Recognition of Finite Simple Groups Whose First Prime Graph Components are $r$-Regular

1 Liangcai Zhang, 2 Wujie Shi, 3 Dapeng Yu and 4 Jin Wang
1, 4 College of Mathematics and Statistics, Chongqing University, Shapingba, Chongqing 401331, China
2, 3 Department of Mathematics and Statistics, Chongqing University of Arts and Sciences, Youngchuan, Chongqing 402160, China
1 zlc213@163.com, 2 wjshi@suda.edu.cn, 3 bird-yu@live.cn, 4 wj19882009@163.com

Abstract. Let $G$ be a finite group and $\pi(G) = \{p_1, p_2, \ldots, p_k\}$. For $p \in \pi(G)$, we put $\deg(p) := |\{q \in \pi(G) : p \sim q \text{ in the prime graph of } G\}|$, which is called the degree of $p$. We also define $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$, which is called the degree pattern of $G$. We say $G$ is $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic finite groups having the same order and degree pattern as $G$. In particular, a 1-fold OD-characterizable group is simply called an OD-characterizable group. In the present paper, we determine all finite simple groups whose first prime graph components are 1-regular and prove that all finite simple groups whose first prime graph components are $r$-regular except $U_4(2)$ are OD-characterizable, where $0 < r < 2$. In particular, $U_4(2)$ is exactly 2-fold OD-characterizable, which improves an earlier obtained result.

2010 Mathematics Subject Classification: 20D05, 20D06, 20D60

Keywords and phrases: Prime graph of a group, $r$-regular, degree pattern, OD-characterization.

1. Introduction

Throughout this paper, groups under consideration are finite. For any group $G$, we denote by $\pi_e(G)$ the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. We associate to $\pi(G)$ a simple graph called prime graph of $G$, denoted by $\Gamma(G)$. The vertex set of this graph is $\pi(G)$, and two distinct vertices $p, q$ are joined by an edge if and only if $pq \in \pi_e(G)$. In this case, we write $p \sim q$ (see [3, 7, 10, 22]). Denote by $s(G)$ the number of connected components of $\Gamma(G)$ and by $\pi_i(G)$ ($i = 1, 2, \ldots, s(G)$) the vertex sets of the distinct connected components of $\Gamma(G)$. When $|G|$ is even, we always assume $2 \in \pi_1(G)$. Denote by $\pi(n)$ the set of all primes dividing $n$, where $n$ is a natural number. Then $|G|$ can be expressed as a product of $m_1, m_2, \ldots, m_{s(G)}$, where $m_i$’s are positive integers with $\pi(m_i) = \pi_i(G)$. These $m_i$’s are called the order components of $G$ (see [4]). Let $OC(G) := \{OC_1(G), OC_2(G), \ldots, OC_{s(G)}(G)\} = \{m_1, m_2, \ldots, m_{s(G)}\}$ be the set of order components of $G$, where $OC_i(G) = m_i$ for $i = 1, 2, \ldots, s(G)$. Meanwhile, let $T(G) = \{\pi_i(G) | i = 1, 2, \ldots, s(G)\}$ be the set of connected components of $G$.
All further unexplained notations are standard and can be found in [13], for instance.

**Definition 1.1.** [13] Let $G$ be a finite group and $\pi(G) = \{p_1, p_2, \ldots, p_s\}$. For $p \in \pi(G)$, put $\deg(p) := |\{q \in \pi(G) | p \sim q\}|$, which is called the degree of $p$. We also put $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_s))$, where $p_1 < p_2 < \cdots < p_s$, which is called the degree pattern of $G$.

Set

$$\Omega_0(G) = \{p \in \pi(G) | \deg(p) = 0\} \quad \text{and} \quad \Omega_q'(G) = \{p \in \pi(G) | \deg(p) \neq 0\}.$$ 

Clearly, $\pi(G) = \Omega_0(G) \cup \Omega_q'(G)$. Since $\deg(p) = 0$ if and only if \{p\} is a connected component of $\Gamma(G)$, we have $|\Omega_0(G)| \leq s(G)$.

**Definition 1.2.** A group $M$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $G$ such that $|G| = |M|$ and $D(G) = D(M)$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group.

**Definition 1.3.** Let $n$ be a natural number. We say that a finite group $G$ is a $K_n$-group if and only if $|\pi(G)| = n$.

**Definition 1.4.** Let $p$ be a prime. A group $G$ is called a $C_{p, p^-}$-group if and only if $p \in \pi(G)$ and the centralizers of its elements of order $p$ in $G$ are $p$-groups.

Up to now, it has been shown that many finite simple groups are OD-characterizable. Here we summarize all known results into the following proposition.

