Some Characterizations of 0-Distributive Semilattices

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Abstract. In this paper we discuss prime down-sets of a semilattice. We give a characterization of prime down-sets of a semilattice. We also give some characterizations of 0-distributive semilattices and a characterization of minimal prime ideals containing an ideal of a 0-distributive semilattice. Finally, we give a characterization of minimal prime ideals of a pseudocomplemented semilattice.

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1. Introduction

Semilattices have been studied by many authors. The class of distributive semilattices is an important subclass of semilattices. We refer the readers to [4, 9, 10] for distributive semilattices. We also refer the monograph [5] for the background of distributive semilattices. The class of 0-distributive semilattices is a nice extension of the class of distributive semilattices. This extension is useful for the study of pseudocomplemented semilattices. For pseudocomplemented semilattices we refer the readers to [2, 3, 5, 6]. We also refer the readers to [7, 8] for 0-distributive semilattices (see [1, 11] for 0-distributive lattices). In this paper we study 0-distributive semilattices. By semilattice we mean meet-semilattice.

A semilattice $S$ with 0 is called 0-distributive if for any $a, b, c \in S$ such that $a \land b = 0 = a \land c$ implies $a \land d = 0$ for some $d \geq b, c$. The pentagonal lattice $P_5$ (see Figure 1) as a semilattice is 0-distributive but the diamond lattice $M_3$ (see Figure 1) as a semilattice is not 0-distributive. A semilattice $S$ is called directed above if for all $x, y \in S$ there exists $z \in S$ such that $z \geq x, y$. Every 0-distributive semilattice is directed above.

Minimal prime ideals and maximal filters play an important role in semilattices. In Section 2, we introduce a notion of minimal prime down-set and maximal filters in semilattices. Here we give a characterization of minimal prime down-sets and maximal filters in semilattices.
Like as a distributive semilattice (or distributive lattice) Stone’s version separation theorem is not true for 0-distributive semilattice. For example, if we consider the pentagonal lattice $P_5$ (see Figure 1) as a 0-distributive semilattice, then $F = \{c\}$ is a filter and $I = \{a\}$, is an ideal such that $F \cap I = \emptyset$ but there is no prime filter containing $F$ and disjoint from $I$.

In Section 3 we discuss Stone’s version separation theorem for 0-distributive semilattices. In this section we give some characterizations of 0-distributive semilattices.

In Section 4 we discuss the pseudocomplementation in semilattices. We close the paper with a characterization of a minimal prime ideals of a pseudocomplemented 0-distributive semilattice.

2. Prime down-sets and maximal filters

Let $S$ be a semilattice. A non-empty subset $D$ of $S$ is called a down-set if $a \in D, b \in S$ with $b \leq a$ implies that $b \in D$. A down-set $D$ of $S$ is called a proper down-set if $D \neq S$. A prime down-set is a proper down-set $P$ of $S$ such that $a \wedge b \in P$ implies $a \in P$ or $b \in P$. A prime down-set $P$ is called minimal if there is a prime down-set $Q$ such that $Q \subseteq P$, then $P = Q$.

**Theorem 2.1.** Any prime down-set of a semilattice contains a minimal prime down-set.

**Proof.** Let $S$ be a semilattice with 0. Let $P$ be a prime down-set of $S$ and let $\mathcal{P}$ be the set of all prime down-sets contained in $P$. Then $\mathcal{P}$ is non-empty since $P \in \mathcal{P}$. Let $\mathcal{C}$ be a chain in $\mathcal{P}$ and let

$$M := \bigcap \{X \mid X \in \mathcal{C} \}.$$ 

We claim that $M$ is a prime down-set. Clearly $M$ is non-empty as $0 \in M$. Let $a \in M$ and $b \leq a$. Then $a \in X$ for all $X \in \mathcal{C}$. Hence $b \in X$ for all $X \in \mathcal{C}$ as $X$ is a down-set. Thus $b \in M$. Now let $x \wedge y \in M$ for some $x, y \in S$. Then $x \wedge y \in X$ for all $X \in \mathcal{C}$. Since $X$ is a prime down-set for all $X \in \mathcal{C}$, we have either $x \in X$ or $y \in X$ for all $X \in \mathcal{C}$. This implies that either $x \in M$ or $y \in M$. Hence $M$ is a prime down-set.

