Asymptotic Distributions of the Generalized and the Dual Generalized Extremal Quotient

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Abstract. Necessary and sufficient conditions for the weak convergence of the generalized and the dual generalized extremal quotient are obtained. The class of possible non-degenerate limit distribution functions of quotient of generalized and its dual extreme order statistics is characterized. Some illustrative examples are obtained.

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1. Introduction

Consider a sequence of independent and identically distributed random variables (rv’s) \( \{X_n: n \geq 1\} \) with distribution function (df) \( F \). Let \( M_n = \max\{X_1, \ldots, X_n\} \) and \( L_n = \min\{X_1, \ldots, X_n\} \). The extremal quotient is defined by \( q_n = M_n/L_n \) (see, Galambos and Simonelli [10]). This statistic is obviously not affected by a change of scale. Therefore, its use may be of interest in cases where the scale plays no role, e.g., in climatic study (see Canard [7]). The extremal quotient has used in several fields, most notably in industrial quality control, life testing, small-area variation analysis and the classical heterogeneity of variance situation. For example, a quality engineer might use this statistic as a basic measurement in controlling the roundness of a circular component in a production process. Also, Wong and Wong [17] used the extremal quotient to test the hypothesis that the population of a sample has an exponential df. The same authors [16] used this statistic for testing the shape parameter of the Weibull df. The limit laws for the extremal quotient were fully characterized by Barakat [5]. Cramer and Kamps [8] have used the extremal quotient in the framework of sequential order statistics. Moreover, some additional relevant references of this statistic are given these.

Generalized order statistics (gos) have been introduced by Kamps [12] as a unification of several models of ascendingly ordered rv’s. The gos \( X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \ldots \).
..., X(n, n, \tilde{m}, k) based on a df \( F \) are defined by their probability density function (pdf)

\[
f^{(\tilde{m}, k)}_{1, 2, \ldots, n}(x_1, \ldots, x_n) = k \left( \prod_{i=1}^{n-1} \gamma_i \right) \left( \prod_{i=1}^{n-1} (1 - F(x_i))^{m_i} \right) (1 - F(x_n))^{k-1} \left( \prod_{i=1}^{n} f(x_i) \right),
\]

on the cone \( \{ (x_1, \ldots, x_n) : x_0 = F^{-1}(0) < x_1 \leq \ldots \leq x_n < F^{-1}(1) = x^0 \} \), where \( x_0 = \inf \{ x : F(x) > 0 \} \geq -\infty \) and \( x^0 = \sup \{ x : F(x) < 1 \} \leq \infty \). The parameters \( \gamma_1, \ldots, \gamma_n \) are defined by \( \gamma_i = k \) and \( \gamma_i = k + n - r + \sum_{j=r}^{n-1} m_j > 0 \), \( r = 1, 2, \ldots, n - 1 \), where \( \tilde{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1} \). Particular choice of the parameters \( \gamma_1, \ldots, \gamma_n \) leads to different models, e.g., ordinary order statistics (oos) (m1 = m2 = \ldots = m_{n-1} = 0, k = 1); order statistics with non-integral sample size \( (m_1 = m_2 = \ldots = m_{n-1} = 0, k = \alpha - n + 1, \text{ and } \alpha > n - 1) \); \( r \)th record values \( (m_1 = m_2 = \ldots = m_{n-1} = -1 \text{ and } k \text{ is any positive integer}) \); and sequential order statistics (sos) \( (m_r = m_{r+1} = \ldots = m_n = 0, \text{ and } k = \alpha_0 \text{ and } \alpha_0 > 0) \).

The concept of dual generalized order statistics (dgos) is introduced in Burkschat et al. [6] to enable a common approach to ascendingly ordered rv’s like reversed order statistics and lower records models. The dgos \( X_{d}(1, n, \tilde{m}, k), X_{d}(2, n, \tilde{m}, k), \ldots, X_{d}(n, n, \tilde{m}, k) \) based on a df \( F \) are defined by their pdf

\[
f^{(d, \tilde{m}, k)}_{1, 2, \ldots, n}(x_1, \ldots, x_n) = k \left( \prod_{i=1}^{n-1} \gamma_i \right) \left( \prod_{i=1}^{n-1} F^{m_i}(x_i) \right) (F^{k-1}(x_n)) \left( \prod_{i=1}^{n} f(x_i) \right),
\]

where \( x^0 = F^{-1}(1) > x_1 \geq \ldots \geq x_n > F^{-1}(0) = x_0 \).

In this work, we consider a wide subclass of gos (dgos), by assuming \( \gamma_j - \gamma_{j+1} = m + 1 \geq 0 \). This subclass is known as m-gos (m-dgos). Clearly, many important practical models of m-gos are included such as oos, order statistics with non-integral sample size, upper record values and sos. The marginal df’s of the \( r \)th and \( (n - r + 1) \)th m-gos (c.f. Nasri-Roudsari [13], see also Barakat [3]) are represented by \( \Phi^{(m, k)}_{r,n}(x) = I_{G(x)}(r, N - r + 1) \) and \( \Phi^{(m, k)}_{n-r+1,n}(x) = I_{G(x)}(N - R + 1, R) \), respectively, where \( I_r(n, m) = 1/B(n, m) \int_0^1 t^{n-1}(1 - t)^m dt \) is the incomplete beta ratio function, \( G(x) = 1 - (1 - F(x))^{m+1}, N = \ell + n - 1, R = \ell + r - 1 \) and \( \ell = k/(n + 1) \). Similarly, by using the results of Kamps [12] and Burkschat et al. [6], the marginal df’s of the \( r \)th and \( (n - r + 1) \)th m-dgos are given by \( \Phi^{(d, m, k)}_{d, n-r+1,n}(x) = I_{F^{m+1}(x)}(N - r + 1, r) \) and \( \Phi^{(d, m, k)}_{d, n-r+1,n}(x) = I_{F^{m+1}(x)}(R, N - R + 1) \), respectively.

The central result of the classical extreme value theory is that the class of possible limit df’s of each of the lower and upper extreme order statistics is restricted to essentially three different types. Namely, for some suitable normalizing constants \( \alpha_0, \alpha_n > 0 \) and \( \beta_0, \beta_n \in \mathbb{R} \), we have

\[(1.1) \quad \Phi^{(0,1)}_{r,n}(\alpha_0 x + \beta_0) = I_{F(\alpha_0 x + \beta_0)}(r, n - r + 1, r) \xrightarrow{w} \frac{w}{n} 1 - \Gamma_r(U_i, \alpha(x)), \text{ } i \in \{1, 2, 3\},\]

if, and only if, \( nF(\alpha_0 x + \beta_0) \rightarrow U_i(\alpha(x)) \), as \( n \rightarrow \infty \), and

\[(1.2) \quad \Phi^{(0,1)}_{n-r+1,n}(\alpha_n x + \beta_n) = I_{F(\alpha_n x + \beta_n)}(n - r + 1, r) \xrightarrow{w} \Gamma_r(V_j, \beta(x)), \text{ } j \in \{1, 2, 3\},\]

if, and only if, \( n(1 - F(\alpha_n x + \beta_n) \rightarrow V_j(\beta(x)) \), as \( n \rightarrow \infty \), where \( \xrightarrow[w]{w} \) denotes the weak convergence, as \( n \rightarrow \infty \), \( \Gamma_r(x) = 1/(\Gamma(r)) \int_x^{\infty} t^{-1} e^{-t} dt \),

Type I: \( U_{1, \alpha}(x) = \begin{cases} (-x)^{-\alpha} & x < 0, \alpha > 0, \\ 0, & x \geq 0 \end{cases} \) Type II: \( U_{2, \alpha}(x) = \begin{cases} x^\alpha & x \geq 0, \alpha > 0, \\ 0, & x < 0 \end{cases} \)
Finally, equivalent necessary and sufficient conditions for (1.7) and (1.8) to be satisfied are that the df $F$ belongs to the domain of attraction of each of the limits (1.6). Normalizing constants can be chosen such that

$$H_{i,\alpha}^{(m,k)}(x) = 1 - \Gamma_r(U_{i,\alpha}(x))$$

and

$$H_{i,\alpha}^{(m,k)}(x) = 1 - \Gamma_r(U_{i,\alpha}(x))$$

(1.4)

The following two theorems, due to Nasri-Roudsari [13] and Nasri-Roudsari and Cramer [14] (see also Barakat [3]), extend the above result to the $X(r,n,m,k), X_d(n-r+1,n,m,k)$, $X(n-r+1,n,m,k)$ and $X_d(r,n,m,k)$.

