Total Colorings of Planar Graphs with Small Maximum Degree

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Abstract. Let G be a planar graph of maximum degree Δ and girth g, and there is an integer t (> g) such that G has no cycles of length from g + 1 to t. Then the total chromatic number of G is Δ + 1 if (Δ, g, t) ∈ { (5, 4, 6), (4, 4, 17) }; or Δ = 3 and (g, t) ∈ { (5, 13), (6, 11), (7, 11), (8, 10), (9, 10) }, where each vertex is incident with at most one g-cycle.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for the terminologies and notations not defined here. Let G be a graph. We use V(G), E(G), Δ(G) and δ(G) (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. For a vertex v ∈ V, let N(v) denote the set of vertices adjacent to v, and let d(v) = |N(v)| denote the degree of v. A k-vertex, a k+-vertex or a k--vertex is a vertex of degree k, at least k or at most k respectively. A k-cycle is a cycle of length k, and a 3-cycle is usually called a triangle.

A total-k-coloring of a graph G is a coloring of V ∪ E using k colors such that no two adjacent or incident elements receive the same color. The total chromatic number χ''(G) of G is the smallest integer k such that G has a total-k-coloring. Clearly, χ''(G) ≥ Δ + 1.

Behzad [1] posed independently the following famous conjecture, which is known as the Total Coloring Conjecture (TCC).

Conjecture 1.1. For any graph G, Δ + 1 ≤ χ''(G) ≤ Δ + 2.

This conjecture was confirmed for a general graph with Δ ≤ 5. But for planar graph, the only open case is Δ = 6 (see [11, 15]). Interestingly, planar graphs with high maximum degree allow a stronger assertion, that is, every planar graph with high maximum degree Δ is (Δ + 1)-totally-colorable. This result was first established in [3] for Δ ≥ 14, which was extended to Δ ≥ 9 (see [12]). For 4 ≤ Δ ≤ 8, it is not known whether the assertion still...
holds true. But there are many related results by adding girth restrictions, see [7–10, 13, 14, 16]. We present our new results in this paper.

**Theorem 1.1.** Let $G$ be a planar graph of maximum degree $\Delta$ and girth $g$, and there is an integer $t (> g)$ such that $G$ has no cycles of length from $g + 1$ to $t$. Then the total chromatic number of $G$ is $\Delta + 1$ if (a) $(\Delta, g, t) = (5, 4, 6)$ or (b) $(\Delta, g, t) = (4, 4, 17)$.

Borodin et al. [6] obtained that if a planar graph $G$ of maximum degree three contains no cycles of length from 3 to 9, then $\chi''(G) = \Delta + 1$. In the following, we make further efforts on the total-colorability of planar graph on the condition that $G$ contains some $k$-cycle, where $k \in \{5, \cdots, 9\}$. We get the following result.

**Theorem 1.2.** Let $G$ be a planar graph of maximum degree 3 and girth $g$, each vertex is incident with at most one $g$-cycle and there is an integer $t (> g)$ such that $G$ has no cycles of length from $g + 1$ to $t$. Then $\chi''(G) = \Delta + 1$ if one of the following conditions holds.

(a) $g = 5$ and $t \geq 13$, (b) $g = 6$ and $t \geq 11$, (c) $g = 7$ and $t \geq 11$, (d) $g = 8$ and $t \geq 10$, (e) $g = 9$ and $t \geq 10$.

We will introduce some more notations and definitions here for convenience. Let $G = (V, E, F)$ be a planar graph, where $F$ is the face set of $G$. The degree of a face $f$, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A $k$-face or a $k^+$-face is a face of degree $k$ or at least $k$, respectively. Let $n_k(v)$ be the number of $k$-vertices adjacent to $v$ and $n_k(f)$ be the number of $k$-vertices incident with $f$.

## 2. Proof of Theorem 1.1

Let $G$ be a minimal counterexample to Theorem 1.1 in terms of the number of vertices and edges. Then every proper subgraph of $G$ is $(\Delta + 1)$-totally-colorable. Firstly, we investigate some structural properties of $G$, which will be used to derive the desired contradiction completing our proof.