**Proposition 1.1.** Let $q$ be a prime power of a prime $p$. A finite non-abelian simple group $G$ is OD-characterizable if $G$ is isomorphic to one of the following groups:

(a) [13, Theorem 1.3 (2)] all sporadic simple groups;
(b) [30, Theorem 3.1]; [26, Theorem 3.1]; [14, Theorem 3.1] all finite simple groups $L_2(q)$ for each $q$ and $2B_2(q)$ for each $q = 2^{2m+1}$;
(c) [28, Theorem 3.5] all finite simple groups with exactly four prime divisors but $A_{10}$; $A_{22}$, $A_p$, $A_{p+1}$ and $A_{p+2}$;
(d) [32, Theorem]; [32, Theorem A]; [14, Theorem 1.5] the alternating groups $A_{16}$, $A_{22}$, $A_p$, $A_{p+1}$ and $A_{p+2}$;
(e) [8, 16, 17, 32] all alternating groups $A_{p+3}$, where $7 \neq p \in \pi(100!)$;
(f) [14, Theorem 1.4] all finite simple $C_{2, 2}$-groups;
(g) [32, Theorem 1] all finite simple groups whose orders are less than $10^8$ but $A_{10}$ and $U_4(2)$;
(h) [2, Theorem 1]; [25, Theorem 1] all finite projective special linear groups $L_p(2)$ and $L_{p+1}(2)$, where $2^p - 1$ is a Mersenne prime;
(i) [2, Theorem 2] the simple groups $L_4(5)$, $L_4(7)$, $U_4(5)$ and $U_4(7)$;
(j) [31, Theorem ]; [29, Theorem 1] the simple groups $L_3(9)$ and $U_3(5)$;
(k) [13, Theorem 1.3(3)] the following simple groups of Lie type:

1. $L_3(q)$ with $|\pi((q^2 + q + 1)/d)| = 1$, where $d = (3, q - 1)$;
2. $U_3(q)$ with $q > 5$ and $|\pi((q^2 - q + 1)/d)| = 1$, where $d = (3, q + 1)$;
3. $2G_2(q)$ with $q = 3^{2m+1}$ ($m \geq 1$) with $|\pi(q + \sqrt{3q} + 1)| = |\pi(q - \sqrt{3q} + 1)| = 1$.

In particular, we have the following proposition.

**Proposition 1.2.** [15, Theorem A], [13, Theorem 1.5] Finite non-abelian simple groups $A_{10}$, $S_6(3)$ and $O_7(3)$ are 2-fold OD-characterizable. Namely, the following statements hold.
(a) If \( G \) is a finite group such that \( |G| = |A_{10}| \) and \( |D(G)| = |D(A_{10})| \), then \( G \cong A_{10} \) or \( J_2 \times Z_3 \).

(b) Let \( M \in \{ S_6(3), O_7(3) \} \). If \( G \) is a finite group such that \( |G| = |M| \) and \( |D(G)| = |D(M)| \), then \( G \cong O_7(3) \) or \( S_6(3) \).

More details about finite 2-fold OD-characterizable simple groups can be found in [1].

**Definition 1.5.** A graph \( \Gamma \) is called an \( r \)-regular graph if every vertex of \( \Gamma \) is adjacent to exactly \( r \) vertices of \( \Gamma \).

In [18], M. Suzuki determined all finite non-abelian simple groups whose first prime graph components are 0-regular, that is, all finite non-abelian simple \( C_{2,2} \)-groups.

**Proposition 1.3.** [18, Theorem] If \( G \) is a finite non-abelian simple \( C_{2,2} \)-groups, then \( G \) is isomorphic to one of the following groups:

(a) \( A_5, A_6, L_3(4) \);

(b) \( L_2(q) \), where \( q \) is a Fermat prime, a Mersenne prime or a prime power of 2;

(c) \( Sz(q) \), where \( q \) is an odd prime power of 2.

In [11], B. Khosravi and E. Fakhraei determined all finite simple groups whose first prime graph components are 2-regular (see Lemma 2.7). In the present paper, we determine all finite simple groups whose first prime graph components are 1-regular and prove that all finite simple groups whose first prime graph components are \( r \)-regular except \( U_4(2) \) are OD-characterizable, where \( 0 \leq r \leq 2 \). In particular, \( U_4(2) \) is exactly 2-fold OD-characterizable, which improves [25, Theorem 1].

2. Lemmas

In this section, we list some basic and known results which will be used later. Let \( n \) be a natural number and \( p \) be a prime. In the sequel, \( n_p \) denotes a prime power of \( p \) such that \( n_p \mid n \) but \( pn_p \nmid n \). Usually, we call \( n_p \) the \( p \)-part of \( n \). Also, \( \varepsilon \) must be one of the symbols + or −.

Suppose \( G \) is a finite group and \( p_1, p_2, \ldots, p_s \in \pi(G) \). We use expression \( p_1 \sim p_2 \sim \cdots \sim p_s \) to mean that vertices \( p_i \) and \( p_{i+1} \) are adjacent to each other in \( \Gamma(G) \) for \( i = 1, 2, \ldots, s-1 \).

Surely \( p_s \sim p_1 \) in this case. On the other hand, \( p_1 \sim p_2 \sim \cdots \sim p_s \sim p_1 \) means that vertices \( p_i \) and \( p_{i+1} \) are adjacent to each other for \( i = 1, 2, \ldots, s-1 \) and \( p_s \sim p_1 \) in \( \Gamma(G) \).

In one of his last papers (see [18]), Suzuki studied the prime graph of finite groups without using the classification and obtained the following result. As far as we know, his paper is the first on the prime graph which does not use the classification.

**Lemma 2.1.** [19, Theorem B] Let \( G \) be a finite simple group whose prime graph \( \Gamma(G) \) is not connected and let \( \Delta \) be a connected component of \( \Gamma(G) \) whose vertex set does not contain \( 2 \). Then \( \Delta \) is a clique.

