The Structure of Some Classes of 3-Dimensional Normal Almost Contact Metric Manifolds

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Abstract. The object of the present paper is to study $\xi$-projectively flat and $\phi$-projectively flat 3-dimensional normal almost contact metric manifolds. An illustrative example is given.

2010 Mathematics Subject Classification: 53C15, 53C40

Keywords and phrases: Normal almost contact metric manifolds, $\xi$-projectively flat, $\phi$-projectively flat, Einstein manifold.

1. Introduction

Let $M$ be an almost contact manifold and $(\phi, \xi, \eta)$ its almost contact structure. This means, $M$ is an odd-dimensional differentiable manifold and $\phi, \xi, \eta$ are tensor fields on $M$ of types $(1,1), (1,0), (0,1)$ respectively, such that

\begin{equation}
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0.
\end{equation}

Let $\mathbb{R}$ be the real line and $t$ a coordinate on $\mathbb{R}$. Define an almost complex structure $J$ on $M \times \mathbb{R}$ by

\[ J\left(X, \frac{\lambda}{dt}\right) = \left(\phi X - \lambda \xi, \eta(X) \frac{d}{dt}\right), \]

where the pair $(X, \lambda \frac{d}{dt})$ denotes a tangent vector to $M \times \mathbb{R}$, $f$ is a smooth function on $M \times R, X$ and $\lambda \frac{d}{dt}$ being tangent to $M$ and $\mathbb{R}$ respectively. $M$ with the structure $(\phi, \xi, \eta)$ is said to be normal if the structure $J$ is integrable [1], [2]. The necessary and sufficient condition for $(\phi, \xi, \eta)$ to be normal is

\[ [\phi, \phi] + 2d \eta \otimes \xi = 0, \]

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$ defined by

\[ [\phi, \phi](X,Y) = [\phi X, \phi Y] + \phi^2[X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \]

for any $X, Y \in T(M)$;

Communicated by Young Jin Suh.

Received: February 5, 2011; Revised: September 1, 2011.
We say that the form η has rank \( r = 2s \) if \((d\eta)^s \neq 0\), and \( \eta \wedge (d\eta)^s = 0 \), and has rank \( r = 2s + 1 \) if \( \eta \wedge (d\eta)^s \neq 0 \) and \((d\eta)^{s+1} = 0\). We also say that \( r \) is the rank of the structure \((\phi, \xi, \eta)\).

A Riemannian metric \( g \) on \( M \) satisfying the condition
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]
for any \( X, Y \in T(M) \), is said to be compatible with the structure \((\phi, \xi, \eta)\). If \( g \) is such a metric, then the quadruple \((\phi, \xi, \eta, g)\) is called an almost contact metric (shortly a.c.m.) structure on \( M \) and \( M \) is an (a.c.m.) manifold. On such a manifold we also have \( \eta(X) = g(X, \xi) \), for any \( X \in T(M) \) and we can always define the 2-form \( \Phi \) by
\[
\Phi(X, Y) = g(X, \phi Y),
\]
where \( X, Y \in T(M) \).

It is no hard to see that if \( \dim M = 3 \), then two Riemannian metrics \( g \) and \( \hat{g} \) are compatible with the same almost contact structure \((\phi, \xi, \eta)\) on \( M \) if and only if \( \hat{g} = \sigma g + (1 - \sigma) \eta \otimes \eta \), for a certain positive function \( \sigma \) on \( M \).

A normal (a.c.m.) structure \((\phi, \xi, \eta, g)\) satisfying additionally the condition \( d\eta = \Phi \) is called Sasakian. Of course, any such structure on \( M \) has rank 3. Also a normal almost contact metric structure satisfying the condition \( d\Phi = 0 \) is said to be quasi-Sasakian [3]. Contact metric manifolds have been studied by several authors [5, 7, 16]. Also if we consider \( M^n \) be a complex \( n \)-dimensional Kaehler manifold and \( M \) a real hypersurface of \( M^n \). We denote by \( \hat{g} \) and \( \hat{J} \) a Kaehler metric tensor and its Hermitian Structure tensor, respectively. For any vector field \( X \) tangent to \( M \), we put
\[
JX = \phi X + \eta(X)N, \quad JN = -\xi,
\]
where \( \phi \) is a \((1,1)\)-type tensor field, \( \eta \) is a 1-form and \( \xi \) is a unit vector field on \( M \). The induced Riemannian metric on \( M \) is denoted by \( g \). Then by the properties of \((\hat{g}, \hat{J})\), we see that the structure \((\phi, \xi, \eta, g)\) is an almost contact metric structure on \( M \). Real hypersurfaces of a complex manifold have been studied by [10, 19] and many others.