**Lemma 2.1.** $G$ is 2-connected and hence, it has no vertices of degree 1 and the boundary $b(f)$ of each face $f$ in $G$ is exactly a cycle (i.e. $b(f)$ cannot pass through a vertex $v$ more than once).

**Lemma 2.2.** [5] $G$ contains no edge $uv$ with $\min\{d(u), d(v)\} \leq \lfloor \Delta/2 \rfloor$ and $d(u) + d(v) \leq \Delta + 1$.

**Lemma 2.3.** [3] The subgraph of $G$ induced by all edges joining 2-vertices to $\Delta$-vertices is a forest.

**Lemma 2.4.** [6] If $\Delta \geq 5$, then no 3-vertex is adjacent two 3-vertices.

Let $G_2$ be the subgraph induced by all edges incident with 2-vertices of $G$. Then $G_2$ is a forest by Lemma 2.3. We root it at a 5-vertex. In this case, every 2-vertex has exactly one parent and exactly one child, which are 5-vertices.

Since $G$ is a planar graph, by Euler’s formula, we have

$$\sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -8 < 0.$$ 

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = d(v) - 4$ if $v \in V$ and $ch(f) = d(f) - 4$ if $f \in F$. It follows that $\sum_{x \in V \cup F} ch(x) < 0$. Note that any
discharging procedure preserves the total charge of $G$. If we can define suitable discharging rules to change the initial charge function $ch$ to the final charge function $ch'$ on $V \cup F$, such that $ch'(x) \geq 0$ for all $x \in V \cup F$, then we get an obvious contradiction. Now we design appropriate discharging rules and redistribute weights accordingly.

For a vertex $v$, we define $f_k(v)$ or $f^+_k(v)$ to be the number of $k$-faces or $k^+$-faces incident with $v$, respectively. To prove (a), our discharging rules are defined as follows.

R11. Each 2-vertex receives 2 from its child.

R12. Each 3-vertex $v$ receives $1/(f^+_2(v))$ from each of its incident $7^+$-faces.

R13. Each $5^+$-vertex receives $1/3$ from each of its incident $7^+$-faces.

Next, we will check $ch'(x) \geq 0$ for all $x \in V \cup F$. Let $f \in F(G)$. If $d(f) = 4$, then $ch'(f) = ch(f) = 0$. Suppose $d(f) = 7$. Then $n_2(f) \leq 4$ by Lemma 2.4. Moreover, every 7-face sends at most 1/2 to its incident 3-vertices by R12 and 1/3 to its incident 5-vertices by R13. So we have $ch'(f) \geq ch(f) - 4 \times 1/2 - 3 \times 1/3 = 0$. Suppose $d(f) \geq 8$. Then $n_3(f) \leq [(2d(f))/3]$ by Lemma 2.2 and Lemma 2.4. Thus, $ch'(f) \geq ch(f) - ((2d(f))/3) \times 1/2 - (d(f) - (2d(f))/3) \times 1/3 \geq (5d(f) - 38)/9 \geq 0$ by R12 and R13.

Let $v \in V(G)$. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 = 0$. If $d(v) = 3$, then $f_2^+(v) \geq 2$ and it follows from R12 that $ch'(v) = ch(v) + f_2^+(v) \times 1/(f_2^+(v)) = 0$. If $d(v) = 4$, then $ch'(v) = ch(v) = 0$. If $d(v) = 5$, then $f_2^+(v) \geq 3$. Moreover, it may be the child of at most one 2-vertex. Thus $ch'(v) \geq ch(v) + 1/3 \times 3 = 0$ by R13. Suppose $d(v) \geq 6$. Then $v$ is incident with at most $\lfloor (d(v))/2 \rfloor$ 4-faces and it may be the parent of at most one 2-vertex. So $ch'(v) \geq ch(v) + (d(v) - \lfloor (d(v))/2 \rfloor) \times 1/3 = (7d(v) - 36)/6 > 0$.