**Theorem 1.1.** Let $m_1 = m_2 = \ldots = m_{n-1} = m > -1$ and $r$ be fixed integer with respect to $n$. Then, there exist normalizing constants $\alpha_{n,m}, \bar{\alpha}_{n,m}$ and $\beta_{n,m}, \bar{\beta}_{n,m}$, for which

$$\Phi_{r,n}^{(m,k)}(x) = \Phi_{r,n}^{(m,k)}(x) = \frac{\alpha_{n,m} + \beta_{n,m}}{n}$$

and

$$\Phi_{d,r,n}^{(m,k)}(x) = \Phi_{d,r,n}^{(m,k)}(x) = \frac{\bar{\alpha}_{n,m} + \bar{\beta}_{n,m}}{n}$$

where $H_{i,\alpha}^{(m,k)}(x)$ and $H_{i,\alpha}^{(m,k)}(x)$ are non-degenerate df’s if, and only if, (1.1) is satisfied. In this case, $H_{i,\alpha}^{(m,k)}(x) = H_{i,\alpha}^{(m,k)}(x) = 1 - \Gamma_r(U_{i,\alpha}(x))$ and $H_{i,\alpha}^{(m,k)}(x) = 1 - \Gamma_r(U_{i,\alpha}^{m+1}(x))$ (and we say that the df $F$ belongs to the domain of attraction of each of the limits $H_{i,\alpha}^{(m,k)}$ and $H_{i,\alpha}^{(m,k)}$, written $F \in D(H_{i,\alpha}^{(m,k)})$ and $F \in D(H_{i,\alpha}^{(m,k)})$, respectively). Moreover, the normalizing constants can be chosen such that $\alpha_{n,m} = \alpha_{\phi(n)}, \beta_{n,m} = \beta_{\phi(n)}, \bar{\alpha}_{n,m} = \alpha_{\psi(n)}$ and $\bar{\beta}_{n,m} = \beta_{\psi(n)}$, where $\phi(n) = n(m+1)$ and $\psi(n) = n^{1/(m+1)}$. Finally, equivalent necessary and sufficient conditions for (1.5) and (1.6) to be satisfied are $N(\alpha_{n,m} + \beta_{n,m}) \rightarrow U_{i,\alpha}(x)$ and $N(\bar{\alpha}_{n,m} + \bar{\beta}_{n,m}) \rightarrow U_{i,\alpha}^{m+1}(x)$, as $n \rightarrow \infty$, respectively.

**Theorem 1.2.** Let $m_1 = m_2 = \ldots = m_{n-1} = m > -1$ and $r$ be fixed integer with respect to $n$. Then, there exist normalizing constants $a_{n,m}, \bar{a}_{n,m}$ and $b_{n,m}, \bar{b}_{n,m}$, for which

$$\Phi_{n-r+1,n}^{(m,k)}(x) = \Phi_{n-r+1,n}^{(m,k)}(x) = \frac{a_{n,m} + b_{n,m}}{n}$$

and

$$\Phi_{d,n-r+1,n}^{(m,k)}(x) = \Phi_{d,n-r+1,n}^{(m,k)}(x) = \frac{\bar{a}_{n,m} + \bar{b}_{n,m}}{n}$$

where $H_{j,\beta}^{(m,k)}(x)$ and $H_{j,\beta}^{(m,k)}(x)$ are non-degenerate df’s if, and only if, (1.2) is satisfied. In this case, $H_{j,\beta}^{(m,k)}(x) = \Gamma_r(V_{j,\beta}^{1}(x))$ and $H_{j,\beta}^{(m,k)}(x) = \Gamma_r(V_{j,\beta}^{1}(x))$ (and we say that the df $F$ belongs to the domain of attraction of each of the limits $H_{j,\beta}^{(m,k)}$ and $H_{j,\beta}^{(m,k)}$, written $F \in D(H_{j,\beta}^{(m,k)})$ and $F \in D(H_{j,\beta}^{(m,k)})$, respectively). Moreover, the normalizing constants can be chosen such that $a_{n,m} = a_{\psi(n)}, b_{n,m} = b_{\psi(n)}, \bar{a}_{n,m} = a_{\phi(n)}$ and $\bar{b}_{n,m} = b_{\phi(n)}$. Finally, equivalent necessary and sufficient conditions for (1.7) and (1.8) to be satisfied are $N(1 - G(a_{n,m} + b_{n,m})) \rightarrow V_{j,\beta}^{m+1}(x)$ and $N(1 - F^{m+1}(\bar{a}_{n,m} + \bar{b}_{n,m})) \rightarrow V_{j,\beta}(x)$, as $n \rightarrow \infty$, respectively.
Lemma 1.1. The normalizing constants $a_n, b_n, \alpha_n$ and $\beta_n$ can be chosen such as:

Type I: $a_n = \lceil n \rceil, b_n = 0$ and $\alpha_n = \lceil n \rceil, \beta_n = 0$, where $x^0 = -x_0 = \infty, \gamma(t) = \inf\{x : F(x) \geq 1 - 1/t \} \uparrow x^0$, and $\gamma(t) = \sup\{x : F(x) \leq 1/t \} \downarrow x_0$, as $n \to \infty$.

Type II: $a_n = \lfloor x^0 - \lceil n \rceil \rfloor, b_n = x^0$ and $\alpha_n = \lfloor x_0 - \lceil n \rceil \rfloor, \beta_n = x_0$, where $-\infty < x_0 < x^0 < \infty$.

Type III: $a_n = \overline{\gamma}(b_n), b_n = \lceil n \rceil, \alpha_n = \overline{\gamma}(b_n), \beta_n = \lceil n \rceil$, where $\overline{\gamma}(t) = 1/(1 - F(t)) \int_{x_0}^{x^0} (1 - F(y))dy, t < x^0 \leq \infty$, and $\overline{\gamma}(t) = 1/(F(t)) \int_{x_0}^{x^0} F(y)dy, -\infty \leq x_0 < t$.

Our aim in this paper is to derive the class of possible non-trivial and trivial limit df’s of the suitably normalized generalized and dual generalized extremal quotient $\tilde{q}_n(r_n, m, k) = A_{n,m}(q_n(m, k) - B_{n,m})$ and $\tilde{q}_n^*(r_n, m, k) = \tilde{A}_{n,m}(q_{d,n}(m, k) - \tilde{B}_{n,m})$, respectively, where $A_{n,m}$, $\tilde{A}_{n,m} > 0, B_{n,m}, \tilde{B}_{n,m} \in \mathbb{R}, q_{n}(m, k) = M_n(m, k)/L_n(m, k) = X(n, n, m, k)/X(1, n, m, k)$, $q_{d,n}(m, k) = M_{d,n}(m, k)/L_{d,n}(m, k) = X_d(n, n, n, m, k)/X_d(1, n, m, k)$ and the trivial convergence takes place, when one of the extremes outweighs the other (see de Haan [9]). Since, any result of dgos can be easily deduced from the corresponding result of gos (as Burkschat et al. [6], and Theorems 1.1, 1.2 have shown), the emphasis of our study will be mainly on gos.

2. The generalized extremal quotient (the case $m > -1$)

In this section and in the sequel the limit df’s $\hat{H}^{(m, k)}_{1, \alpha}$ and $\overline{H}^{(m, k)}_{i, \beta}$ ($\hat{H}^{d(m, k)}_{i, \beta}$ and $\overline{H}^{d(m, k)}_{i, \alpha}$), $i = 1, 2, 3$, are considered as the limit df’s of the minimum and maximum df’s, respectively, e.g., in the sequel the limit df’s $\hat{H}^{(m, k)}_{1, \alpha}$ and $\overline{H}^{(m, k)}_{i, \beta}$, $i = 1, 2, 3$, are defined in Theorems 1.1 and 1.2, with $r = 1$. The following two theorems fully characterize the possible non-trivial and trivial limit df’s of $q_n^*(m, k)$.

