Two-Fluid Nonlinear Mathematical Model for Pulsatile Blood Flow Through Stenosed Arteries

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Abstract. Pulsatile flow of blood through mild stenosed narrow arteries is analyzed by treating the blood in the core region as Casson fluid and the plasma in the peripheral layer as Newtonian fluid. Perturbation method is used to solve the coupled implicit system of non-linear differential equations. The expressions for velocity, wall shear stress, plug core radius, flow rate and resistance to flow are obtained. The effects of pulsatility, stenosis, peripheral layer and non-Newtonian behavior of blood on these flow quantities are discussed. It is found that the pressure drop, plug core radius, wall shear stress and resistance to flow increase with the increase of the yield stress or stenosis size while all other parameters held constant.

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1. Introduction

Hemodynamics plays a vital role in the development, progression and treatment of arterial stenosis [4, 13, 16]. Arteries are narrowed by the development of atherosclerotic plaques that protrude into the lumen, resulting arterial stenosis. When an obstruction developed in an artery, one of the most serious consequences is the increased resistance and the associated reduction of the blood flow to the particular vascular bed supplied by the artery. Thus, the presence of a stenosis leads to the serious circulatory disorder. Hence, the mathematical modeling of blood flow through stenosed arteries is very important.

Several researchers studied the blood flow characteristics in the presence of stenosis [1, 5, 7, 10]. The assumption of Newtonian behavior of blood is acceptable for high shear rate flow through larger arteries [13]. But, blood, being a suspension of cells in plasma, exhibits non-Newtonian behavior at low shear rate ($\gamma < 10/\text{sec}$) in small diameter arteries (0.02 mm–0.1 mm) [6, 9]. In diseased state, the actual flow is distinctly pulsatile [8, 9]. Several researchers have studied the non-Newtonian behavior and pulsatile flow of blood through stenosed arteries [3, 7, 10, 13].

Srivastava and Saxena [15] and Misra and Pandey [6] propounded that for blood flowing through small vessels, there is an erythrocyte-free plasma (Newtonian) layer adjacent to the
vessel wall and a core layer of a suspension of all erythrocytes (non-Newtonian). Accepting this idea, several studies [2, 6, 15] revealed that the existence of the peripheral layer has some significance in the flow characteristics of the arterial system. Chaturani and Ponnallagam Samy [3] and Scott Blair [12] pointed out that Casson fluid model is well suited and simple to apply for blood flow problems. Many researchers [2, 3, 6, 15] have used Casson fluid model for mathematical modeling of blood flow through narrow arteries at low shear rates for different flow situations. Hence, in this paper, we have studied the pulsatile flow of a two-fluid model for blood through stenosed narrow arteries (of diameters 0.02mm – 0.2mm) at low shear rates ($\dot{\gamma} < 10/\text{sec}$), assuming the suspension of all the erythrocytes in the core region of the blood vessel as a Casson fluid and the plasma in the peripheral layer as a Newtonian fluid.

2. Mathematical formulation

Consider an axially symmetric, laminar, pulsatile and fully developed flow of blood in the axial direction ($\bar{z}$) through a circular artery with an axially symmetric mild stenosis. It is assumed that the blood is represented by a two-fluid model with a central layer (core region) of suspension of all the erythrocytes as a Casson fluid and a peripheral layer of plasma as a Newtonian fluid. In order to idealize the present two-fluid model, we have assumed the wall of the artery to be rigid. The geometry of the arterial stenosis is shown in Figure 1. The cylindrical polar coordinates ($\bar{r}, \phi, \bar{z}$) are used to study the flow, where $\bar{r}$ and $\phi$ are the radial coordinate and the azimuthal angle, respectively. Since, the stenosis present in the artery is considered to be mild, the radial transport of the blood is negligible [3] and thus, we have neglected the radial velocity of the blood in this study. Hence, in the present study, the flow of blood is considered to be unidirectional and is in the axial direction.