In a recent paper [14], Olszak studied the curvature properties of normal almost contact manifold of dimension three with several examples. De, Yıldız and Funda [9] studied locally \( \phi \)-symmetric normal (a.c.m.) manifolds of dimension 3. Also De and Kalam [8] recently characterized certain curvature conditions on 3-dimensional normal almost contact manifolds. Since at each point \( p \in M \) the tangent space \( T_p(M) \) can be decomposed into the direct sum \( T_p(M) = \phi(T_p(M)) \oplus \{\xi_p\} \), where \( \{\xi_p\} \) is the 1-dimensional linear subspace of \( T_p(M) \) generated by \( \xi_p \), the conformal curvature tensor \( C \) is a map
\[
C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) \oplus \{\xi_p\}, \quad p \in M.
\]
One has the following well known particular cases: (1) the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero; (2) the projection of the image of \( C \) in \( \{\xi_p\} \) is zero; and (3) the projection of the image of \( C |_{\phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M))} \) in \( \phi(T_p(M)) \) is zero. An (a.c.m.) manifold satisfying the cases (1), (2) and (3) is said to be conformally symmetric [11], \( \xi \)-conformally flat [20] and \( \phi \)-conformally flat [4] respectively.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let \( M \) be a \( n \)-dimensional Riemannian manifold. If there exist an one-to-one correspondence between each coordinate neighborhood of \( M \) and a domain in Euclidian space such that any geodesic of the
Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n \geq 3$, $M$ is locally projectively flat if and only if the well known projective curvature tensor $P$ vanishes. Here $P$ is defined by [13]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} \{S(Y, Z)X - S(X, Z)Y\},$$

for $X, Y, Z \in T(M)$, where $R$ is the curvature tensor and $S$ is the Ricci tensor. In fact, $M$ is projectively flat (that is $P = 0$) if and only if the manifold is of constant curvature [17, pp. 84–85]. Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

The present paper is devoted to study $\xi$-projectively flat and $\phi$-projectively flat normal (a.c.m.) metric manifold of dimension 3. After preliminaries in section 3, we prove that a compact 3-dimensional normal (a.c.m.) manifold is $\xi$-projectively flat if and only if the manifold is $\beta$-Sasakian. In the next section, it is proved that a 3-dimensional normal (a.c.m.) manifold is $\phi$-projectively flat if and only if it is an Einstein manifold provided $\alpha, \beta = constant$. Finally we cited of a normal almost contact metric manifold.

2. Preliminaries

For a normal (a.c.m.) structure $(\phi, \xi, \eta, g)$ on $M$, we have [14]

$$\nabla_X \xi = \alpha\{X - \eta(X)\xi\} - \beta\phi X,$$

where $2\alpha = \text{div}\xi$ and $2\beta = \text{tr}(\phi \nabla \xi)$, $\text{div}\xi$ is the divergence of $\xi$ defined by $\text{div}\xi = \text{trace}\{X \rightarrow \nabla_X \xi\}$ and $\text{tr}(\phi \nabla \xi) = \text{trace}\{X \rightarrow \phi \nabla_X \xi\}$. As a consequence of (2.1) we have

$$(\nabla_X \phi)(Y) = g(\phi \nabla_X \xi, Y)\xi - \eta(Y)\phi \nabla_X \xi$$

$$= \alpha\{g(\phi X, Y)\xi - \eta(Y)\phi X\} + \beta\{g(X, Y)\xi - \eta(Y)X\},$$

$$R(X, Y)\xi = \{Y\alpha + (\alpha^2 - \beta^2)\eta(Y)\} \phi^2 X - \{X\alpha + (\alpha^2 - \beta^2)\eta(X)\} \phi^2 Y$$

$$+ \{Y\beta + 2\alpha\beta\eta(Y)\}\phi X - \{X\beta + 2\alpha\beta\eta(X)\}\phi Y,$$

$$S(X, Y) = \left(\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2\right)g(X, Y) - \left\{\frac{r}{2} + \xi\alpha + 3(\alpha^2 - \beta^2)\right\}\eta(X)\eta(Y)$$

$$- \eta(Y)X\alpha + \eta(X)Y\alpha - \{\eta(Y)(\phi X)\beta + \eta(X)(\phi Y)\beta\},$$

$$S(Y, \xi) = -Y\alpha - (\phi Y)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2)\}\eta(Y),$$

$$\xi\beta + 2\alpha\beta = 0,$$

where $R$ denotes the curvature tensor and $S$ is the Ricci tensor.