Theorem 2.1. (Non-trivial types).

Part 1. If $F \in D(\hat{H}^{(m, k)}_{1, \alpha})$ and $F \in D(\overline{H}^{(m, k)}_{2, \beta})$, then

$$P(q_n^*(m, k) \leq q) \xrightarrow{w} Q^{(m, k)}_{1, 1, \beta, \alpha}(q) = \begin{cases} 1, & q \geq 0, \\ 1 - \int_0^\infty \Gamma_\ell ((|q|^{y^{-1/\alpha}})^{(m+1)\beta})e^{-y}dy, & q < 0, \end{cases}$$

Moreover, If $F \in D(\hat{H}^{(m, k)}_{1, \alpha})$, $F \in D(\overline{H}^{(m, k)}_{2, \beta})$ and $x^0 = 0$, then

$$P(q_n^*(m, k) \leq q) \xrightarrow{w} Q^{(m, k)}_{2, 1, \beta, \alpha}(q) = \begin{cases} 0, & q < 0, \\ 1 - \int_0^\infty \Gamma_\ell ((|q|^{y^{-1/\alpha}})^{(m+1)\beta})e^{-y}dy, & q \geq 0. \end{cases}$$

Finally, if $F \in D(\hat{H}^{(m, k)}_{2, \alpha})$, $F \in D(\overline{H}^{(m, k)}_{1, \beta})$ and $x_0 = 0$, then

$$P(q_n^*(m, k) \leq q) \xrightarrow{w} Q^{(m, k)}_{2, 1, \beta, \alpha}(q) = \begin{cases} 0, & q < 0, \\ \int_0^\infty \Gamma_\ell ((|q|^{y^{1/\alpha}})^{(m+1)\beta})e^{-y}dy, & q \geq 0. \end{cases}$$

In the above three cases, we can take $A_{n,m} = a_{n,m}/\alpha_{n,m}$ and $B_{n,m} = 0$.

Part 2. Let $F \in D(\hat{H}^{(m, k)}_{2, \alpha})$ and $F \in D(\overline{H}^{(m, k)}_{2, \beta})$, where $x_0, x^0 \neq 0$ and $\alpha = \beta (m + 1)$. Then, the df of $q_n^*(m, k)$ converges weakly to a non-degenerate df if, and only if,

$$\eta = \lim_{n \to \infty} \frac{a_{n,m}}{\alpha_{n,m}} \text{ exists, with } 0 \leq \eta \leq \infty.$$
The convergence is non-trivial if, and only if, $0 < \eta < \infty$. In this case we can take $A_{n,m} = a_{n,m}/|\beta_{n,m}|$ and $B_{n,m} = b_{n,m}/|\beta_{n,m}|$. The non-trivial types are

**Type I:** $Q_{2,2,\beta,\alpha}(q) = \int_{-\infty}^{\infty} (H_{2,\beta}^{(m)}(\eta | x_0/x_0 | (q-z))) d(1-H_{2,\beta}^{(m)}(-z))$, if $x_0 < 0$.

**Type II:** $1 - Q_{2,2,\beta,\alpha}(-q)$, if $x_0 > 0$.

**Type III:** $Q_{2,2,\beta,\alpha}^{(m,k):3}(q) = 1 - Q_{2,2,\beta,\alpha}^{(m,k):2}(-q)$, if $x_0 < 0 < x_0^*$, where $Q_{2,2,\beta,\alpha}^{(m,k):2}(q) = \int_{-\infty}^{\infty} (H_{2,\beta}^{(m)}(\eta | x_0/x_0 | (q-z))) d(1-H_{2,\beta}^{(m)}(-z))$.

**Part 3.** Let $F \in D(H_{3,0}^{(m,k)})$ and $F \in D(H_{3,0}^{(m,k)})$. In order that $q_n^{(m,k)}$ converges weakly to a non-degenerate df, it is necessary and sufficient that

\begin{equation}
\zeta = \lim_{n \to \infty} \frac{1}{a_{n,m}} \left| \frac{a_{n,m}}{b_{n,m}} \right| \end{equation}

exists, with $0 \leq \zeta < \infty$. The convergence is non-trivial if, and only if, $0 < \zeta < \infty$. The limit type in this case is given by $Q_{3,0}^{(m,k)}(q) = \overline{H}_{3,0}^{(m,k)}(\zeta^{-1}q) \ast \overline{H}_{3,0}^{(m,k)}(q)$, where \(\ast\) denotes the convolution operator,

\[
\overline{H}_{3,0}^{(m,k)}(q) = \left\{ \begin{array}{ll}
\overline{H}_{3,0}^{(m,k)}(q), & x_0 \geq 0, \\
1 - \overline{H}_{3,0}^{(m,k)}(-q), & x_0 < 0,
\end{array} \right.
\]

and $H_{3,0}^{(m,k)}(q) = \left\{ \begin{array}{ll}
H_{3,0}^{(m,k)}(q), & x_0 \leq 0, \\
1 - H_{3,0}^{(m,k)}(-q), & x_0 > 0.
\end{array} \right.$

**Corollary 2.1.** Let $F \in D(H_{3,0}^{(m,k)})$, $F \in D(H_{3,0}^{(m,k)})$ and $x_0, x_0 \in (0, \infty) \setminus \{0\}$. Then $q_n^{(m,k)}$ converges weakly to a non-degenerate df if, and only if, the condition (2.1) is satisfied. In this case, we can take $A_{n,m} = a_{n,m}/|\beta_{n,m}|$ and $B_{n,m} = b_{n,m}/|\beta_{n,m}|$. Moreover, the non-trivial types are

**Type I:** $Q_{3,0}^{(m,k):1}(q) = \int_{-\infty}^{\infty} (H_{3,0}^{(m,k)}(\eta | x_0 / x_0 | (q-z))) d(1-H_{3,0}^{(m,k)}(-z))$, if $x_0 < 0$.

**Type II:** $1 - Q_{3,0}^{(m,k):1}(-q)$, if $x_0 > 0$.

**Type III:** $Q_{3,0}^{(m,k):3}(q) = 1 - Q_{3,0}^{(m,k):2}(-q)$, if $x_0 < 0 < x_0^*$, where $Q_{3,0}^{(m,k):2}(q) = \int_{-\infty}^{\infty} (H_{3,0}^{(m,k)}(\eta | x_0 / x_0 | (q-z))) d(H_{3,0}^{(m,k)}(z))$.

**Theorem 2.2 (Trivial types).** **Part 1.** If $F \in D(H_{3,0}^{(m,k)})$ and $F \in D(H_{1,\beta}^{(m,k)})$, then

\[
P(q_n^{(m,k)} \leq q) \xrightarrow{w} Q_{1,3,\beta,0}^{(m,k)}(q) = \left\{ \begin{array}{ll}
\overline{H}_{1,\beta}^{(m,k)}(q), & x_0 > 0, \\
1 - \overline{H}_{1,\beta}^{(m,k)}(-q), & x_0 < 0.
\end{array} \right.
\]

Moreover, if $F \in D(H_{2,\alpha}^{(m,k)})$, $F \in D(H_{1,\beta}^{(m,k)})$ and $x_0 \neq 0$, then the df of $q_n^{(m,k)}$ converges weakly to the trivial df $Q_{1,2,\beta,\alpha}^{(m,k)}(q) = Q_{1,3,\beta,0}^{(m,k)}(q)$. Finally, the normalizing constants in the above two cases can be chosen as $A_{n,m} = a_{n,m}/|\beta_{n,m}|$ and $B_{n,m} = 0$.