Thus by applying the dual form of Zorn’s Lemma to $\mathcal{P}$, there is a minimal member of $\mathcal{P}$.

Let $S$ be a semilattice. A non-empty subset $F$ of $S$ is called a filter if

(i) $a, b \in F$ implies $a \wedge b \in F$

(ii) $a \in S, b \in F$ with $a \geq b$ implies $a \in F$. 

![Figure 1. 0-distributive and non-0-distributive semilattices](image-url)
A filter $F$ of a semilattice $S$ is called a **proper filter** if $F \neq S$. A **maximal filter** $F$ of $S$ is a proper filter which is not contained in any other proper filter, that is, if there is a proper filter $G$ such that $F \subseteq G$, then $F = G$.

Following result is due to [8].

**Lemma 2.1.** Let $M$ be a proper filter of $S$ with 0. Then $M$ is maximal if and only if for all $a \in S \setminus M$, there is some $b \in M$ such that $a \wedge b = 0$.

Now we have the following result.

**Theorem 2.2.** Let $F$ be a non-empty proper subset of a semilattice $S$. Then $F$ is a filter if and only if $S \setminus F$ is a prime down-set.

**Proof.** Let $F$ be a filter of a semilattice $S$. Let $x \in S \setminus F$ and $y \leq x$. Then $x \notin F$ and hence $y \notin F$ as $F$ is a filter. This implies $y \in S \setminus F$. Thus $S \setminus F$ is a down-set. Since $F$ is a filter $S \setminus F \neq S$. Thus $S \setminus F$ is a proper down-set. To prove $S \setminus F$ is a prime down-set, let $a, b \in S$ such that $a \wedge b \in S \setminus F$. Then $a \wedge b \notin F$ and hence either $a \notin F$ or $b \notin F$ as $F$ is filter. This implies either $a \in S \setminus F$ or $b \in S \setminus F$. Therefore, $S \setminus F$ is a prime down-set.

Conversely, let $S \setminus F$ be a prime down-set and $x, y \in F$. Then clearly, $x, y \notin S \setminus F$ and hence $x \wedge y \notin S \setminus F$ as $S \setminus F$ is a prime down-set. Thus $x \wedge y \notin F$. Suppose $x \in F$ and $x \leq y$. Then $x \notin S \setminus F$. Since $S \setminus F$ is a down-set, we have $y \notin S \setminus F$. Hence $y \in F$. This implies $F$ is a filter.

**Theorem 2.3.** Let $F$ be a non-empty subset of a semilattice $S$. Then $F$ is a maximal filter if and only if $S \setminus F$ is a minimal prime down-set.

**Proof.** Let $F$ be a maximal filter and $S \setminus F$ is not a minimal prime down-set. Then there exists a prime down-set $I$ such that $I \subseteq S \setminus F$ which implies $F \subseteq S \setminus I$ which contradict to the maximality of $F$. Hence $S \setminus F$ is minimal prime down-set.

Conversely, let $S \setminus F$ be a minimal prime down-set and $F$ is not a maximal filter. Thus there exists a proper filter $G$ such that $F \subseteq G$ which implies $S \setminus G \subseteq S \setminus F$ which contradict the minimality of $S \setminus F$. Hence $F$ is a maximal filter.

### 3. Minimal prime ideals

Let $S$ be a semilattice. A down-set $I$ of $S$ is called an **ideal** if $a, b \in I$ implies the existence of $c \in I$ such that $a, b \leq c$. The set of all ideals of $S$ is denoted by $\mathcal{I}(S)$. An ideal $I$ of $S$ is called a **proper ideal** if $I \neq S$. A **prime ideal** $P$ is a proper ideal of $S$ such that $a \wedge b \in P$ implies either $a \in P$ or $b \in P$. A prime ideal $P$ is called **minimal** if there is a prime ideal $Q$ such that $Q \subseteq P$, then $P = Q$. A filter $F$ of $S$ is called a **prime filter** if $F \neq S$ and $S \setminus F$ is a prime ideal.