**Lemma 2.2.** [13, Corollary 7.6] Let \( G \) be a finite non-abelian simple group. All connected components of the prime graph \( \Gamma(G) \) are cliques if and only if \( G \) is one of the following groups:

(a) sporadic groups \( M_{11}, M_{22}, J_1, J_2, J_3, HS \);

(b) alternating groups \( A_n \), where \( n = 5, 6, 7, 9, 12, 13 \);

(c) groups of Lie type \( A_1(q) \), where \( q > 3; A_2(4); A_2(q), \) where \( (q-1)_3 \neq 3, q+1 = 2^k; A_3(3); A_5(2); A_2(q), \) where \( (q-1)_3 \neq 3, q-1 = 2^k; C_3(2), C_2(q), \) where \( q > 2; D_4(2); D_4(q); B_2(q), \) where \( q = 2^{2k+1}; G_2(q), \) where \( q = 3^k \).
The first connected component of $\Gamma(G)$, where $q$ denotes an odd prime and $p$ denotes a prime power.

**Lemma 2.4.** [12, Lemma 2] Let $q$ be a prime power. Then $|\pi(q^2 - 1)| \leq 2$ if and only if $q \in \{2, 3, 4, 5, 7, 8, 9, 17\}$.

**Lemma 2.5.** [13, Lemma 2] Let $G$ and $M$ be finite groups such that $|G| = |M|$ and $D(G) = D(M)$. In addition, we suppose one of the following conditions holds:

(a) $|\Omega_0^*(M)| = 0$;

(b) $|\Omega_0^*(M)| = 2$;

(c) $|\Omega_0^*(M)| \geq 3$, and there exists a vertex $p \in \pi(M)$ such that $\deg(p) \geq |\Omega_0^*(M)| - 2$.

Then $OC(G) = OC(M)$.

**Lemma 2.6.** [24, Theorem 3.3] If $G$ is a finite group such that $OC(G) = OC(U_4(2))$, then $G \cong U_4(2)$ or $(U \times V)F$, where $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $V \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $F \cong \mathbb{Z}_5 : \mathbb{Z}_4$, a split extension of $\mathbb{Z}_5$ by $\mathbb{Z}_4$. In particular, $U_4(2)$ is 2-recognizable by its order components.

**Lemma 2.7.** [11, Main Theorem] Let $G$ be a finite non-abelian simple group. Then $\pi_1(G)$, the first connected component of $\Gamma(G)$, is 2-regular if and only if $G$ is isomorphic to one of the following groups:

(a) $A_9; J_1, J_2, J_3, HS; S_6(2); U_3(9), U_4(3); ^3D_4(2); G_2(9); O^+_8(2)$.

(b) $L_2(q)$, where $4 | (q + 1)$ and $|\pi(q + 1)| = 3$.

(c) $S_4(q)$, where $q = 4, 5, 7, 8, 9, 17$.

The following lemmas suggest that some finite simple groups can be uniquely characterized by their order components.

**Lemma 2.8.** [9, Theorem; 27, Lemma 2.6] Let $S_4(q)$ be the projective symplectic simple group, where $q > 3$ is a prime power. If $G$ is a finite group such that $OC(G) = OC(S_4(q))$, then $G \cong S_4(q)$.

**Lemma 2.9.** [4, Theorem 1] Let $q$ be a prime power. If $G$ is a finite group such that $OC(G) = OC(G_2(q))$, then $G \cong G_2(q)$.

Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. That is, $t(G)$ is a maximal number of vertices in independent sets of $\Gamma(G)$ and is called an
independence number of the graph. By analogy we denote by \( t(r,G) \) the maximal number of vertices in independent sets of \( \Gamma(G) \) containing the prime \( r \). We call this number an \( r \)-independence number. Denote by \( \rho(G) \) (by \( \rho(r,G) \)) some independence set in \( \Gamma(G) \) (containing \( r \)) with maximal number of vertices and \( |\rho(G)| = t(G) \). \( |\rho(r,G)| = t(r,G) \) (see [21]).

**Lemma 2.10.** [20, Theorem] Let \( G \) be a finite group such that \( t(G) \geq 3 \) and \( t(2,G) \geq 2 \). Then the following assertions hold.

(a) There is a finite non-abelian simple group \( S \) such that \( S \leq G = G/K \leq \text{Aut}(S) \) for the maximal normal soluble subgroup \( K \) of \( G \). Furthermore, \( t(S) \geq t(G) - 1 \).

(b) One of the following statements holds:

1. \( S \cong A_7 \) or \( L_2(q) \) for some odd prime power \( q \) and \( t(S) = t(2,S) = 3 \);

2. For every prime \( \pi(G) \) non-adjacent to 2 in \( \Gamma(G) \) a Sylow \( r \)-subgroup of \( G \) is isomorphic to a Sylow \( r \)-subgroup of \( S \). In particular, \( t(2,S) \geq t(2,G) \).

Given a prime \( p \), we denote by \( \Pi_p \) the class of all finite non-abelian simple groups \( G \) such that \( p \in \pi(G) \subseteq \{2,3,5,\ldots,p\} \). It is clear that the set of all finite non-abelian simple groups is the disjoint union of the finite sets \( \Pi_p \) for all primes \( p \). Observe that the first two sets \( \Pi_2 \) and \( \Pi_3 \) are trivially empty, whereas the sets \( \Pi_p \) \( (p \geq 5) \) are always non-empty because they contain some generic elements.