**Part 2.** If $F \in D(H_{1,\alpha}^{(m,k)})$ and $F \in D(H_{3,0}^{(m,k)})$, then

\[
P(q_n^{(m,k)} \leq q) \xrightarrow{w} Q_{3,1,0,\alpha}^{(m,k)}(q) = \left\{ \begin{array}{ll}
(1 - e^{-q\alpha})I_{[0,\infty)}(q), & x_0 < 0, \\
e^{-q\alpha}I_{(-\infty,0)}(q) + I_{[0,\infty)}(q), & x_0 > 0.
\end{array} \right.
\]

Moreover, if $F \in D(H_{1,\alpha}^{(m,k)})$, $F \in D(H_{2,\beta}^{(m,k)})$ and $x_0 \neq 0$, then the df of $q_n^{(m,k)}$ converges weakly to the trivial df $Q_{2,1,\beta,\alpha}^{(m,k)}(q) = Q_{3,1,0,\alpha}^{(m,k)}(q)$. In the above two cases we can take $A_{n,m} = \cdots$
\[ \frac{[b_{n,m}]}{a_{n,m}} \text{ and } B_{n,m} = 0. \]

**Part 3.** Let \( F \in D(H_{3,0}^{(m,k)}) \) and \( F \in D(H_{3,β}^{(m,k)}) \). Then the df of \( q_{n}^{*}(m,k) \) converges weakly to the trivial df

\[
Q_{2,3,β,0}(q) = \begin{cases} 
1 - H_{3,0}^{(m,k)}(-q), & \text{if } x^0 = 0, \text{ with } A_{n,m} = a_{n,m}, B_{n,m} = 0, \\
H_{3,0}^{(m,k)}(q), & \text{if } x^0 < 0, \text{ with } A_{n,m} = \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = \frac{b_{n,m}}{[b_{n,m}]}, \\
1 - H_{3,0}^{(m,k)}(-q), & \text{if } x^0 > 0, \text{ with } A_{n,m} = \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = \frac{b_{n,m}}{[b_{n,m}]}. 
\end{cases}
\]

**Part 4.** Let \( F \in D(H_{2,α}^{(m,k)}) \) and \( F \in D(H_{3,0}^{(m,k)}) \). Then the df of \( q_{n}^{*}(m,k) \) converges weakly to the trivial df

\[
Q_{3,2,0,α}(q) = \begin{cases} 
1 - I_{(0,∞)}(-q), & \text{if } x^0 = 0, \text{ with } A_{n,m} = b_{n,m} - \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = 0, \\
I_{(0,∞)}(q), & \text{if } x^0 < 0, \text{ with } A_{n,m} = \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = \frac{b_{n,m}}{[b_{n,m}]} - \frac{b_{n,m}}{[b_{n,m}]}, \\
I_{(0,∞)}(-q), & \text{if } x^0 > 0, \text{ with } A_{n,m} = \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = \frac{b_{n,m}}{[b_{n,m}]} - \frac{b_{n,m}}{[b_{n,m}]}.
\end{cases}
\]

**Part 5.** Let \( F \in D(H_{2,α}^{(m,k)}) \) and \( F \in D(H_{3,0}^{(m,k)}) \). Then the df of \( q_{n}^{*}(m,k) \) converges weakly to the trivial df

\[
Q_{2,2,β,α}(q) = \begin{cases} 
e^{-q}I_{(0,∞)}(-q), & \text{if } x^0 = 0, \text{ with } A_{n,m} = b_{n,m} - \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = 0, \\
1 - H_{2,β}^{(m,k)}(-q), & \text{if } x^0 < 0, \text{ with } A_{n,m} = \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = \frac{b_{n,m}}{[b_{n,m}]}, \\
H_{2,β}^{(m,k)}(q), & \text{if } x^0 > 0, \text{ with } A_{n,m} = \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = \frac{b_{n,m}}{[b_{n,m}]}, \\
H_{2,α}^{(m,k)}(-q), & \text{if } x^0 > 0, \text{ with } A_{n,m} = \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = \frac{b_{n,m}}{[b_{n,m}]}, \\
1 - H_{2,α}^{(m,k)}(-q), & \text{if } x^0 > 0, \text{ with } A_{n,m} = \frac{a_{n,m}}{[b_{n,m}]} - \frac{a_{n,m}}{[b_{n,m}]^2}, B_{n,m} = \frac{b_{n,m}}{[b_{n,m}]}.
\end{cases}
\]

**2.1. Proofs**

The proof of the preceding two theorems depends on the following lemmas.

**Lemma 2.1.** Let \( F \in D(H_{3,0}^{(m,k)}) \), \( F \in D(H_{3,β}^{(m,k)}) \) and \( q_{n}^{*}(m,k) \) converges weakly to a non-degenerate df \( Q \), then \( a_{n,m}^{-1}b_{n,m}^{-1} |b_{n,m}^{-1} - \beta_{n,m}| \rightarrow \zeta \), as \( n \rightarrow \infty \), \( 0 < \zeta \leq \infty \).

**Proof.** Suppose that the df of \( q_{n}^{*}(m,k) \) converges weakly to a non-degenerate limit df. Take \( A_{n,m} = a_{n,m}/[b_{n,m}] \) and \( B_{n,m} = b_{n,m}/[b_{n,m}][b_{n,m}]^{-1} \).

Then

\[
q_{n}^{*}(m,k) = \frac{M_{n}^{*}(m,k) - (a_{n,m}^{-1}b_{n,m}^{-1})L_{n}^{*}(m,k)}{[b_{n,m}]^{-1}L_{n}(m,k)}
\]

where \( M_{n}^{*}(m,k) = a_{n,m}^{-1}(M_{n}(m,k) - b_{n,m}) \) and \( L_{n}^{*}(m,k) = a_{n,m}^{-1}(L_{n}(m,k) - \beta_{n,m}) \). On the other hand, we have \( b_{n,m} \uparrow x^0 \) and \( \beta_{n,m} \downarrow x_0 \), as \( n \rightarrow \infty \) (c.f. Gnedenko [11]). Thus, on account of Lemma 3.3 in Barakat [5], \( |\beta_{n,m}|^{-1}L_{n}(m,k) \xrightarrow{P} 1 \), if \( x_0 \geq 0 \), \( |\beta_{n,m}|^{-1}L_{n}(m,k) \xrightarrow{P} -1 \), if \( x_0 < 0 \), where \( \xrightarrow{P} \) denotes the convergence in probability, as \( n \rightarrow \infty \).

After some algebra, we get, for sufficiently large \( n \), the following representation

\[
q_{n}^{*}(m,k) \xrightarrow{w} S(\frac{M_{n}(m,k) - b_{n,m}}{a_{n,m}}) + S(\frac{a_{n,m}}{b_{n,m}^{-1}}L_{n}(m,k) - \beta_{n,m})/a_{n,m}, \quad (1)
\]
where “$X_n \xrightarrow{w} Y_n$” means that the rv’s $X_n$ and $Y_n$ have the same limit df,

$$S_0 = \begin{cases} 
-1, & x^0 > 0, \\
+1, & x^0 \leq 0, 
\end{cases} \quad \text{and} \quad S_0 = \begin{cases} 
1, & x_0 \geq 0, \\
-1, & x_0 < 0. 
\end{cases}$$

Now, in view of Theorems 1.1 and 1.2, we have $F \in D(H_{3.0}^{(0,1)})$ and $F \in D(\overline{H}_{3.0}^{(0,1)})$. Therefore, by replacing $n$ in $(\alpha_n, \beta_n)$ and $(\alpha_n, b_n)$ respectively by $\phi(n)$ and $\psi(n)$, we can easily see that the remaining part of the proof is the same as the proof of Lemma 3.4 in Barakat [5].

**Remark 2.1.** In the proof of Lemma 2.1, if we take $A_{n,m} = |b_{n,m}/\beta_{n,m}^2$ and $B_{n,m} = b_{n,m}/\beta_{n,m}$, we get the asymptotic relation

$$q_n^*(m,k) = \frac{w}{n} S_0(\mid b_{n,m}^{-1} \beta_{n,m} \mid a_{n,m} \alpha_{n,m}^{-1})(\frac{M_n(m,k) - b_{n,m}}{a_{n,m}}) + S_0(\frac{L_n(m,k) - \beta_{n,m}}{\alpha_{n,m}}). \quad (II)$$

**Remark 2.2.** If $F \in D(H_{2.0}^{(m,k)})$, $F \in D(\overline{H}_{2.0}^{(m,k)})$ and $x_0, x^0 \neq 0$, the asymptotic representations (I) and (II) hold with $\beta_{n,m} = x_0$ and $b_{n,m} = x^0$.