We shall often use the following lemma in this paper.

**Lemma 3.1.** Let $S$ be a directed above semilattice with 0. If $S$ is not 0-distributive, then the set

$$F := \{x \in S \mid x \geq a \wedge y \neq 0 \text{ for all } y \geq b, c\},$$

where $a, b, c \in S$ such that $a \wedge b = a \wedge c = 0$, is a proper filter.

**Proof.** Since $S$ is not 0-distributive, there are $p, q, r \in S$ such that $p \wedge q = p \wedge r = 0$ and $p \wedge d \neq 0$ for all $d \geq q, r$. Now we have $p \geq p \wedge d$. Thus $p \in F$. Hence $F$ is non-empty. Clearly $0 \notin F$. It is enough to show that $F$ is a filter. Let $x \in F$ and $z \geq x$. Then $x \geq a \wedge y$ for
all $y \geq b, c$ and by transitivity $z \geq a \land y$ for all $y \geq b, c$. Hence $z \in F$. Again let $x,z \in F$. Then $x \geq a \land y$ and $z \geq a \land y$ for all $y \geq b, c$. Thus $x \land z \geq a \land y$ for all $y \geq b, c$. Hence $x \land z \in F$. This implies $F$ is a filter.

Now we have the following result.

**Theorem 3.1.** Every maximal filter of a 0-distributive semilattice is a prime filter.

**Proof.** Let $S$ be a 0-distributive semilattice. Again let $Q$ be a maximal filter of $S$. We shall show that $Q$ is prime. It is sufficient to show that $S \setminus Q$ is a prime ideal. By Theorem 2.3 we have $S \setminus Q$ is a minimal prime down-set. Now let $x, y \in S \setminus Q$. Then by Lemma 2.1 we have $a \land x = 0 = b \land y$ for some $a, b \in Q$. Let $c = a \land b$. Clearly $c \land x = 0 = c \land y$ and $c \in Q$. Hence by the 0-distributivity of $S$ there exists $z \in S$ such that $z \geq x, y$ and $c \land z = 0$. Hence $z \in S \setminus Q$. Thus $S \setminus Q$ is a prime ideal which implies $Q$ is prime. \hfill \Box

Let $A$ be non-empty subset of a semilattice $S$ with 0. Set

$$A^\perp := \{x \in S \mid a \land x = 0 \text{ for all } a \in A\}.$$  

Then $A^\perp$ is called the annihilator of $A$. If $A = S$ then $A^\perp = S^\perp = \{0\}$. For $a \in S$, the annihilator of $\{a\}$ is simply denoted by $a^\perp$ and hence $a^\perp = \{x \in S \mid a \land x = 0\}$. We can easily show that

$$A^\perp = \bigcap_{a \in A} a^\perp.$$  

Let $S$ be a semilattice with 0. An ideal $I$ of $S$ is called an annihilator ideal if $I = A^\perp$ for some non-empty subset $A$ of $S$.

Our aim is to prove a Stone’s version separation theorem for 0-distributive semilattices. The following result due to [8, Theorem 7].

**Theorem 3.2.** Let $S$ be a semilattice with 0. Then $S$ is 0-distributive if and only if for any filter $F$ of $S$ such that $F \cap x^\perp = \emptyset (x \in S)$, there exists a prime filter containing $F$ and disjoint from $x^\perp$.

Our conjecture is:

**Conjecture 3.1.** Let $S$ be a directed above semilattice with 0. Then $S$ is 0-distributive if and only if for any filter $F$ and any annihilator ideal $I$ of $S$ such that $F \cap I = \emptyset$, there exists a prime filter containing $F$ and disjoint from $I$.