**Lemma 2.11.** [6, 23, Table 1] Let \( \Pi \) be a given set of prime numbers and \( p \) be a prime. If \( G \in \Pi_p \) and \( \pi(G) \subseteq \Pi \), where \( p = 5,7,13,17,29,73 \), then \( G \) is isomorphic to one of the following groups exhibited in the following tables. In particular,

(a) either \( \pi(\text{Out}(G)) \subseteq \{2,3\} \) or \( \pi(\text{Out}(G)) = \emptyset \);

(b) \( 3 \notin \pi(G) \) if and only if \( G = Sz(8) \).

3. Recognition of finite simple groups whose first prime graph components are \( r \)-regular, where \( 0 \leq r \leq 2 \)

In this section, we prove the main results–Theorems 3.4 and 3.8. In the investigation, we should pay special attention to the subtle changes of the prime graph component itself, the
Proof. It is clear from [13, Table 1].

Theorem 3.2. If $G$ is an alternating simple group whose first prime graph component is 1-regular, then $G$ is isomorphic to $A_7$.

Proof. Since $\pi_1(G)$ is 1-regular, it follows that $|\pi_1(G)| = 2$. Hence $\pi_1(G)$ is a clique. Moreover, $\pi_i(G)$ is also a clique for $i = 1, 2, \ldots, s(G)$ by Lemma 2.1. By Lemma 2.2(b), we have $G \cong A_7$.

Theorem 3.3. Let $G$ be a finite non-abelian simple groups of Lie type over the finite field of order $q = p^s$, where $p$ is a prime number and $s$ is a natural number. If the first prime graph component of $G$ is 1-regular, then $G$ is isomorphic to one of the following finite simple groups.

(a) $L_3(3), U_3(3), U_4(2), G_2(3)$.
(b) $A_1(q)$, where $q$ is a prime power such that $3 < q \equiv 1 \pmod{4}$ and $|\pi(q - 1)| = 2$.

Proof. Since $\pi_1(G)$ is 1-regular, it follows that $|\pi_1(G)| = 2$. Hence $\pi_1(G)$ is a clique. Moreover, $\pi_i(G)$ is also a clique for $i = 1, 2, \ldots, s(G)$ by Lemma 2.1. By Lemma 2.2(c), we have
G is isomorphic to one of the following groups: \( A_1(q) \), where \( q > 3 \); \( A_2(4) \); \( A_2(q) \), where \( (q-1)3 \neq 3, q+1 = 2^k \); \( 2A_3(3) \); \( 2A_4(2) \); \( 2A_2(1) \), where \( (q-1)3 \neq 3, q-1 = 2^k \); \( C_3(2) \), \( C_2(q) \), where \( q > 2 \); \( D_4(2) \); \( D_3(4) \); \( B_3(q) \), where \( q = 2^{2k+1} \); \( G_2(q) \), where \( q = 3^k \).

Case 1. Suppose \( G \cong A_2(q) \) \( \left( (q-1)3 \neq 3, q+1 = 2^k \right) \), \( 2A_2(q) \) \( \left( (q-1)3 \neq 3, q-1 = 2^k \right) \), \( C_2(q) \) \( (q > 2) \) or \( G_2(q) \) \( (3q) \).

Clearly \( \pi(q(q^2-1)) \subseteq \pi(G) \) by [1, table Lemma 1, 2.3]. Recall that \( |\pi(G)| = 2 \) and we have \( |\pi(q^2-1)| \leq 1 \). By Lemma 2.4, \( q \in \{2,3\} \). By a simple check, we obtain the candidates are \( L_3(3), U_3(3), G_2(3) \) and \( S_4(3) \cong U_4(2) \cong 2A_3(2) \).

Case 2. If \( G \in \{ A_2(4), 3A_3(3), 3A_5(2), C_3(2), D_4(2), 3D_4(2) \} \), then no candidate arises in this case.

Case 3. If \( G \cong 2B_2(q) \) \( (q = 2^{2n+1} > 2) \) or \( A_1(q) \) \( (4q) \), then \( OC_1(G) = q^2 \) or \( q \) from Table 1, which implies that the first prime graph component of \( G \) can not be 1-regular.

Case 4. If \( G \cong A_1(q) \), where \( q \) is a prime power such that \( 3 < q \equiv 1 \) (mod 4), then \( OC_1(A_1(q)) = q - 1 \) from Table 1. Since \( G \) is a simple group whose first prime graph component is 1-regular, it follows that \( G \cong A_1(q) \), where \( q \) is a prime power such that \( 3 < q \equiv 1 \) (mod 4) and \( |\pi(q-1)| = 2 \).

By Theorems 3.1–3.3, we have the following theorem.

**Theorem 3.4.** Let \( G \) be a finite non-abelian simple group. The first prime graph component of \( G \) is 1-regular if and only if \( G \) is isomorphic to one of the following finite simple groups.

1. \( A_7, M_{11}, M_{22}, L_3(3), U_3(3), U_4(2), G_2(3) \).
2. \( A_1(q) \), where \( q \) is a prime power such that \( 3 < q \equiv 1 \) (mod 4) and \( |\pi(q-1)| = 2 \).

**Theorem 3.5.** Let \( G \) be a finite non-abelian simple group whose first prime graph component is 1-regular. Then the following statements hold.

1. If \( G \cong U_4(2) \), then \( G \) is OD-characterizable.
2. If \( |G| = |U_4(2)| \) and \( D(G) = D(U_4(2)) \), then \( G \cong U_4(2) \) or \( (U \times V)F \), where \( U \cong Z_3 \times Z_2 \times Z_2 \times Z_2 \), \( V \cong Z_3 \times Z_3 \times Z_3 \times Z_3 \) and \( F \cong Z_5 : Z_4 \). Namely, \( U_4(2) \) is exactly 2-fold OD-characterizable.