**Lemma 2.2.** For any $\epsilon > 0$, as $n \rightarrow \infty$, we have

(i) If $F \in D(H_{1.0}^{(m,k)})$, then $a_{n,m}n^{-(\beta(m+1))^{-1} + \epsilon} \rightarrow \infty$ and $a_{n,m}n^{-(\beta(m+1))^{-1} - \epsilon} \rightarrow 0$;

(ii) If $F \in D(H_{1.0}^{(m,k)})$, then $a_{n,m}n^{-(\alpha)^{-1} + \epsilon} \rightarrow \infty$ and $a_{n,m}n^{-(\alpha)^{-1} - \epsilon} \rightarrow 0$;

(iii) If $F \in D(H_{2.0}^{(m,k)})$, then $a_{n,m}n^{(\beta(m+1))^{-1} + \epsilon} \rightarrow \infty$ and $a_{n,m}n^{(\beta(m+1))^{-1} - \epsilon} \rightarrow 0$;

(iv) If $F \in D(H_{2.0}^{(m,k)})$, then $a_{n,m}n^{(\alpha)^{-1} + \epsilon} \rightarrow \infty$ and $a_{n,m}n^{(\alpha)^{-1} - \epsilon} \rightarrow 0$;

(v) If $F \in D(\overline{H}_{3.0}^{(m,k)})$, then $a_{n,m}n^{+\epsilon} \rightarrow \infty$ and $a_{n,m}n^{-\epsilon} \rightarrow 0$;

(vi) If $F \in D(\overline{H}_{3.0}^{(m,k)})$, then $a_{n,m}n^{+\epsilon} \rightarrow \infty$ and $a_{n,m}n^{-\epsilon} \rightarrow 0$.

**Proof.** In view of Theorems 1.1 and 1.2 and by replacing $n$ in $\alpha_n$ and $a_n$ respectively by $\phi(n)$ and $\psi(n)$, we can easily see that the proof of this lemma is exactly the same as the proof of Lemma 3.5 in Barakat [5].

**Lemma 2.3.** For any $\epsilon > 0$, as $n \rightarrow \infty$, we have

(i) If $F \in D(H_{3.0}^{(m,k)})$, then $\overline{d}_{n,m}n^{+\epsilon} \rightarrow \infty$ and $\overline{d}_{n,m}n^{-\epsilon} \rightarrow 0$, where $\overline{d}_{n,m} = a_{n,m}/|b_{n,m}|$.

(ii) If $F \in D(H_{3.0}^{(m,k)})$, then $\overline{d}_{n,m}n^{+\epsilon} \rightarrow \infty$ and $\overline{d}_{n,m}n^{-\epsilon} \rightarrow 0$, where $\overline{d}_{n,m} = a_{n,m}/|\beta_{n,m}|$.

**Proof.** Again since $F \in D(H_{3.0}^{(m,k)})$ and $F \in D(\overline{H}_{3.0}^{(m,k)})$, then in view of Theorems 1.1, 1.2, we get $F \in D(H_{3.0}^{(0,1)})$ and $F \in D(\overline{H}_{3.0}^{(0,1)})$. Therefore, by using Lemma 3.6 in Barakat [5], we get $\overline{d}_{n,m}n^{+\epsilon} \rightarrow \infty$; $\overline{d}_{n,m}n^{-\epsilon} \rightarrow 0$ and $\overline{d}_{n,m}n^{+\epsilon} \rightarrow \infty$; $\overline{d}_{n,m}n^{-\epsilon} \rightarrow 0$, where, $\overline{d}_{n} = a_{n}/|b_{n}|$ and $\overline{d}_{n} = a_{n}/|\beta_{n}|$. Then by replacing $n^{1/(m+1)}$ and $n(m+1)$ instead of $n$ we get the result.

**Proof of Theorem 2.1.** The proof of Part 1 follows, after some algebra, by using Lemma 1.1, Lemma 3.1 in Barakat [5] and Khinchine’s convergence to types theorem. To prove Part 2, we use Remark 2.2 with Lemma 2.1 and the result of de Haan [9, Theorem 2]). In this case, it is easy to show that $q_n^*(m,k)$ converges to a non-degenerate df if, and only if, (2.1) is satisfied with $0 < \eta < \infty$. Finally, to prove Part 3, we use Lemmas 2.1, 2.2 and the result of de Haan [9, Theorem 2]. It is easy in this case to show that $q_n^*(m,k)$ converges to a non-degenerate df if, and only if, (2.2) is satisfied, with $0 < \zeta < \infty$. 


Proof of Corollary 2.1. Under the assumption of the corollary, we have \( \lim_{n \to \infty} a_{n,m}/\alpha_{n,m} = \eta \). Therefore, the proof follows by using the representation (I) and by applying the result of Theorem 2.1, Part 3.

Proof of Theorem 2.2. Under the assumptions of Parts 1 and 2 and by using Lemma 1.1, Lemma 3.3 in Barakat [5], we get after some algebra the two representations

\[
q_n^*(m,k) \left\{ \frac{M_n(m,k)}{a_{n,m}}, \quad x_0 > 0, \quad \text{and} \quad \frac{M_n(m,k)}{a_{n,m}}, \quad x_0 < 0. \right. \]

respectively. The proof of the first two parts follows by using the above two representations. The proof of Part 3 follows, by virtue of Lemmas 1.1, 2.1, 2.2, 2.3 and the representation in (II), if \( x^0 \neq 0 \), or by virtue of the relation \( q_n^*(m,k) \left\{ \frac{M_n(m,k)}{a_{n,m}}, \right. \) if \( x^0 = 0 \). The proof of Part 4 follows, by virtue of Lemmas 1.1, 2.1, 2.2, 2.3 and the representation in (I), if \( x_0 \neq 0 \), or by virtue the relation \( q_n^*(m,k) \left\{ \frac{M_n(m,k)}{a_{n,m}}, \right. \) if \( x_0 = 0 \). Finally, the proof of Part 5 can be obtained after some algebra from the asymptotic representations (I) or (II) by choosing the normalizing constants, as stated in this part.

The class of possible non-trivial and trivial limit df’s of \( q_{d,n}(m,k) \) can be obtained in a simple way as Theorems 2.1 and 2.2, e.g.,

Theorem 2.3 (Non-trivial types for \( q_{d,n}(m,k) \)).

**Part 1.** Let \( F \in D(H_{1,\alpha}^{d(m,k)}) \) and \( F \in D(H_{1,\beta}^{d(m,k)}) \). Then the df of \( P(q_{d,n}(m,k) \leq q) \left\{ \frac{w}{n}, \right. \) \( \tilde{\alpha}_{n,m} = \alpha_{n,m}/\tilde{\alpha}_{n,m} \) and \( \tilde{\beta}_{n,m} = 0. \) Moreover, if \( F \in D(H_{1,\alpha}^{d(m,k)}) \), \( F \in D(H_{2,\beta}^{d(m,k)}) \) and \( x_0 = 0 \), then \( P(q_{d,n}(m,k) \leq q) \left\{ \frac{w}{n}, \right. \) \( Q_{2,1,\alpha,\beta}(q). \) Finally, if \( F \in D(H_{2,\alpha}^{d(m,k)}) \), \( F \in D(H_{1,\beta}^{d(m,k)}) \) and \( x^0 = 0 \). then \( P(q_{d,n}(m,k) \leq q) \left\{ \frac{w}{n}, \right. \) \( Q_{1,2,\alpha,\beta}(q). \) \)

In the above three cases, we can take \( \tilde{\alpha}_{n,m} = \alpha_{n,m}/\tilde{\alpha}_{n,m} \) and \( \tilde{\beta}_{n,m} = 0. \)

**Part 2.** Let \( F \in D(H_{2,\alpha}^{d(m,k)}) \), \( F \in D(H_{2,\beta}^{d(m,k)}) \), where \( x_0, x^0 \neq 0 \) and \( \beta = \alpha(m+1) \). Then \( q_{d,n}(m,k) \) converges weakly to a non-degenerate df, it is necessary and sufficient that \( \tau = \lim_{n \to \infty} \tilde{\alpha}_{n,m}/\tilde{\alpha}_{n,m} \) exists, with \( 0 < \tau < \infty \). The convergence is non-trivial if, and only if, \( 0 < \tau < \infty \). Moreover, we can take \( \tilde{\alpha}_{n,m} = \tilde{\alpha}_{n,m}/\tilde{\beta}_{n,m} \) and \( \tilde{\beta}_{n,m} = \tilde{\beta}_{n,m}/\tilde{\beta}_{n,m} \). The non-trivial types are

**Type I:** \( Q_{2,2,\alpha,\beta}(q) = \int_{-\infty}^{q} Q_{2,2,\alpha,\beta}(\tau |x_0/x^0|(q - z)) d(1 - H_{2,\beta}^{d(m,k)}(-z)), \) if \( x_0 > 0 \).