On the other hand, the curvature tensor in a 3-dimensional Riemannian manifold always satisfies

$$\tilde{R}(X, Y, Z, W) = g(X, W)S(Y, Z) - g(X, Z)S(Y, W) + g(Y, Z)S(X, W)$$

$$- g(Y, W)S(X, Z) - \frac{r}{2} [g(X, W)g(Y, Z) - g(X, Z)g(Y, W)],$$

where $\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$ and $r$ is the scalar curvature.
From (2.3) we can derive that
\[
\tilde{R}(\xi, Y, Z, \xi) = -(\xi \alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z) - (\xi \beta + 2\alpha \beta)g(Y, \phi Z).
\]

By (2.5), (2.7) and (2.8) we obtain for \(\alpha, \beta = \text{constant},\)
\[
S(Y, Z) = \left(\frac{r}{2} + \alpha^2 - \beta^2\right)g(\phi Y, \phi Z) - 2(\alpha^2 - \beta^2)\eta(Y)\eta(Z).
\]

Applying (2.9) in (2.7) we get
\[
R(X, Y)Z = \left(\frac{r}{2} + 2(\alpha^2 - \beta^2)\right)\{g(Y, Z)X - g(X, Z)Y\} + g(X, Z)\left\{\left(\frac{r}{2} + 3(\alpha^2 - \beta^2)\right)\eta(Y)\xi\right\} - \left\{\left(\frac{r}{2} + 3(\alpha^2 - \beta^2)\right)\eta(Y)\eta(Z)X - g(Y, Z)\left\{\left(\frac{r}{2} + 3(\alpha^2 - \beta^2)\right)\eta(X)\xi\right\} + \left(\frac{r}{2} + 3(\alpha^2 - \beta^2)\right)\eta(X)\eta(Z)Y.
\]

From (2.6) it follows that if \(\alpha, \beta = \text{constant},\) then the manifold is either \(\beta\)-Sasakian, or \(\alpha\)-Kenmotsu [12] or cosymplectic [1].

**Proposition 2.1.** A 3-dimensional normal almost contact metric manifold with \(\alpha, \beta = \text{constant}\) is either \(\beta\)-Sasakian, or \(\alpha\)-Kenmotsu or cosymplectic.

**Definition 2.1.** An almost \(C(\lambda)\)-manifold \(M\) is an almost co-Hermitian manifold such that the Riemannian curvature tensor satisfies the following property:
there exist \(\lambda \in R\) such that for all \(X, Y, Z, W \in T(M)\):
\[
R(X, Y, Z, W) = R(X, Y, \phi Z, \phi W) + \lambda\left\{-g(X, Z)g(Y, W) + g(X, W)g(Y, Z) + g(X, \phi Z)g(Y, \phi W) - g(X, \phi W)g(Y, \phi Z)\right\}.
\]

A normal almost \(C(\lambda)\)-manifold is a \(C(\lambda)\)-manifold. If we take \(\lambda = -\alpha^2\) for \(\alpha > 0\), then we get \(C(-\alpha^2)\)-manifold.

We note that \(\beta\)-Sasakian manifold are quasi-Sasakian [3]. They provide examples of \(C(\lambda)\)-manifolds with \(\lambda \geq 0\).

An \(\alpha\)-Kenmotsu manifold is a \(C(-\alpha^2)\)-manifold [12].
Cosymplectic manifolds provide a natural setting for time dependent mechanical systems as they are locally product of a Kaehler manifold and a real line or a circle [6].