**Proof.** By Theorem 3.4, \( G \cong A_7, M_{11}, M_{22}, L_3(3), U_3(3), U_4(2), G_2(3) \) or \( A_1(q) \), where \( q \) is a prime power such that \( 3 < q \equiv 1 \) (mod 4) and \( |\pi(q-1)| = 2 \). If \( G \in \{ A_7, M_{11}, M_{22}, L_3(3), U_3(3), G_2(3) \} \), then \( |G| < 10^6 \) and so \( G \) is OD-characterizable by Proposition 1.5(2).

If \( G \cong A_1(q) \), where \( q \) is a prime power such that \( 3 < q \equiv 1 \) (mod 4) and \( |\pi(q-1)| = 2 \), then \( G \) is OD-characterizable by Proposition 1.5(b).

If \( |G| = |U_4(2)| = 2^6 \cdot 3^4 \cdot 5 \) and \( D(G) = D(U_4(2)) = (1,1,0) \), then \( OC(G) = OC(U_4(2)) \) by Lemma 2.5(b). Therefore \( G \cong U_4(2) \) or \( (U \times V)F \), where \( U \cong Z_2 \times Z_2 \times Z_2 \times Z_2 \), \( V \cong Z_3 \times Z_3 \times Z_3 \times Z_3 \) and \( F \cong Z_5 : Z_4 \), by Lemma 2.6. Namely, \( U_4(2) \) is exactly 2-fold OD-characterizable.

**Remark 3.1.** Theorem 3.5 improves [25, Theorem 1], in which it has been proved only that \( U_4(2) \) is \( k \)-fold OD-characterizable, where \( k \geq 2 \).

**Theorem 3.6.** If \( G \) is a finite non-abelian simple group whose first prime graph component is 2-regular, then \( G \) is OD-characterizable.

**Proof.** By Lemma 2.7, \( G \) is isomorphic to one of the following groups:
(a) $A_9; J_1, J_2, J_3, HS; S_6(2); U_3(9), U_4(3); 3D_4(2); G_2(9); O_{13}^+(2)$.
(b) $L_2(q)$, where $4 \mid (q+\varepsilon 1)$ and $|\pi(q+\varepsilon 1)| = 3$.
(c) $S_4(q)$, where $q = 4, 5, 7, 8, 9, 17$.

Case 1. If $G \in \{A_9, J_1, J_2, J_3, HS, S_4(4), S_4(5), S_6(2), U_3(9), U_4(3)\}$, then $|G| < 10^8$ and so $G$ is $OD$-characterizable by Proposition 1.5(g).

Case 2. If $G \in \{3D_4(2), S_4(7), S_4(9), O_{13}^+(2)\}$, $G$ is a finite simple $K_4$-group and so $G$ is $OD$-characterizable by Proposition 1.5(c).

Case 3. If $G \cong L_2(q)$, where $4 \mid (q+\varepsilon 1)$ and $|\pi(q+\varepsilon 1)| = 3$, then $G$ is $OD$-characterizable by Proposition 1.5(b).

Case 4. If $G$ is a finite group such that $|G| = |S_4(8)| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ and $D(G) = D(S_4(8)) = (2, 2, 1, 2, 1)$, then $\Gamma(G) = \{2 \sim 3 \sim 7 \sim 2; 5 \sim 13\}, \{5 \sim 2 \sim 3 \sim 7 \sim 13\}, \{5 \sim 2 \sim 7 \sim 3 \sim 13\}, \{5 \sim 3 \sim 2 \sim 7 \sim 13\}, \{5 \sim 3 \sim 7 \sim 2 \sim 13\}, \{5 \sim 7 \sim 3 \sim 2 \sim 13\}$ or $\{5 \sim 7 \sim 2 \sim 3 \sim 13\}$.

Subcase 1. If $\Gamma(G) = \{2 \sim 3 \sim 7 \sim 2; 5 \sim 13\}$, then $OC(G) = OC(S_4(8))$ and so $G \cong S_4(8)$ by Lemma 2.8.

Subcase 2. If $\Gamma(G) = \{5 \sim 2 \sim 3 \sim 7 \sim 13\}, \{5 \sim 2 \sim 7 \sim 3 \sim 13\}, \{5 \sim 3 \sim 2 \sim 7 \sim 13\}, \{5 \sim 7 \sim 2 \sim 3 \sim 13\}$ or $\{5 \sim 7 \sim 2 \sim 3 \sim 13\}$, we claim that $G \cong S_4(8)$, which is a contradiction.

Clearly, $t(G) = 3$ and $t(2, G) \geq 2$ in any case. By Lemma 2.10, there is a finite non-abelian simple group $S$ such that $S \trianglelefteq \bar{G} = G/K \trianglelefteq Aut(S)$ for the maximal normal solvable subgroup $K$ of $G$.

Step 1. $K$ is a $\{2, 3\}$-group.