The necessary conditions of a directed above semilattice to be 0-distributive is given below, but unfortunately, we could not prove or disprove the condition is sufficient or not.

**Theorem 3.3.** Let $S$ be a directed above semilattice with 0. If for any filter $F$ and any annihilator ideal $I$ of $S$ such that $F \cap I = \emptyset$, there exists a prime filter containing $F$ and disjoint from $I$, then $S$ is 0-distributive.

**Proof.** Suppose the condition holds. If $S$ is not 0-distributive, then there are $a, b, c \in S$ such that $a \land b = 0 = a \land c$ and $a \land d \neq 0$ for all $d \geq b, c$ (such $d$ exists as $S$ is directed above). Let

$$F := \{x \in S \mid x \geq a \land y \text{ for all } y \geq b, c\}. $$

Then by Lemma 3.1, we have $F$ is a proper filter.

Let $I$ be an annihilator ideal such that $a \land d \notin I$ (such annihilator exists as $a \land d \notin S^\perp$). We shall show that $I \cap F = \emptyset$. If $x \in I \cap F$, then $x \geq a \land y$ for all $y \geq b, c$ which implies $a \land d \in I$.
Some Characterizations of 0-Distributive Semilattices

Let $S$ be a semilattice. For $a \in S$, the ideal $\langle a \rangle$ is called the ideal generated by $a$. It can be easily seen that $(a^+) = a^+$ for any $a \in S$. An ideal $I$ of $S$ is called an $\alpha$-ideal if $(i^+) = I$ for any $i \in I$.

Now we shall give some characterizations of 0-distributive semilattice. The following lemma is due to [1].

**Lemma 3.2.** Every proper filter of a semilattice with 0 is contained in a maximal filter.

We have the following result which is a generalization of [1, Theorem 3.1].

**Theorem 3.4.** Let $S$ be a semilattice with 0. Then the following statements (i)–(iv) are equivalent and any one of them implies (v) and (vi).

(i) $S$ is 0-distributive;
(ii) every maximal filter of $S$ is prime;
(iii) every minimal prime down-set of $S$ is a minimal prime ideal;
(iv) every proper filter of $S$ is disjoint from a minimal prime ideal;
(v) for each element $a \in S$ such that $a \neq 0$, there is a minimal prime ideal not containing $a$;
(vi) each element $a \in S$ such that $a \neq 0$ is contained in a prime filter.

**Proof.** (i)⇒(ii). This follows by the Lemma 3.1.

(ii)⇒(iii). Let $N$ be a minimal prime down-set. Then by Lemma 2.3 we have $S \setminus N$ is a maximal filter. Hence by (ii) $S \setminus N$ is a prime filter. Thus $N$ is a prime ideal.

(iii)⇒(iv). Let $F$ be a proper filter of $S$. By Lemma 3.2 there is a maximal filter $M$ such that $F \subseteq M$. Hence by Lemma 2.3 we have $S \setminus M$ is a minimal prime down-set. Thus by (iii) $S \setminus M$ is a minimal prime ideal. Clearly, $F \cap (S \setminus M) = \emptyset$.

(iv)⇒(i). Suppose $S$ is not 0-distributive. Then there are $a, b, c \in S$ such that $a \land b = a \land c = 0$ and $a \land d \neq 0$ for all $d \geq b, c$. Now set

$$F = \{ x \in S \mid x \geq a \land y \text{ for all } y \geq b, c \}.$$  

Then by Lemma 3.1, we have $F$ is a proper filter and hence by (iv) there exists a prime ideal $Q$ such that $F \cap Q = \emptyset$. Thus $a \land p \notin Q$ for any $p \geq b, c$. This implies $a, p \notin Q$ for any $p \geq b, c$. Now $a \notin Q$ implies $b, c \in Q$. Then there is $m \geq b, c$ such that $m \in Q$ which is a contradiction. Therefore, $a \land d = 0$ for some $d \geq b, c$ and hence $S$ is 0-distributive.