**Type II:** \( 1 - Q_{2,2,\alpha,\beta}(q), \) if \( x^0 < 0 \).

**Type III:** \( Q_{2,2,\alpha,\beta}(q) = 1 - Q_{2,2,\alpha,\beta}(q), \) if \( x_0 < 0 > x^0 \),

where \( Q_{2,2,\alpha,\beta}(q) = \int_{-\infty}^{\infty} \left( \tau |x_0/x^0|(q - z) \right) dH_{2,\beta}^{d(m,k)}(z) \).

**Part 3.** Let \( F \in D(H_{3,0}^{d(m,k)}) \) and \( F \in D(H_{3,0}^{d(m,k)}) \). In order that \( q_{d,n}(m,k) \) converges weakly to a non-degenerate df, it is necessary and sufficient that \( \xi = \lim_{n \to \infty} \tilde{\alpha}_{n,m}/\tilde{\alpha}_{n,m} \) exists, with \( 0 < \xi < \infty \). The convergence is non-trivial if, and only if, \( 0 < \xi < \infty \). Moreover, the limit type in this case is given by \( Q_{3,0}^{d(m,k)}(q) = \int_{-\infty}^{q} Q_{3,0}^{d(m,k)}(\xi |x_0/x|) dH_{3,0}^{d(m,k)}(q), \) where

\[
H_{3,0}^{d(m,k)}(q) = \left\{ \begin{array}{ll}
H_{3,0}^{d(m,k)}(q), & x^0 \leq 0, \\
1 - H_{3,0}^{d(m,k)}(-q), & x^0 > 0,
\end{array} \right.
\]

and \( Q_{3,0}^{d(m,k)}(q) = \left\{ \begin{array}{ll}
Q_{3,0}^{d(m,k)}(q), & x_0 \geq 0, \\
1 - Q_{3,0}^{d(m,k)}(-q), & x_0 < 0.
\end{array} \right. \)
3. The generalized extremal quotient (the case $m = -1$, i.e., record values)

The upper (lower) record model can be obtained as a special case of gos (dgos) model by putting $m = -1$ and $k = 1$. In this section we consider the limit behavior of the statistic

$$q^*_{n(-1,1)} = C_n^{-1}(q_n(-1,1) - D_n),$$

where $C_n > 0$ and $D_n$ are suitable sequences of normalizing constants. Beside the definition of the record values based on the concept of the gos, the upper record values (or simply a record) can be defined as an observation $X_j$, such that $X_j > \max(X_1,\ldots,X_{j-1})$. By convention $X_1$ is a record value. The indices at which record values occur are given by the rv’s $T_n = \min\{j : j > T_{n-1}, X_j > X_{j-1}, n > 1\}$ and $T_1 = 1$.

Thus, the record value sequence $\{R_n\}$ is then defined by $R_n = X_{T_n}$, $n \geq 1$. Consequently, the record extremal quotient is defined by $q_n(-1,1) = R_n/R_1 = R_n/X_1$. Therefore, we expect that the limit df of $q^*_n$ will depend on the population df $F(x)$. The explicit form of the df of $R_n$ is given by

$$P(R_n \leq x) = \begin{cases} 1 - \Gamma_n(H(x)), & \text{if } n > 1, \\ F(x), & \text{if } n = 1, \end{cases}$$

where $H(x) = -\log(1 - F(x))$ is the hazard function of the df $F$ (see Arnold et al. [2]).

Resnick [15] showed that the possible limiting record value distributions of the suitably normalized record $R^*_n = c_n^{-1}(R_n - d_n)$, $c_n > 0$, $d_n \in \mathbb{R}$, are

$$\mathcal{H}^{-1,1}_{i,\beta}(x) = \mathcal{N}(\log(\mathcal{H}^{0,1}_{i,\beta}(x))) = \mathcal{N}(V_{i,\beta}(x)), \ i = 1,2,3,$$

where $\mathcal{N}(\cdot)$ is the standard normal distribution, $\mathcal{H}^{0,1}_{i,\beta}$ is an maximum value distribution and the functions $V_{i,\beta}$, $i = 1,2,3$, are defined in (1.4). In this case we say that $F$ is in the domain of record attract of $\mathcal{H}^{-1,1}_{i,\beta}$ and write $F \in D_R(\mathcal{H}^{-1,1}_{i,\beta})$. The following theorem due to Resnick [15] (see Arnold et al., [2]) is a basic tool of our study in this section.

**Theorem 3.1 (Duality Theorem).** If an associated df $F_a$ is defined by $F_a = 1 - \exp(-\sqrt{H(x)})$ and $\Psi_F(n) = \inf\{y : F(y) > 1 - e^{-n}\} = F^{-1}(1 - e^{-n}) \to x^0$, as $n \to \infty$, then the following limit implications hold:

(i) $F \in D_R(\mathcal{H}^{-1,1}_{1,\alpha})$ if, and only if, $F_a \in D(\mathcal{H}^{0,1}_{1,\alpha/2})$ and in this case we may use as normalizing constants $c_n = \Psi_F(n)$ and $d_n = 0$;

(ii) $F \in D_R(\mathcal{H}^{-1,1}_{2,\alpha})$ if, and only if, $F_a \in D(\mathcal{H}^{0,1}_{2,\alpha/2})$. In this case $F^{-1}(1) = x^0$ is necessarily finite (see Lemma 1.1) and we may use as normalizing constants $c_n = x^0 - \Psi_F(n)$ and $d_n = x^0$;

(iii) $F \in D_R(\mathcal{H}^{-1,1}_{3,0})$ if, and only if, $F_a \in D(\mathcal{H}^{0,1}_{3,0})$ and in this case we may use as normalizing constants $c_n = \Psi_F(n + \sqrt{n}) - \Psi_F(n)$ and $d_n = \Psi_F(n)$.

The following theorem fully characterizes the possible limit df’s of $q^*_n(-1,1)$.

**Theorem 3.2.** Let $C_n > 0$ and $D_n$ be suitable normalizing constants. Furthermore, let $q^*_n(-1,1) = C_n^{-1}(q_n(-1,1) - D_n)$. Then, we have the following implications:

(i) If $F \in D_R(\mathcal{H}^{-1,1}_{1,\alpha})$, then

$$P(q^*_n(-1,1) \leq q) \xrightarrow{n \to \infty} \begin{cases} F(0) + \int_0^\infty \mathcal{N}(x \log qx) \, dx, & \text{if } q \geq 0, \\ F(0) - \int_{-\infty}^0 \mathcal{N}(x \log qx) \, dx, & \text{if } q \leq 0, \end{cases}$$

with $C_n = c_n = \Psi_F(n)$ and $D_n = -1$.

(ii) If
(a) $F \in D_{R}(\overline{H}_{1:1},)$, $x^0 > 0$ or
(b) $F \in D_{R}(\overline{H}_{1:1}),$ $0 < x^0 < \infty$ or
(c) $F \in D_{R}(\overline{H}_{1:1}),$ $x^0 = \infty, \frac{\Psi_{F}(n+\sqrt{n})}{\Psi_{F}(n)} \rightarrow 1$, as $n \rightarrow \infty$, then

\[ P(q^*_n(-1, 1) \leq q) \xrightarrow{w} P(W \leq q + 1), \]

where $W = 1/X_1$, with $C_n = d_n$ and $D_n = d_n$.