Assume that $5 \in \pi(K)$. We claim that $13 \notin \pi(K)$ and so $5 \in \pi(K) \subseteq \{2, 3, 5, 7\}$. Otherwise, we may suppose that $T$ is a Hall $\{5, 13\}$-subgroup of $K$. It is easy to see that $T$ is a cyclic subgroup of order $5 \cdot 13$. Thus $5 \cdot 13 \in \pi_e(G) \subseteq \pi_e(G)$, a contradiction. Let $R \in \text{Syl}_5(K)$. By Frattini argument $G = KN_G(R)$. Therefore, the normalizer $N_G(R)$ contains an element of order $13$, say $x$. Thus $\langle x \rangle R$ is a cyclic subgroup of $G$ of order $5 \cdot 13$. Hence, $5 \cdot 13 \in \pi_e(G)$, a contradiction. Thus $5 \notin \pi(K)$. Similarly, we can prove that $13 \notin \pi(K)$, either.

If $5 \notin \pi(K), 13 \notin \pi(K)$ and $7 \in \pi(K)$, then $7 \in \pi(K) \subseteq \{2, 3, 7\}$. Let $R \in \text{Syl}_7(K)$. By Frattini argument $G = KN_G(R)$. Therefore, the normalizer $N_G(R)$ contains two elements of orders $5$ and $13$, say $\langle x \rangle$ and $\langle y \rangle$, respectively. Thus $\langle x \rangle R$ and $\langle y \rangle R$ are both nilpotent subgroups of $G$ of order $5 \cdot 7^i$ and $7^i \cdot 13$ for $i = 1$ or $2$, respectively. Hence, $5 \cdot 7 \in \pi_e(G)$ and $7 \cdot 13 \in \pi_e(G)$, a contradiction.

In a word, $K$ is a $\{2, 3\}$-group.

Step 2. $S$ is isomorphic to $A_5, A_6, U_4(2), L_2(7), L_2(8), A_7, L_2(13), A_8, A_9, L_3(3), U_3(3), L_2(27), L_3(4), S_8(8), 3D_4(2), L_2(64), S_6(2)$ or $S_4(8)$. Since $|G| = |S_4(8)| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$, it follows that $\pi(S) \subseteq \{2, 3, 5, 7, 13\}$. By Tables 2 and 3, $S \cong A_5, A_6, U_4(2), L_2(7), L_2(8), A_7, L_2(13), A_8, A_9, L_3(3), U_3(3), L_2(27), L_3(4), S_8(8), 3D_4(2), L_2(64), S_6(2)$ or $S_4(8)$.

Step 3. $G \cong S_4(8)$, which is a contradiction.

If

\[ \cong A_5, A_6, U_4(2), L_2(7), L_2(8), A_7, A_8, A_9, U_3(3), L_3(4) \text{ or } S_6(2), \]
then $13 \notin \pi(S)$. Since $S \subseteq \overline{G} \subseteq \text{Aut}(S)$ and $\pi(\text{Out}(S)) = \emptyset$ or $\pi(\text{Out}(S)) \subseteq \{2, 3\}$ by Lemma 2.11, it follows that $13 \in \pi(K)$, which is a contradiction since $K$ is a $\{2, 3\}$-group by Step 1.

If $S \cong L_2(13)$, $L_3(3)$, $L_2(27)$, $S_5(8)$ or $L_2(64)$, then $|K| \leq 7$. Since $S \subseteq \overline{G} \subseteq \text{Aut}(S)$ and $\pi(\text{Out}(S)) \subseteq \{2, 3\}$, it follows that $7 \in \pi(K)$, which is a contradiction since $K$ is a $\{2, 3\}$-group by Step 1.

If $S \cong D_4(2)$, then $5 \notin \pi(D_4(2))$. Since $D_4(2) \cong \overline{G} \subseteq \text{Aut}(D_4(2))$ and $\pi(\text{Out}(S)) = \{3\}$, it follows that $5 \in \pi(K)$, which is a contradiction since $K$ is a $\{2, 3\}$-group by Step 1.

Therefore $S \cong S_4(8)$. Thus $S_4(8) \subseteq \overline{G} \subseteq \text{Aut}(S_4(8))$. Hence $|K| = 1$ and so $G \cong S_4(8)$, a contradiction.

**Case 5.** If $G$ is a finite group such that $|G| = |S_4(17)| = 2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29$ and $D(G) = D(S_4(17)) = (2, 1, 2, 1, 1)$, then $\Gamma(G) = \{2 \sim 3 \sim 17 \sim 2; 5 \sim 29\}$, $\{5 \sim 2 \sim 3 \sim 17 \sim 29\}$, $\{5 \sim 3 \sim 2 \sim 17 \sim 19\}$, $\{5 \sim 2 \sim 17 \sim 3 \sim 29\}$, $\{5 \sim 17 \sim 3 \sim 2 \sim 29\}$ or $\{5 \sim 17 \sim 2 \sim 3 \sim 29\}$.

**Subcase 1.** If $\Gamma(G) = \{2 \sim 3 \sim 17 \sim 2; 5 \sim 29\}$, then $OC(G) = OC(S_4(17))$ and so $G \cong S_4(17)$ by Lemma 2.8.

**Subcase 2.** If $\Gamma(G) = \{5 \sim 2 \sim 3 \sim 17 \sim 29\}$, $\{5 \sim 2 \sim 17 \sim 3 \sim 29\}$, $\{5 \sim 3 \sim 17 \sim 2 \sim 29\}$, $\{5 \sim 17 \sim 3 \sim 2 \sim 29\}$ or $\{5 \sim 17 \sim 2 \sim 3 \sim 29\}$, we claim that $G \cong S_4(17)$, which is a contradiction.