(v)⇒(vi). Let $a \in S$ such that $a \neq 0$. Then $\langle a \rangle$ is a proper filter. Then by (iv) $\langle a \rangle$ is disjoint from a minimal prime ideal $N$ of $S$. Thus $a \notin N$.

Now we have following result which is a generalization of [1, Lemma 1.8].
Lemma 3.3. Let $A$ be a non-empty subset of a semilattice $S$ with 0. Then $A^\perp$ is the intersection of all the minimal prime down-set not containing $A$.

Proof. Let $S$ be a semilattice with 0 and $\emptyset \neq A \subseteq S$. Suppose

$$X := \bigcap \{P \mid A \nsubseteq P \text{ and } P \text{ is a minimal prime down-set}\}$$

Let $x \in A^\perp$. Then $x \land y = 0$ for all $y \in A$. This implies there is $z \notin P$ such that $x \land z = 0 \in P$. As $P$ is prime, we have $x \in P$. Hence $x \in X$.

Conversely let, $x \in X$. If $x \notin A^\perp$. Then $x \land q \neq 0$ for some $q \in A$. Let $D = [x \land q]$. Then $0 \notin D$. Hence, $D \neq S$. Then by Lemma 3.2 we have $D \subseteq M$ for some maximal filter $M$. Hence by Lemma 2.3 we have $S \setminus M$ is a minimal prime down-set. Now $x \notin S \setminus M$ as $x \in D$ implies $x \in M$. Moreover $A \nsubseteq S \setminus M$ as $q \in A$ but $q \in M$ implies $q \notin S \setminus M$, which is a contradiction to $x \in X$. Hence $x \in A^\perp$. Thus the lemma is proved.

Theorem 3.5. Let $S$ be a 0-distributive semilattice. If $A$ is a non-empty subset of $S$ and $F$ is a proper filter intersecting $A$, there is a minimal prime ideal containing $A^\perp$ and disjoint from $F$.

Proof. Let $S$ be a directed above semilattice with 0. Again let $A$ be a non-empty subset of $S$ and $F$ be a proper filter such that $F \cap A \neq \emptyset$. Then Lemma 2.2 $S \setminus F$ is a prime down-set and by Lemma 2.1 $N \subseteq S \setminus F$ for some minimal prime down-set $N$. Clearly, $N \cap F = \emptyset$. Also $A \nsubseteq S \setminus F$ and so $A \nsubseteq N$. By Lemma 3.3 $A^\perp \subseteq N$. Since $S$ is 0-distributive, by theorem 3.4(iv) $N$ is a minimal prime ideal.

4. Pseudocomplementation for 0-distributive semilattices

Let $S$ be a semilattice with 0. An element $d \in S$ is called the pseudocomplement of $x \in S$, if $x \land d = 0$ and $y \in S, x \land y = 0$ implies $y \leq d$.

A semilattice $S$ is said to be pseudocomplemented if each element of $S$ has a pseudocomplement. The pseudocomplement of 0 is the largest element 1. Thus a pseudocomplemented semilattice contains both the smallest and the largest element.

Theorem 4.1. Every pseudocomplemented semilattice is 0-distributive but the converse is not true.

Proof. Let $S$ be a pseudocomplemented semilattice. Suppose $a, b, c \in S$ with $a \land b = 0 = a \land c$. By the definition of pseudocomplemented, $b \leq a^\ast, c \leq a^\ast$ and $a \land a^\ast = 0$. Thus $S$ is a 0-distributive semilattice.

To prove the converse is not true, consider the semilattice, $\mathcal{M}_2$ shown in the Figure 2, which is clearly 0-distributive but not pseudocomplemented as $a^\ast$ does not exist.

Theorem 4.2. Let $S$ be a pseudocomplemented semilattice and let $J$ be an ideal of $S$. Then a prime ideal $P$ containing $J$ is a minimal prime ideal containing $J$ if and only if for each $x \in P$ there is $y \in S \setminus P$ such that $x \land y \in J$.

