c$, $\pi w(X_3) = \pi \chi(X_3) = \omega$ and $\chi(X_3, x) = 2^c$ for all $x \in \mathbb{R}$. Hence $P_\pi(X) \Rightarrow P''(X)$ (so $P'_\pi(X) \Rightarrow P''(X)$ and $P''_\pi(X) \Rightarrow P''(X)$).

Example 10. Let $C$ be the Cantor ternary set, and $\mathcal{I}$ be an ideal of subsets of $C$ with $\text{cf}(\mathcal{I}) = 2^c$ (see Theorem 4). Define a topology $T$ on $\mathbb{R}$ by a complete system of the neighbourhoods. If $x \in C$ then neighbourhoods of $x$ are of the form $(x - \delta, x + \delta) \setminus I$ where $\delta > 0$, and $I \in \mathcal{I}$, $x \notin I$. If $x \notin C$ then the neighbourhoods of $x$ are of the form $(x - \delta, x + \delta)$ where $\delta > 0$. Let $X_4 = (\mathbb{R}, T)$. Then $X_4$ is card-homogeneous, $\Delta(X_4) = c$, and the set $A = \mathbb{R} \setminus C$ is dense in $X_4$. We have $\chi(X_4, x) = \omega$ for all $x \in A$, and $\chi(X_4, x) = 2^c$ for all $x \in C$. Moreover $\pi w(X_4) = \pi \chi(X_4) = \omega$. Hence $P''(X) \Rightarrow P'(X)$.

Theorem 11. In the class of card-homogeneous spaces the following relations hold

$$
\begin{align*}
P(X) & \iff P'(X) \iff P''(X) \\
P_\pi(X) & \iff P'_\pi(X) \iff P''_\pi(X)
\end{align*}
$$

Moreover, the implications $P'(X) \Rightarrow P''(X)$ and $P''(X) \Rightarrow P'_\pi(X)$ do not reverse.
Proof: Example 10 shows that $P''(X) \rightsquigarrow P'(X)$, and Example 9 yields $P''(X) \rightsquigarrow P''(X)$.

The proof of $P'(X) \rightarrow P(X)$: Suppose that for each $x \in X$, $\mathcal{B}(x)$ is a base of $X$ at a point $x$ such that $|\mathcal{B}(x)| \leq |X|$. Then $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}(x)$ is a base of $X$ with $|\mathcal{B}| \leq |X|$. In a similar way we prove the implication $P''(X) \rightarrow P'(X)$.

**Remark 12.** Theorem 11 solves a problem which follows Remark 4 in [BT].

**Theorem 13.** If $X$ is a dense-in-itself metrizable space then $P'(X)$ is true and the following relations hold

\[
\begin{array}{c}
P(X) \longrightarrow P'(X) \Longleftrightarrow P''(X) \\
\downarrow \quad \downarrow \\
P_\pi(X) \longrightarrow P'_\pi(X) \Longleftrightarrow P''_\pi(X)
\end{array}
\]

Moreover, the implications $P(X) \rightarrow P'(X)$ and $P_\pi(X) \rightarrow P'_\pi(X)$ do not reverse.

**Proof:** Observe that if $X$ is metrizable and dense in itself then $\Delta(X) \geq \omega$ and $\chi(X) = \omega$. Thus $P'(X)$ holds, and consequently $P''(X)$, $P'_\pi(X)$ and $P''_\pi(X)$ hold too. Example 7 shows that $P'(X) \rightsquigarrow P(X)$ and $P'(X) \rightarrow P_\pi(X)$ (so $P'_\pi(X) \rightarrow P''_\pi(X)$).

To prove the implication $P_\pi(X) \rightarrow P(X)$ fix a $\pi$-base $\mathcal{B}$ of $X$ with $|\mathcal{B}| \leq \Delta(X)$. For each $B \in \mathcal{B}$ choose an $x_B \in B$. Then the set $D = \{x_B: B \in \mathcal{B}\}$ is dense in $X$ and $|D| \leq \Delta(X)$, thus the family of all open balls with the center at $x \in D$ and radii $1/n$, $n \in \mathbb{N}$, forms a base of $X$ of size $\leq \Delta(X)$.

**Corollary 14.** In the class of metrizable card-homogeneous spaces all six considered conditions hold.

4. $P''_\pi(X)$ implies MR($X$)

**Lemma 15** ([BT, Lemma 5]). For every dense-in-itself topological space $X$ with $|X| = \kappa$ there exist pairwise disjoint open and card-homogeneous sets $G_\alpha$, $\alpha < \kappa$, such that $X = \text{cl}(\bigcup_{\alpha < \kappa} G_\alpha)$.