Clearly, $t(G) = 3$ and $t(2, G) \geq 2$ in any case. By Lemma 2.10, there is a finite non-abelian simple group $S$ such that $S \subseteq \overline{G} = G/K \subseteq \text{Aut}(S)$ for the maximal normal solvable subgroup $K$ of $G$.

**Step 1.** $K$ is a $\{2, 3\}$-group.

Assume that $5 \in \pi(K)$. We claim that $29 \notin \pi(K)$ and so $5 \in \pi(K) \subseteq \{2, 3, 5, 17\}$. Otherwise, we may suppose that $T$ is a Hall $\{5, 29\}$-subgroup of $K$. It is easy to see that $T$ is a cyclic subgroup of order $5 \cdot 29$. Thus $5 \cdot 29 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Let $R \in \text{Syl}_5(K)$. By Frattini argument $G = KN_G(R)$. Therefore, the normalizer $N_G(R)$ contains an element of order 29, say $x$. Thus $\langle x \rangle$ is a cyclic subgroup of $G$ of order $5 \cdot 29$. Hence, $5 \cdot 29 \in \pi_e(G)$, a contradiction. Thus $5 \notin \pi(K)$. Similarly, we can prove that $29 \notin \pi(K)$, either.

If $5 \notin \pi(K)$, $29 \notin \pi(K)$ and $17 \in \pi(K)$, then $17 \in \pi(K) \subseteq \{2, 3, 17\}$. Let $|K|_{17} \neq 17^4$ and $R \in \text{Syl}_{17}(K)$. By Frattini argument $G = KN_G(R)$. Therefore, the normalizer $N_G(R)$ contains two elements of orders 5 and 29, say $x$ and $y$, respectively. Thus $\langle x \rangle$ $R$ and $\langle y \rangle R$ are both nilpotent subgroups of $G$ of order $5 \cdot 17^4$ and $17^4 \cdot 29$ for $i = 1, 2, 3$, respectively. Hence, $5 \cdot 17 \in \pi_e(G)$ and $17 \cdot 29 \in \pi_e(G)$, a contradiction. Hence $|K|_{17} = 17^4$ and so $17 \notin \pi(G/K)$. Since $S \subseteq \overline{G} = G/K \subseteq \text{Aut}(S)$, it follows that $17 \notin \pi(S)$. Thus $\pi(S) \subseteq \{2, 3, 5, 29\}$. From Tables 2 and 4, $S \cong A_5, A_6$ or $U_4(2)$. Since $|G| = 2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29$ and $\text{Out}(S) \subseteq \{2, 3\}$, it follows that $29 \in \pi(K)$, a contradiction.

In a word, $K$ is a $\{2, 3\}$-group.

**Step 2.** $S$ is isomorphic to $A_5, A_6, U_4(2), L_2(17), L_2(16), L_2(17^2)$ or $S_4(17)$.

Since $|G| = |S_4(17)| = 2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29$, it follows that $\pi(S) \subseteq \{2, 3, 5, 17, 29\}$. From Tables 2 and 4, $S \cong A_5, A_6, U_4(2), L_2(17), L_2(16), L_2(17^2)$ or $S_4(17)$.

**Step 3.** $G \cong S_4(17)$, which is a contradiction.

If $S \cong A_5, A_6, U_4(2), L_2(17), L_2(16)$ or $L_2(17^2)$, then $|S|_{17} \leq 17^2$. Since $S \subseteq \overline{G} \subseteq \text{Aut}(S)$ and $\pi(\text{Out}(S)) \subseteq \{2, 3\}$, it follows that $17 \in \pi(K)$, which is a contradiction since $K$ is a $\{2, 3\}$-group by Step 1.
Therefore $S \cong S_4(17)$. Thus $S_4(17) \leq \overline{G} \leq \text{Aut}(S_4(17))$. Hence $|K| = 1$ and so $G \cong S_4(17)$, a contradiction.

**Case 6.** If $G$ is a finite group such that $|G| = |G_2(9)| = 2^8 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 73$ and $D(G) = D(G_2(9)) = \langle 2, 2, 2, 1, 1, 0 \rangle$, then $\Gamma(G) = \{2 \cdot 2 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 2 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 3 \cdot 2 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 5 \cdot 3 \cdot 2 \cdot 13 \cdot 73 \} \}$. From Tables 2 and 3, we claim that $\overline{G} = G_2(9)$ and so $G \cong G_2(9)$ by Lemma 2.9.

**Subcase 1.** If $\Gamma(G) = \{2 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 2 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 3 \cdot 2 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 5 \cdot 3 \cdot 2 \cdot 13 \cdot 73 \} \}$. From Tables 2 and 3, we claim that $G \cong G_2(9)$, which is a contradiction.

Similarly, we can prove that $G \cong G_2(9)$, which is a contradiction.

Clearly, $t(G) = 4 > 3$ and $t(2, G) \geq 3$ in any case. By Lemma 2.10, there is a finite non-abelian simple group $S$ such that $S \leq \overline{G} = G/K \leq \text{Aut}(S)$ for the maximal normal solvable subgroup $K$ of $G$.