Note that (a) implies that (b) is true if $\Delta \geq 5$. So it suffice to prove (b) by assuming $\Delta = 4$. Since $G$ is a planar graph, by Euler’s formula, we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -12 < 0.$$ 

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = 2d(v) - 6$ if $v \in V$ and $ch(f) = d(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F} ch(x) < 0$.

To prove (b), we construct the new charge $ch'(x)$ on $G$ as follows.

R21. Each $d(f)(d(f) \geq 18)$-face gives $1 - 6/(d(f))$ to its incident vertices.

R22. Each 2-vertex gets $3/2$ from its child and $1/2$ from its parent.

R23. Let $f$ be a 4-face. If $f$ is incident with a 2-vertex, then it gets $2/3$ from each of its incident $3^+$-vertices. If $f$ is incident with no 2-vertices, then it gets $1/2$ from each of its incident vertices.

The rest of this paper is devoted to checking $ch'(x) \geq 0$ for all $x \in V \cup F$. Let $f \in F(G)$. If $d(f) = 4$, then $ch'(f) = ch(f) = \max\{2/3 \times 3, 1/2 \times 4\} = 0$. If $d(f) \geq 18$, then $ch'(f) = ch(f) - r \times (1 - 6/r) = 0$ by R21.

Let $v \in V(G)$. If $d(v) = 2$, then $ch'(v) = ch(v) + 3/2 + 1/2 = 0$ by R22. If $d(v) = 3$, then $f_{18}^+(v) \geq 2$ and $f_4(v) \leq 1$, and it follows from R21 and R23 that $ch'(v) = ch(v) + 2 \times 2/3 - 2/3 > 0$. Suppose that $d(v) = 4$. Then $ch(v) = 2 \times 4 - 6 = 2$. If $n_2(v) \geq 1$, then $v$ sends at most $(n_2(v) + 2)/2$ to all its adjacent 2-vertices by R22. If $3 \leq n_2(v) \leq 4$, then $f_4(v) \leq 1$ by Lemma 2.3, and it follows that $ch'(v) \geq ch(v) - (n_2(v) + 2)/2 + 2/3 \times 3 - 2/3 = (14 - n_2(v) \times 3)/6 > 0$ by R21 and R23. If $1 \leq n_2(v) \leq 2$, then $f_4(v) \leq 2$, and it follows that $ch'(v) \geq ch(v) - (n_2(v) + 2)/2 + 2/3 \times 2 - 2/3 \times 2 = (2 - n_2(v))/2 \geq 0$. If $n_2(v) = 0$, we have $f_4(v) \leq 2$. Moreover, each 4-face incident with $v$ contains no 2-vertices. By R23, we have $ch'(v) \geq ch(v) + 2/3 \times 2 - 1/2 \times 2 > 0$. Now we complete the proof of Theorem 1.1.
3. Proof of Theorem 1.2

A 3(k)-vertex is a 3-vertex adjacent to exactly k 2-vertices. Let G be a minimal counterexample to Theorem 1.2 in terms of the number of vertices and edges. By minimality of G, it has the following result.

**Lemma 3.1.** [6]

(a) no 2-vertex is adjacent to two 2-vertices;
(b) no 2-vertex is adjacent to a 2-vertex and a 3(2)-vertex;
(c) no 3-vertex is adjacent to three 2-vertices.

Let $G_{23}$ be the bipartite subgraph of G comprising V and all edges of G that join a 2-vertex to a 3-vertex. Then $G_{23}$ has no isolated 2-vertices by Lemma 3.1(a), and the maximum degree is at most 2 by Lemma 3.1(c), and any component of $G_{23}$ is a path with more than one edges must end in 2-3-vertices by Lemma 3.1(b). It follows that $n_3 \geq n_2$. So we can find a matching M in G saturating all 2-vertices. If $uv \in M$ and $d(u) = 2, v$ is called the 2-master of u. Each 2-vertex has one 2-master and each vertex of degree $\Delta$ can be the 2-master of at most one 2-vertex.