### 3. 3-dimensional \(\xi\)-projectively flat normal almost contact metric manifolds

\(\xi\)-conformally flat \(K\)-contact manifolds have been studied by Zhen, Cabrerizo and Fernandez [20]. In this section we study \(\xi\)-projectively flat normal (a.c.m.) manifold. Analogous to the definition of \(\xi\)-conformally flat (a.c.m.) manifold we define \(\xi\)-projectively flat (a.c.m.) manifolds.

**Definition 3.1.** A normal almost contact metric manifold \(M\) is called \(\xi\)-projectively flat if the condition \(P(X, Y)\xi = 0\) holds on \(M\), where projective curvature tensor \(P\) is defined by (1.3).
Putting \( Z = \xi \) in (1.3) and using (2.3) and (2.5), we get
\[
P(X, Y)\xi = -\frac{1}{2}\{(Y\alpha)X - (X\alpha)Y\} + \{(Y\alpha)\eta(X) - (X\alpha)\eta(Y)\}\xi
\]
(3.1)
\[
+ (Y\beta)\phi X - (X\beta)\phi Y + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\}
\]
\[
+ \frac{1}{2}\{(\phi Y)\beta X - (\phi X)\beta Y + (\xi\alpha)(\eta(Y)X - \eta(X)Y)\}.
\]

Now assume that \( M \) is a compact 3-dimensional \( \xi \)-projectively flat normal (a.c.m.) manifold. Then from (3.1) we can write
\[
-\frac{1}{2}\{(Y\alpha)X - (X\alpha)Y\} + \{(Y\alpha)\eta(X) - (X\alpha)\eta(Y)\}\xi
\]
(3.2)
\[
+ (Y\beta)\phi X - (X\beta)\phi Y + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\}
\]
\[
+ \frac{1}{2}\{(\phi Y)\beta X - (\phi X)\beta Y + (\xi\alpha)(\eta(Y)X - \eta(X)Y)\} = 0.
\]

Putting \( Y = \xi \) in (3.2) and using (2.6), we obtain
\[
(X\alpha)\xi + (\phi X)\beta - (\xi\alpha)\eta(X)\xi = 0
\]
which implies
\[
(X\alpha) + (\phi X)\beta - (\xi\alpha)\eta(X) = 0.
\]
(3.3)

Now (3.3) can be written as
\[
(X\alpha) + g(\text{grad} \beta, \phi X) - (\xi\alpha)\eta(X) = 0.
\]
(3.4)

Differentiating (3.4) covariantly along \( Y \), we get
\[
\nabla_Y(X\alpha) + g(\nabla_Y\beta, \phi X) + g(\text{grad} \beta, (\nabla_Y\phi)X)
\]
(3.5)
\[
- Y(\xi\alpha)\eta(X) - (\xi\alpha)(\nabla_Y\eta)(X) = 0.
\]

Hence, by antisymmetrization with respect to \( X \) and \( Y \), we have from (3.5)
\[
g(\nabla_Y\beta, \phi X) - g(\nabla_X\beta, \phi Y) + g(\text{grad} \beta, (\nabla_Y\phi)X) - g(\text{grad} \beta, (\nabla_X\phi)Y)
\]
\[
- Y(\xi\alpha)\eta(X) + X(\xi\alpha)\eta(Y) - (\xi\alpha)(\nabla_Y\eta)(X) - (\nabla_X\eta)(Y)\} = 0.
\]

This implies
\[
g(\nabla_Y\beta, \phi X) - g(\nabla_X\beta, \phi Y) + \{(\nabla_Y\phi)X\beta - (\nabla_X\phi)Y\beta\}
\]
(3.6)
\[
- Y(\xi\alpha)\eta(X) + X(\xi\alpha)\eta(Y) + 2(\xi\alpha)d\eta(X, Y) = 0.
\]

Using (2.2) and \( d\eta = \beta\Phi \) [14], (3.6) yields
\[
g(\nabla_Y\beta, \phi X) - g(\nabla_X\beta, \phi Y) + \{2\alpha g(\phi Y, X)\xi - \alpha(\eta(X)\phi Y - \eta(Y)\phi X)
\]
(3.7)
\[
- \beta(\eta(X)Y - \eta(Y)X)\beta - Y(\xi\alpha)\eta(X) - X(\xi\alpha)\eta(Y)\} + 2\beta(\xi\alpha)\Phi(X, Y) = 0.
\]