**Step 1.** $K$ is a $\{2, 3\}$-group.

Assume $7 \in \pi(K)$. We claim that $p \notin \pi(K)$, where $p \in \{13, 73\}$. Otherwise, we may suppose that $T$ is a Hall $\{7, p\}$-subgroup of $K$. It is easy to see that $T$ is a cyclic subgroup of order $7p$. Thus $7p \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Therefore $7 \in \pi(K) \subseteq \{2, 3, 5, 7\}$. Let $R \in Syl_7(K)$. By Frattini argument $G = K\text{N}_G(R)$. Therefore, the normalizer $N_G(R)$ contains an element of order $p$, say $x$. Thus $\langle x \rangle R$ is a cyclic subgroup of $G$ of order $7p$. Hence, $7p \in \pi_e(G)$, a contradiction. Thus $7 \notin \pi(K)$. Similarly, we can prove that $p \notin \pi(K)$, either.

If $7 \notin \pi(K)$, $13 \notin \pi(K)$, $73 \notin \pi(K)$ and $5 \in \pi(K)$, then $5 \in \pi(K) \subseteq \{2, 3, 5\}$. Let $R \in Syl_5(K)$. By Frattini argument $G = K\text{N}_G(R)$. Therefore, the normalizer $N_G(R)$ contains an element of order $73$, say $x$, respectively. Thus $\langle x \rangle R$ and $\langle y \rangle R$ is a nilpotent subgroups of $G$ of order $5^i \cdot 73$ for $i = 1, 2$. Hence, $5 \cdot 73 \in \pi_e(G)$, a contradiction.

In a word, $K$ is a $\{2, 3\}$-group.

**Step 2.** $S$ is isomorphic to $A_5, A_6, U_4(2), L_2(7), L_2(8), U_3(3), A_7, L_2(49), U_3(5), L_3(4), A_8, A_9, J_2, A_{10}, U_4(3), L_3(3), L_2(25), U_3(4), L_4(3), L_2(13), L_2(27), G_2(3), S_3(8), L_2(64), L_3(9), U_3(9), L_2(3^6)$.

Since $|G| = |G_2(9)| = 2^8 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 73$, it follows that $\pi(S) \subseteq \{2, 3, 5, 7, 13, 73\}$. From Tables 2 and 3, $S \cong A_5, A_6, U_4(2), L_2(7), L_2(8), U_3(3), A_7, L_2(49), U_3(5), L_3(4), A_8, A_9, J_2, A_{10}, U_4(3), L_3(3), L_2(25), U_3(4), L_4(3), L_2(13), L_2(27), G_2(3), S_3(8), L_2(64)$ or $L_3(9), U_3(9), L_2(3^6)$ or $G_2(9)$. from $\Gamma(G) = \{2 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 2 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 3 \cdot 2 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 73, \{7 \cdot 5 \cdot 3 \cdot 2 \cdot 13 \cdot 73 \} \}$. From Tables 2 and 3, we claim that $G \cong G_2(9)$, which is a contradiction.

**Step 3.** $G \cong G_2(9)$, a contradiction.

If $S \cong A_5, A_6, U_4(2), L_2(7), L_2(8), U_3(3), A_7, L_2(49), U_3(5), L_3(4), A_8, A_9, J_2, A_{10}, U_4(3), L_3(3), L_2(25), U_3(4), L_4(3), L_2(13), L_2(27), G_2(3), S_3(8), L_2(64)$ or $L_3(9), U_3(9), L_2(3^6)$ or $G_2(9)$, then $73 \notin \pi(S)$. Since $S \cong \overline{G} \leq \text{Aut}(S)$ and $\pi(\text{Out}(S)) \subseteq \{2, 3\}$, it follows that $73 \notin \pi(K)$, which is a contradiction since $K$ is a $\{2, 3\}$-group by Step 1.

If $S \cong U_3(9)$, then $13 \notin \pi(S)$. Since $U_3(9) \cong \overline{G} \leq \text{Aut}(U_3(9))$ and $\pi(\text{Out}(U_3(9))) = \{2\}$, it follows that $13 \in \pi(K)$, which is a contradiction since $K$ is a $\{2, 3\}$-group by Step 1.

If $S \cong L_2(3^6)$, then $|L_2(3^6)| = 5$. Since $L_2(3^6) \cong \overline{G} \leq \text{Aut}(L_2(3^6))$ and $\pi(\text{Out}(L_2(3^6))) = \{2, 3\}$, it follows that $5 \in \pi(K)$, which is a contradiction since $K$ is a $\{2, 3\}$-group by Step 1.

Therefore $S \cong G_2(9)$. Thus $G_2(9) \cong \overline{G} \leq \text{Aut}(G_2(9))$. Hence $|K| = 1$ and so $G \cong G_2(9)$, a contradiction.
By Proposition 1.5(f), all finite simple \( C_{2,2} \)-groups are OD-characterizable. Hence all finite non-abelian simple groups whose first prime graph components are 0-regular are OD-characterizable. Together with Theorems 3.3 and 3.7, we obtain the following theorem.

**Theorem 3.7.** Let \( G \) be a finite non-abelian simple group whose first prime graph components are 0-regular. Then \( G \) is OD-characterizable.

**Question 3.8.** Let \( G \) be a finite group such that \( |G| = |M| \) and \( D(G) = D(M) \), where \( M \) is an almost simple group. Is \( \Gamma(G) = \Gamma(M) \)?

**Acknowledgement.** This work is partly supported by the NNSF of China (11271301, 11171364 and 10871032), the Natural Science Foundation Project of CQ CSTC (No. 2010BB 9206) and National Science Foundation for Distinguished Young Scholars of China (No. 11001226).

**References**