**Proof.** First, we notice that the condition $x^0 > 0$, in Part (ii), guarantees that the scale normalizing constant $C_n = d_n$ will be positive (at least for large $n$), namely, $d_n = x^0 > 0$, in Part (a) and $C_n = d_n = \Psi_F(n) \rightarrow x^0 > 0$, as $n \rightarrow \infty$, in Part (b)). Now, it is easy to check the validity of the representation

\[ q^*_n(-1, 1) \xrightarrow{w} \begin{cases} \frac{R^*_n}{X_1}, & \text{if } C_n = c_n, D_n = d_n = 0, \\ \frac{c_n d_n^{-1} - (X_1 - 1)}{X_1}, & \text{if } C_n = d_n, D_n = d_n, \end{cases} \]

where $R^*_n = c_n^{-1}(R_n - d_n)$. The implication (i) follows from the first part of (3.1), Theorem 3.1, Lemma 3.1 in Barakat [5] and from the independency between $R_n$ and $R_n$, if $s - r \rightarrow \infty$, as $n \rightarrow \infty$ (see Barakat [4]). The implication (ii) follows from the second part of (3.1) and Theorem 3.1 (note that Theorem 3.1 implies $c_n d_n^{-1} \rightarrow 0$, as $n \rightarrow \infty$, in Parts (a) and (b), while the condition $(\Psi_F(n + \sqrt{n})/\Psi_F(n) \rightarrow 1$, as $n \rightarrow \infty$, implies $c_n d_n^{-1} \rightarrow 0$, as $n \rightarrow \infty$, in Part (c)).

**Example 3.1.** For the Weibull, $F_1(x) = P(X_1 \leq x) = 1 - e^{-x^c}, x > 0$, and the Logistic $F_2(x) = P(X_2 \leq x) = e^x/(1 + e^x), \forall x$, distributions, we can easily show that $\Psi_{F_1}(u) = u^1/c$ and $\Psi_{F_2}(u) = \log(e^u - 1)$, respectively. Therefore $(\Psi_{F_1}(n + \sqrt{n})/\Psi_{F_1}(n) = (1 + 1/\sqrt{n})^{1/c} \rightarrow 1$, as $n \rightarrow \infty$, and $(\Psi_{F_2}(n + \sqrt{n})/\Psi_{F_2}(n) = (\log(e^{n+\sqrt{n}} - 1))/\log(e^{n-1}) \rightarrow 1$, as $n \rightarrow \infty$. Thus, for both distributions, we get $P(q^*_n(-1, 1) \leq q) \xrightarrow{w} P(W_i \leq q + 1)$, where $W_i = 1/X_i, i = 1, 2$.

**4. Applications**

In this section, within some illustrative examples, we show that the domains of attraction of the non-trivial types of the generalized extremal quotient are non-empty. Some of these examples individually express intersecting facts. In all the following examples, the normalizing constants can be found in Table 4.1.

**Example 4.1 (Standard Cauchy Distribution).** It can be shown that $\alpha_n^{-1}(M_n - b_n)$ and $\alpha_n^{-1}(L_n - \beta_n)$ weakly converge to $\overline{H}_{1:1}$ and $\overline{H}_{1:1}^{(0,1)}$, respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.1, Part 1, we get

\[ P(q^*_n(m, k) \leq q) \xrightarrow{w} Q^{(m,k)}_{1,1,1,1}(q) = \begin{cases} 1, & q \geq 0, \\ 1 - \int_0^\infty \Gamma_{e^k/|q|}((k/|q|)^{(m+1)})e^{-y} dy, & q < 0, \end{cases} \]

**Example 4.2 (Pareto Distribution).** It can be shown that, for the Pareto distribution $F(x) = (1 - x^{-\sigma})I_{[1, \infty)}(x), \sigma > 0, \alpha_n^{-1}(M_n - b_n)$ and $\alpha_n^{-1}(L_n - \beta_n)$ weakly converge to $\overline{H}_{1:1}^{(0,1)}$ and $\overline{H}_{1:1}^{(0,1)}$, respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.1, Part 1, we get

\[ P(q^*_n(m, k) \leq q) \xrightarrow{w} Q^{(m,k)}_{1,1,1,1}(q) = \begin{cases} 1, & q \geq 0, \\ 1 - \int_0^\infty \Gamma_{e^k/|q|}((k/|q|)^{(m+1)}\sigma)e^{-y} dy, & q < 0, \end{cases} \]
Example 4.3 (Uniform Distribution). For the uniform \((-\theta, \theta), (-\theta, 0)\) and \((0, \theta)\) distributions, it can be shown that \(a_n^{-1}(M_n - b_n)\) and \(\alpha_n^{-1}(L_n - \beta_n)\) weakly converge to \(\overline{H}_{2,1}^{(0,1)}\) and \(H_{2,1}^{(0,1)}\), respectively. Therefore, for the uniform \((-\theta, \theta)\) distribution, in view of Theorems 1.1, 1.2, we can easily show that

\[
\eta = \lim_{n \to \infty} \frac{a_{n,m}}{\alpha_{n,m}} = \begin{cases} 
1, & \text{if } m = 0, \\
\infty, & \text{if } m > 0.
\end{cases}
\]

Thus, by using Theorem 2.1, Part 2, we get the non-trivial convergence \(P(q_n^*(0,k) \leq q) \xrightarrow{w} \mathcal{Q}_{2,1,1}^{(0,k);3}(q) = 1 - \mathcal{Q}_{2,2,1,1}^{(0,k);2}(q)\), where \(\mathcal{Q}_{2,2,1,1}^{(0,k);2}(q) = \int_{-\infty}^{\infty} \mathcal{H}_{2,1}^{(0,k)}(q-z)d\overline{H}_{2,1}^{(0,1)}(z)\). Moreover, for the uniform \((-\theta, 0)\) and \((0, \theta)\) distributions, by using Theorem 2.2, Part 5, the df of the statistic \(q_n^*(m,k)\) weakly converges to the trivial types \(1 - \overline{H}_{2,1}^{(m,k)}(-q)\) and \(e^{-q^{-1}}I_{[0,\infty)}(q)\), respectively.

Example 4.4 (Beta(\(\alpha, \beta\) Distribution). For the beta distribution \(F(x; \alpha, \beta), 0 < x < 1\), \(\alpha, \beta > 0\), it can be shown that \(a_n^{-1}(M_n - b_n)\) and \(\alpha_n^{-1}(L_n - \beta_n)\) weakly converge to \(\overline{H}_{2,1}^{(0,1)}\) and \(H_{2,1}^{(0,1)}\), respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.2, Part 5 (i), we have the trivial convergence (since \(x_0 = 0\)) \(P(q_n^*(m,k) \leq q) \xrightarrow{w} e^{-q^{-1}}I_{[0,\infty)}(q)\). Clearly, the same result holds for the power distribution \(F(x; \alpha, 1)\).

Example 4.5 (Standard Normal, Logistic, Laplace and Log-Normal Distributions). It is known that (see Ahsanullah and Nevzorov [1]), for the above four distributions, we have \(a_n^{-1}(M_n - b_n)\) and \(\alpha_n^{-1}(L_n - \beta_n)\) weakly converge to \(\overline{H}_{3,0}^{(0,1)}\) and \(H_{3,0}^{(0,1)}\), respectively. Therefore, in view of Theorems 1.1 and 1.2, and after some algebra, we get

\[\zeta = \lim_{n \to \infty} a_{n,m}^{-1} \alpha_{n,m} \mid b_{n,m} \beta_{n,m}^{-1} = \begin{cases} 
\frac{4 + M \log M}{4M}, & \text{for the normal distribution}, \\
\frac{1}{M}, & \text{for the logistic and Laplace distributions}, \\
\frac{1}{\sqrt{M}}, & \text{for the log-normal distribution},
\end{cases}\]

where \(M = m + 1\). Thus, in view of Theorem 2.1, Part 3, we get \(P(q_n^*(m,k) \leq q) \xrightarrow{w} \left(1 - \mathcal{H}_{3}^{(m,k)}(-\zeta^{-1}q)\right) \ast \left(1 - \overline{H}_{3}^{(m,k)}(-q)\right)\), where \(\zeta\) is given by (4.1). This example reveals the following interesting facts:

1. The extremal quotient for the logistic and Laplace distributions weakly converges to the same limit df.
2. In the case of ordinary order statistics, i.e., \(m = 0, k = 1\), the extremal quotient, for the normal, logistic, Laplace and log-normal distributions, weakly converges to the same limit df, as the limit df of the sample range, i.e., \(\overline{H}_{3}^{(0,1)}(q) \ast H_{3}^{(0,1)}(q)\).