Let \( \{e_1, e_2, \xi\} \) be an orthonormal \( \phi \)-basis where \( \phi e_1 = -e_2 \) and \( \phi e_2 = e_1 \). Taking \( Y = e_1 \) and \( X = e_2 \) in (3.7), we find that
\[
g(\nabla_{e_1}\beta, e_1) + g(\nabla_{e_2}\beta, e_2) = 2\alpha(\xi\beta) + 2\beta(\xi\alpha).
\]
(3.8)

On the other hand (2.6) yields \( g(\text{grad} \beta, \xi) = -2\alpha\beta \), whence by covariant differentiation we get, on account of (2.1)
\[
g(\nabla_{\xi}\beta, \xi) = -2\alpha(\xi\beta) - 2\beta(\xi\alpha).
\]
(3.9)
Denoting by $\triangle$ the Laplacian defined by $\triangle = \text{div} \, \text{grad}$, in view of (3.8) and (3.9) we have $\triangle \beta = 0$. Since $M$ is compact, $\beta$ is a constant. Now if $\beta \neq 0$, (2.6) implies $\alpha = 0$. This implies $M$ is a $\beta$-Sasakian manifold. Conversely, if $M$ is a $\beta$-Sasakian manifold, then from (3.1) it is easy to see that $P(X,Y)\xi = 0$. Hence we can state the following:

**Theorem 3.1.** A compact 3-dimensional normal almost contact metric manifold is $\xi$-projectively flat if and only if it is a $\beta$-Sasakian manifold.

4. 3-dimensional $\phi$-projectively flat normal almost contact metric manifolds

 Analogous to the definition of $\phi$-conformally flat contact metric manifold [4], we define $\phi$-projectively flat normal almost contact metric manifold. In this connection we can mention the work of Ozgur [15] who has studied $\phi$-projectively flat Lorentzian Para-Sasakian manifolds.

**Definition 4.1.** A 3-dimensional normal almost contact metric manifold satisfying the condition

$$\phi^2 P(\phi X, \phi Y) \phi Z = 0$$

is called $\phi$-Projectively flat.

Let us assume that $M$ is a 3-dimensional $\phi$-projectively flat normal (a.c.m.) manifold. It can be easily seen that $\phi^2 P(\phi X, \phi Y) \phi Z = 0$ holds if and only if

$$g(P(\phi X, \phi Y) \phi Z, \phi W) = 0,$$

for $X, Y, Z, W \in T(M)$.

Using (1.3) and (1.1), $\phi$-projectively flat means

$$g(R(\phi X, \phi Y) \phi Z, \phi W) = \frac{1}{2} \{S(\phi Y, \phi Z) g(\phi X, \phi W) - S(\phi X, \phi Z) g(\phi Y, \phi W)\}.$$  

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in $M$ and using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.1) and summing up with respect to $i$, then we have

$$\sum_{i=1}^{2} g(R(\phi e_i, \phi Y) \phi Z, \phi e_i) = \frac{1}{2} \sum_{i=1}^{2} \{S(\phi Y, \phi Z) g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z) g(\phi Y, \phi e_i)\}.$$  

It can be easily verified that

$$\sum_{i=1}^{2} g(R(\phi e_i, \phi Y) \phi Z, \phi e_i) = S(\phi Y, \phi Z) + (\xi \alpha + \alpha^2 - \beta^2) g(\phi Y, \phi Z),$$  

$$\sum_{i=1}^{2} g(\phi e_i, \phi e_i) = 2, \quad \sum_{i=1}^{2} S(\phi e_i, \phi Z) g(\phi Y, \phi e_i) = S(\phi Y, \phi Z).$$  

So using (1.2) and (2.4), the equation (4.2) becomes

$$\left(\frac{r}{2} + 3(\xi \alpha + \alpha^2 - \beta^2)\right) \{g(Y, Z) - \eta(Y) \eta(Z)\} = 0,$$

which gives $r = -6(\xi \alpha + \alpha^2 - \beta^2)$. So we state the following:

**Proposition 4.1.** The scalar curvature $r$ of a 3-dimensional $\phi$-projectively flat normal almost contact metric manifold is $-6(\xi \alpha + \alpha^2 - \beta^2)$.
Also if \( r = -6(\xi \alpha + \alpha^2 - \beta^2) \), it follows from (2.4) that the manifold is an Einstein manifold provided \( \alpha, \beta = \text{constant} \). Hence we can state the following:

**Proposition 4.2.** A 3-dimensional \( \phi \)-projectively flat normal almost contact metric manifold is an Einstein manifold, provided \( \alpha, \beta = \text{constant} \).