**Theorem 16.** For each dense-in-itself topological space $X$, the condition $P''_\pi(X)$ implies MR($X$).

**Proof:** The proof of this theorem is analogous to the proof of Theorem 6 in [BT]. Let $X_0$ be a dense subset of $X$ with $\pi\chi(X,x) \leq \Delta(X)$ for each $x \in X_0$. By Lemma 15 there exists a family of pairwise disjoint open and card-homogeneous sets $G_\alpha$, $\alpha < |X|$, such that $X = \text{cl}(\bigcup_{\alpha} G_\alpha)$. Then $P''_\pi(G_\alpha)$ for each $\alpha$ and, by Theorem 11, $P_\pi(G_\alpha)$ holds for $\alpha < |X|$. By Fact 2, all $G_\alpha$ are maximally resolvable. Note that $\Delta(G_\alpha) \geq \Delta(X)$, so $G_\alpha$ can be decomposed into dense
subsets $D_{\alpha,\beta}$, $\beta < \Delta(X)$. Put $D_\beta = \bigcup_{\alpha < |X|} D_{\alpha,\beta}$ for $\beta < \Delta(X)$. Then the sets $D_\beta$ are pairwise disjoint and dense in $X$. \qed

5. MR($X$) for card-homogeneous spaces does not imply $P_\pi(X)$

We shall prove that the implication given in Fact 2 cannot be reversed.

**Theorem 17.** There exists a card-homogeneous topological space $X$ which is maximally resolvable but does not satisfy condition $P_\pi(X)$.

**Proof:** We will construct $X$ as a countable dense subspace of the Cantor cube $\{0,1\}^\omega$. (The existence of such subspaces follows from Hewitt-Marczewski-Pondiczery Theorem [E].) Let $\mathcal{B}$ be a countable base of the space $\{0,1\}^\omega$, let $\mathcal{B}$ be the family of all finite subsets of pairwise disjoint sets from $\mathcal{B}$, and let $\mathcal{G}$ be the family of all functions $g: A \to \{0,1\}$, such that:

1. $(\exists B_A \in \mathcal{B}) A = \bigcup B_A$;
2. $(\forall B \in B_A) g|B$ is constant.

The family $\mathcal{G}$ is countable, so put $\mathcal{G} = \{g_n: n < \omega\}$. Let $\{g_{n,m}: n, m < \omega\}$ be a sequence such that $g_{n,m} = g_n$ for $n, m < \omega$. Fix a bijection $\varphi: \omega \to \omega \times \omega$, $\varphi = (\varphi_1, \varphi_2)$, and choose inductively a one-to-one sequence $f_n: \{0,1\}^\omega \to \{0,1\}$ with

$$g_{\varphi(n)} \subset f_n$$

for each $n$.

Let $X = \{f_n: n < \omega\}$ and, for $m < \omega$, $X_m = \{f_k \in X: \varphi_2(k) = m\}$. Then all $X_m$’s are dense in $\{0,1\}^\omega$. Indeed, fix an $m < \omega$ and a basic open set $U \subset \{0,1\}^\omega$. There exists a function $\psi_U: T \to \{0,1\}$ where $T$ is a finite subset of $\{0,1\}^\omega$, with $f \in U$ iff $\psi_U \subset f$. Since $\{0,1\}^\omega$ is a Hausdorff space, there is $n$ with $\psi_U \subset g_n$. Let $k = \varphi^{-1}(n, m)$. Then $f_k \in X_m \cap U$.

Thus $X$ is a countable dense subspace of $\{0,1\}^\omega$. Moreover $X$ is card-homogeneous, $\Delta(X) = \omega$, and, since $X_m$ are pairwise disjoint, $X$ is maximally resolvable. Finally, observe that $X$ has no countable $\pi$-base, thus $P_\pi(X)$ does not hold. Indeed, suppose that $\{V_n: n < \omega\}$ is a $\pi$-base of $X$. We may assume that all $V_n$ are of the form $U_n \cap X$ where $U_n$ is a basic open set in $\{0,1\}^\omega$ determined by a function $\psi_n: T_n \to \{0,1\}$ with $T_n$ being a finite subset of $\{0,1\}^\omega$ (i.e., $f \in U_n$ iff $\psi_n \subset f$). Fix $t_0 \in \{0,1\}^\omega \setminus \bigcup_n T_n$. Then $H = \{f \in X: f(t_0) = 0\}$ is non-empty open in $X$, and no $V_n$ is contained in $H$. \qed

**References**


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(Received March 18, 2004, revised November 2, 2004)