Proof. Let $P$ be a prime ideal of $S$ containing $J$ such that the given condition holds. We shall show that $P$ is a minimal prime ideal containing $J$. Let $K$ be a prime ideal containing $J$ such that $K \subseteq P$. Let $x \in P$. Then there is $y \in S \setminus P$ such that $x \land y \in J$. Hence $x \land y \in K$ as $K$ containing $J$. Since $K$ is prime and $y \notin K$ implies $x \in K$. Hence $P \subseteq K$. Thus $K = P$. Therefore, $P$ is a minimal prime ideal containing $J$. 
Conversely, let \( P \) be a minimal prime ideal containing \( J \). Let \( x \in P \). Suppose for all \( y \in S \setminus P, x \land y \notin J \). Set \( D = (S \setminus P) \lor \{x\} \). We claim that \( 0 \notin D \). For if \( 0 \in D \), then \( 0 = q \land x \) for some \( q \in S \setminus P \). Thus, \( x \land q = 0 \in J \) which is a contradiction. Therefore, \( 0 \notin D \). Since \( \{0\} = 1^\perp \) by Theorem 3.2, there is a prime filter \( Q \) such that \( D \subseteq Q \) and \( 0 \notin Q \). Let \( M = S \setminus Q \).

Then by the definition of prime filter of a semilattice, \( M \) is a prime ideal. We claim that \( M \cap D = \emptyset \). If \( a \in M \cap D \), then \( a \in M \) and hence \( a \notin Q \). Thus \( a \notin D \) which is a contradiction. Hence \( M \cap D = \emptyset \). Therefore, \( M \cap (S \setminus P) = \emptyset \) and hence \( M \subseteq P \). Also \( M \neq P \), because \( x \in D \) implies \( x \in Q \) and hence \( x \notin M \) but \( x \in P \). This shows that \( P \) is not minimal which is a contradiction. Hence the given condition holds.

We enclose the paper with the following useful characterization of minimal prime ideal.

**Theorem 4.3.** Let \( S \) be a pseudocomplemented semilattice and let \( P \) be a prime ideal of \( S \). Then the followings are equivalent:

(i) \( P \) is minimal.

(ii) \( x \in P \) implies that \( x^* \notin P \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( P \) be a minimal prime ideal and let \( x^* \in P \) for some \( x \in P \). Set \( D = (S \setminus P) \lor \{x\} \). We claim that \( 0 \notin D \). For if \( 0 \in D \), then \( 0 = q \land x \) for some \( q \in S \setminus P \), which implies \( q \leq x^* \in P \) which is a contradiction. Therefore, \( 0 \notin D \). Since \( \{0\} = 1^\perp \) by Theorem 3.2, there is a prime filter \( Q \) such that \( D \subseteq Q \) and \( 0 \notin Q \). Let \( M = S \setminus Q \). Then by the definition of prime filter of a semilattice, \( M \) is a prime ideal. We claim that \( M \cap D = \emptyset \). If \( a \in M \cap D \), then \( a \in M \) and hence \( a \notin Q \). Thus \( a \notin D \) which is a contradiction. Hence \( M \cap D = \emptyset \). Therefore, \( M \cap (S \setminus P) = \emptyset \) and hence \( M \subseteq P \). Also \( M \neq P \), because \( x \in D \) implies \( x \in Q \) and hence \( x \notin M \) but \( x \in P \). This shows that \( P \) is not minimal which is a contradiction. Hence (ii) holds.

(ii) \( \Rightarrow \) (i). Let \( P \) be a prime ideal of \( S \) such that (ii) holds. We shall show that \( P \) is a minimal prime ideal. Let \( K \) be a prime ideal satisfying (ii) such that \( K \subseteq P \). Let \( x \in P \). Then \( x \land x^* = 0 \in K \). Since \( K \) is prime and \( x^* \notin K \) implies \( x \in K \). Hence \( P \subseteq K \). Thus \( K = P \). Therefore, \( P \) is a minimal prime ideal.

**References**