Example 4.6 (Exponential(\(\sigma\))). It can be shown that \(a_n^{-1}(M_n - b_n)\) and \(\alpha_n^{-1}(L_n - \beta_n)\) weakly converge to \(\overline{H}_{3,0}^{(0,1)}\) and \(H_{2,1}^{(0,1)}\), respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.2, Part 4, we get the trivial convergence (since \(x_0 = 0\)) \(P(q_n^*(m,k) \leq q) \xrightarrow{w} e^{-q^{-1}}I_{[0,\infty)}(q)\).

Example 4.7 (Rayleigh(\(\sigma\))). For the Rayleigh distribution \(F(x) = \left(1 - e^{-x^2/\sigma^2}\right)I_{[0,\infty)}(x)\), \(\sigma > 0\), it can be shown that \(a_n^{-1}(M_n - b_n)\) and \(\alpha_n^{-1}(L_n - \beta_n)\) weakly converge to \(\overline{H}_{3,0}^{(0,1)}\) and
respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.2, Part 4, we get the trivial convergence (since $x_0 = 0$) $P(q_n^*(m,k) \leq q) \xrightarrow{w}{\frac{w}{n}} e^{-q^{-2}I_{[0,\infty)}(q)}$.

**Example 4.8** (The sos Model). Consider a sos $X(r,n,1,1)$ (in this case $m = k = 1, \gamma_i = 1 + 2(n - i), i \in \{1, 2, ..., n - 1\}, \gamma_n = k = 1$ and $\ell = 1/2$), with $\alpha_i = 2 - 1/(n - i + 1), i \in \{1, 2, ..., n - 1\}$. Thus, Theorems 2.1, Part 1, and 3.1, Part 1 yield, the implications:

If $F \in D(H_{1,\alpha}^{(1,1)}), D(H_{2,\beta}^{(1,1)})$, and $F \in D(H_{1,\alpha}^{d(1,1)}), D(H_{2,\beta}^{(1,1)})$, then

$$P(q_n^*(1,1) \leq q) \xrightarrow{w}{\frac{w}{n}} Q_{1,1,\alpha}(q) = \begin{cases} 1, \\ -1 + 2 \int_0^\infty N(\sqrt{2} |q|^{\beta/\alpha} y^{-\beta/\alpha} e^{-y} dy, \quad q \geq 0, \\ -1 + 2 \int_0^\infty N(\sqrt{2} |q|^{\beta/\alpha} y^{-\beta/\alpha} e^{-y} dy, \quad q < 0, \end{cases}$$

and

$$P(q_d,n^*(1,1) \leq q) \xrightarrow{w}{\frac{w}{n}} Q_{d(1,1)}^{d(1,1)}(q) = \begin{cases} 1, \\ -1 + 2 \int_0^\infty N(\sqrt{2} |q|^{\beta/\alpha} y^{-\beta/\alpha} e^{-y} dy, \quad q \geq 0, \end{cases}$$

respectively. Moreover, if $x^0 = 0$, $F \in D(H_{1,\alpha}^{(1,1)}), D(H_{2,\beta}^{(1,1)})$ and $F \in D(H_{1,\alpha}^{d(1,1)}), D(H_{2,\beta}^{d(1,1)})$, $x^0 = 0$, then

$$P(q_n^*(1,1) \leq q) \xrightarrow{w}{\frac{w}{n}} Q_{2,1,\alpha}(q) = \begin{cases} 0, \\ -1 + 2 \int_0^\infty N(\sqrt{2} |q|^{\beta/\alpha} y^{-\beta/\alpha} e^{-y} dy, \quad q \geq 0, \end{cases}$$

and

$$P(q_d,n^*(1,1) \leq q) \xrightarrow{w}{\frac{w}{n}} Q_{d(1,1)}^{d(1,1)}(q) = \begin{cases} 0, \\ -1 + 2 \int_0^\infty N(\sqrt{2} |q|^{\beta/\alpha} y^{-\beta/\alpha} e^{-y} dy, \quad q \geq 0, \end{cases}$$

respectively. Finally, if $x_0 = 0$, $F \in D(H_{1,\alpha}^{(1,1)}), D(H_{1,\beta}^{(1,1)})$ and $F \in D(H_{2,\beta}^{d(1,1)}), D(H_{1,\beta}^{d(1,1)})$, $x_0 = 0$, then

$$P(q_n^*(1,1) \leq q) \xrightarrow{w}{\frac{w}{n}} Q_{1,2,\alpha}(q) = \begin{cases} 0, \\ 2 - 2 \int_0^\infty N(\sqrt{2} |q|^{\beta/\alpha} y^{-\beta/\alpha} e^{-y} dy, \quad q \geq 0, \end{cases}$$

and

$$P(q_d,n^*(1,1) \leq q) \xrightarrow{w}{\frac{w}{n}} Q_{d(1,1)}^{d(1,1)}(q) = \begin{cases} 0, \\ 2 - 2 \int_0^\infty N(\sqrt{2} |q|^{\beta/\alpha} y^{-\beta/\alpha} e^{-y} dy, \quad q \geq 0, \end{cases}$$

respectively.
Table 1. The normalizing constants of some popular distributions.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>((a_n, \beta_n))</th>
<th>((\alpha_n, \beta_n))</th>
<th>((A_{n,m}, B_{n,m}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Cauchy</td>
<td>((n/\pi, 0))</td>
<td>((\cot \pi/n, 0))</td>
<td>((a_{n,m}/b_{n,m}, 0))</td>
</tr>
<tr>
<td>Pareto</td>
<td>(n^{1/\sigma}, 0)</td>
<td>((n/(n+1))^{1/\sigma}, 0)</td>
<td>((a_{n,m}/b_{n,m}, 0))</td>
</tr>
<tr>
<td>Beta((\alpha, \beta))</td>
<td>((\beta/(cn))^{1/\beta}, 1)</td>
<td>(((c \alpha)/n)^{1/\alpha}, 0)</td>
<td>((b_{n,m}/a_{n,m}, 0))</td>
</tr>
<tr>
<td>Power((\alpha))</td>
<td>(n^{-1/\alpha}, 1)</td>
<td>((1/(\alpha n), 0))</td>
<td>((b_{n,m}/a_{n,m}, 0))</td>
</tr>
<tr>
<td>Standard Normal</td>
<td>((\tau_n, \rho_n))</td>
<td>((\tau_n, -\rho_n))</td>
<td>((a_{n,m}/</td>
</tr>
<tr>
<td>Logistic</td>
<td>((1, \log n))</td>
<td>((1, -\log n))</td>
<td>((a_{n,m}/</td>
</tr>
<tr>
<td>Laplace</td>
<td>((1, \log n/2))</td>
<td>((1, -\log n/2))</td>
<td>((a_{n,m}/</td>
</tr>
<tr>
<td>log-Normal</td>
<td>((\tau_ne^{\rho_n}, e^{\rho_n}))</td>
<td>((\tau_ne^{-\rho_n}, e^{-\rho_n}))</td>
<td>((a_{n,m}/</td>
</tr>
<tr>
<td>Exponential((\sigma))</td>
<td>((1/\sigma, 1/\sigma \log n))</td>
<td>((1/\sigma, 1/\sigma \log 4\pi))</td>
<td>((b_{n,m}/a_{n,m}, 0))</td>
</tr>
<tr>
<td>Rayleigh((\sigma))</td>
<td>((\sigma/2(\log n)^{-1/2}, \sigma(\log n)^{1/2}))</td>
<td>((\sigma/\sqrt{n}, 0))</td>
<td>((b_{n,m}/a_{n,m}, 0))</td>
</tr>
<tr>
<td>Uniform ((-\theta, \theta))</td>
<td>((2\theta/n, \theta))</td>
<td>((2\theta/n, -\theta))</td>
<td>((a_{n,m}/</td>
</tr>
<tr>
<td>Uniform ((0, \theta))</td>
<td>((\theta/n, \theta))</td>
<td>((\theta/n, 0))</td>
<td>((b_{n,m}/a_{n,m}, 0))</td>
</tr>
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<td>Uniform ((-\theta, 0))</td>
<td>((\theta/n, 0))</td>
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<td>((a_{n,m}/</td>
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</table>

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References