It is known \([18]\) that a 3-dimensional Einstein manifold is a manifold of constant curvature. Also \( M \) is projectively flat if and only if it is of constant curvature \([17]\). Now trivially, projectively flatness implies \( \phi \)-projectively flat. Hence using Proposition 4.2 we can state the following:

**Theorem 4.1.** A 3-dimensional normal almost contact metric manifold is \( \phi \)-projectively flat if and only if it is an Einstein manifold, provided \( \alpha, \beta = \text{constant} \).

### 5. Example of a 3-dimensional normal almost contact metric manifold

We consider the 3-dimensional manifold \( M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\} \), where \( (x,y,z) \) are standard coordinate of \( \mathbb{R}^3 \).

The vector fields

\[
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z}
\]

are linearly independent at each point of \( M \).

Let \( g \) be the Riemannian metric defined by

\[
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,
\]

\[
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,
\]

that is, the form of the metric becomes

\[
g = \frac{dx^2 + dy^2 + dz^2}{z^2}.
\]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in T(M) \). Let \( \phi \) be the \((1,1)\) tensor field defined by

\[
\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.
\]

Then using the linearity of \( \phi \) and \( g \), we have

\[
\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]

for any \( Z, W \in T(M) \).

Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines an almost contact metric structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to the metric \( g \). Then we have

\[
[e_1, e_3] = e_1 e_3 - e_3 e_1 = z \frac{\partial}{\partial x} \left( z \frac{\partial}{\partial z} \right) - z \frac{\partial}{\partial z} \left( z \frac{\partial}{\partial x} \right) = z^2 \frac{\partial^2}{\partial x \partial z} - z^2 \frac{\partial^2}{\partial z \partial x} - z \frac{\partial}{\partial x} = -e_1.
\]

Similarly

\[
[e_1, e_2] = 0 \quad \text{and} \quad [e_2, e_3] = -e_2.
\]

The Riemannian connection \( \nabla \) of the metric \( g \) is given by

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]),
\]

(5.1)
which is known as Koszul’s formula.

Using (5.1) we have

\[(5.2)\]
\[2g(\nabla_{e_1} e_3, e_1) = -2g(e_1, e_1) = 2g(-e_1, e_1).\]

Again by (5.1)

\[(5.3)\]
\[2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(-e_1, e_2)\]

and

\[(5.4)\]
\[2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3).\]

From (5.2), (5.3) and (5.4) we obtain

\[2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X),\]

for all \(X \in T(M)\). Thus

\[\nabla_{e_1} e_3 = -e_1.\]

Therefore, (5.1) further yields

\[(5.5)\]
\[\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = e_3,\]

\[\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_1 = 0,\]

\[\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.\]

(5.5) tells us that the manifold satisfies (2.1) for \(\alpha = -1\) and \(\beta = 0\) and \(\xi = e_3\). Hence the manifold is a normal almost contact metric manifold with \(\alpha, \beta\) constants.

It is known that

\[(5.6)\]
\[R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.\]

With the help of the above results and using (5.6) it can be easily verified that

\[R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,\]

\[R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_3)e_2 = e_3, \quad R(e_1, e_3)e_2 = 0,\]

\[R(e_1, e_2)e_1 = e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = e_3.\]

From the above expressions of the curvature tensor we obtain

\[S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -2.\]

Similarly, we have

\[S(e_2, e_2) = S(e_3, e_3) = -2.\]

Therefore,

\[r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6.\]

We note that here \(\alpha, \beta\) and \(r\) are all constants. It is sufficient to check

\[S(e_i, e_i) = -2 = -2(\alpha^2 - \beta^2)g(e_i, e_i),\]

for all \(i = 1, 2, 3\) and \(\alpha = -1, \beta = 0\). Hence \(M\) is an Einstein manifold. Therefore \(M\) is \(\phi\)-projectively flat. Thus Theorem 4.1 is verified.

**Acknowledgement.** The authors are thankful to the referees for their valuable comments and suggestions for the improvement of this paper.
References