Since G is a planar graph, by Euler’s formula, we have

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12 < 0.$$ 

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = d(v) - 6$ if $v \in V$ and $ch(f) = 2d(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F}ch(x) < 0$. Note that any discharging procedure preserves the total charge of G. If we can define suitable discharging rules to change the initial charge function $ch$ to the final charge function $ch'$ on $V \cup F$, such that $ch'(x) \geq 0$ for all $x \in V \cup F$, then we get an obvious contradiction. Now we design appropriate discharging rules and redistribute weights accordingly.

R31. Each $d(f)(d(f) \geq 5)$-face gives $2 - 6/(d(f))$ to its incident vertices.

R32. Each 2-vertex receives $3 - 12/(t+1) - 6/g$ from its 2-master.

Let $ch'(x)$ be the new charge obtained by the above rules for all $x \in V \cup F$. If $f \in F(G)$, then $ch'(f) = ch(f) - d(f) \times (2d(f) - 6)/(d(f)) = 0$ by R31. Let $v \in V(G)$. Suppose $d(v) = 3$. Then v can be the 2-master of at most one 2-vertex, and v sends at most $3 - 12/(t+1) - 6/g$ to 2-vertex by R32. In addition, If $v$ is incident with a g-face, then the other faces incident with $v$ are two $(t+1)^+$-faces, for G has no cycles of length from g + 1 to t. Thus, v receives $(2 - 6/g)$ from its incident $g$-face and $(2 - 6/(t+1))$ from each of its incident $(t+1)^+$-face by R31. So $ch'(v) \geq ch(v) + 2(2 - 6/(t+1)) + (2 - 6/g) - (3 - 12/(t+1) - 6/g) = 0$ for all g and t. Otherwise, v is incident with three $(t+1)^+$-faces, then $ch'(v) \geq ch(v) + 3(2 - 6/(t+1)) - (3 - 12/(t+1) - 6/g) = 6/g - 6/(t+1) > 0$, for $t + 1 > g$. Suppose $d(v) = 2$. Then $v$ receives at most $3 - 12/(t+1) - 6/g$ from its 2-master by R31. If $v$ is incident with a g-face, since G has no cycles of length from g + 1 to t, then the other face incident with $v$ is a $(t+1)^+$-face, and it follows that $ch'(v) \geq ch(v) + (2 - 6/(t+1)) + (2 - 6/g) + (3 - 12/(t+1) - 6/g) = 0$ for all g and t. Otherwise, v is incident with two $(t+1)^+$-faces, then $ch'(v) \geq ch(v) + 2(2 - 6/(t+1)) + (3 - 12/(t+1) - 6/g) = 3 - 24/(t+1) - 6/g > 0$.

From the above, we can see that $ch'(f) = ch(f) - d(f) \times (2d(f) - 6)/(d(f)) = 0$ for all $f \in F(G)$. Suppose $d(v) = 3$. So $ch'(v) \geq ch(v) + 2(2 - 6/(t+1)) + (2 - 6/g) - (3 - 12/(t+1) - 6/g) = 0$ for all g and t. When $v$ is incident with three $(t+1)^+$-faces, then
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\[ ch'(v) \geq ch(v) + 3(2 - 6/(t+1)) - (3 - 12/(t+1) - 6/g) = 6/g - 6/(t+1) > 0, \text{ for } t+1 > g. \]
Suppose \( d(v) = 2 \). If \( v \) is incident with a \( g \)-face and a \((t+1)^+\)-face, then
\[ ch'(v) \geq ch(v) + (2 - 6/(t+1)) + (2 - 6/g) + (3 - 12/(t+1) - 6/g) = 0 \]
for all \( g \) and \( t \). When \( v \) is incident with two \((t+1)^+\)-faces, then
\[ ch'(v) \geq ch(v) + 2(2 - 6/(t+1)) + (3 - 12/(t+1) - 6/g) = 3 - 24/(t+1) - 6/g. \]
So when \( g = 5 \), then \( t \geq 13 \); when \( g = 6 \), then \( t \geq 11 \); when \( g = 7 \), then \( t \geq 11 \); when \( g = 8 \), then \( t \geq 10 \); when \( g = 9 \), then \( t \geq 10 \), and it follows that \( ch'(v) \geq 0 \).

Our proof of Theorem 1.2 is now complete.

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References