The principle of conservation of mass for one dimensional fluid flow in a deformable tube gives the following equations of continuity in the core region and peripheral layer region [8]:

\begin{equation}
\frac{\partial \bar{R}_1}{\partial \bar{t}} + \bar{R}_1 \frac{\partial \bar{u}_C}{\partial \bar{z}} + \bar{u}_C \frac{\partial \bar{R}_1}{\partial \bar{z}} = 0, \quad \text{in } 0 \leq \bar{r} \leq \bar{R}_1(\bar{z}),
\end{equation}

\begin{equation}
\frac{\partial \bar{R}}{\partial \bar{t}} + \bar{R} \frac{\partial \bar{u}_N}{\partial \bar{z}} + \bar{u}_N \frac{\partial \bar{R}}{\partial \bar{z}} = 0, \quad \text{in } \bar{R}_1(\bar{z}) \leq \bar{r} \leq \bar{R}(\bar{z}),
\end{equation}

where $\bar{R}_1 (= \bar{R}_1(\bar{z}))$ is the radius of the core region of the stenosed artery; $\bar{R}(= \bar{R}(\bar{z}))$ is the radius of the artery with peripheral layer; $\bar{u}_C$ is the velocity of the fluid (Casson fluid) in the core region and $\bar{u}_N$ is the velocity of the fluid (Newtonian fluid) in the peripheral layer region of the fluid flow. Since, the blood flow in narrow arteries at low shear rates is a slow flow, the viscous forces dominate over the inertial forces and thus, the magnitude of the convective terms are negligibly small. Hence, in the present study, we have neglected the convective terms in the momentum equations:

\begin{equation}
\bar{\rho}_C \frac{\partial \bar{u}_C}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \bar{\tau}_C) \quad \text{in } 0 \leq \bar{r} \leq \bar{R}_1(\bar{z}),
\end{equation}

\begin{equation}
\bar{\rho}_N \frac{\partial \bar{u}_N}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \bar{\tau}_N) \quad \text{in } \bar{R}_1(\bar{z}) \leq \bar{r} \leq \bar{R}(\bar{z}),
\end{equation}
where the shear stress $\tau = |\bar{\tau}| = -\bar{\tau}_r$ (since $\tau = \bar{\tau}_C$ or $\tau = \bar{\tau}_N$); $\bar{p}$ is the pressure; $\bar{u}_C$ and $\bar{u}_N$ are the axial velocity of the fluid in the core region and peripheral region, respectively; $\bar{\tau}_C$ and $\bar{\tau}_N$ are the shear stress of the Casson fluid and Newtonian fluid, respectively; $\bar{\rho}_C$ and $\bar{\rho}_N$ are the densities of the Casson fluid and Newtonian fluid respectively; $\bar{t}$ is the time. The relations between the shear stress and strain rate of the fluids in motion in the core region (Casson fluid) and peripheral region (Newtonian fluid) are given by

\begin{align*}
\sqrt{\bar{\tau}_C} &= \sqrt{-\bar{\mu}_C \frac{\partial \bar{u}_C}{\partial \bar{r}}} + \sqrt{\bar{\tau}_y} \quad \text{if } \bar{\tau}_C \geq \bar{\tau}_y \text{ and } \bar{R}_P \leq \bar{r} \leq \bar{R}_1(\bar{z}), \\
\frac{\partial \bar{u}_C}{\partial \bar{r}} &= 0 \quad \text{if } \bar{\tau}_C \leq \bar{\tau}_y \text{ and } 0 \leq \bar{r} \leq \bar{R}_P, \\
\bar{\tau}_N &= -\bar{\mu}_N \frac{\partial \bar{u}_N}{\partial \bar{r}} \quad \text{if } \bar{R}_1(\bar{z}) \leq \bar{r} \leq \bar{R}(\bar{z}),
\end{align*}

where $\bar{\mu}_C$ and $\bar{\mu}_N$ are the viscosities of the Casson fluid and Newtonian fluid, respectively; $\bar{\tau}_y$ is the yield stress; $\bar{R}_P$ is the plug core radius. The geometry of the stenosis in the peripheral region and core region are given by

\begin{align*}
\bar{R}(\bar{z}) &= \begin{cases} 
\bar{R}_0, & \text{in the normal artery region,} \\
\bar{R}_0 - \frac{\delta_P}{\bar{\mu}} \left\{ 1 + \cos \left[ \frac{2\pi}{\bar{L}_0} (\bar{z} - \bar{d} - \frac{\bar{L}_0}{2}) \right] \right\}, & \text{in } \bar{d} \leq \bar{z} \leq \bar{d} + \bar{L}_0,
\end{cases} \\
\bar{R}_1(\bar{z}) &= \begin{cases} 
\beta \bar{R}_0, & \text{in the normal artery region,} \\
\beta \bar{R}_0 - \frac{\delta_C}{\bar{\mu}} \left\{ 1 + \cos \left[ \frac{2\pi}{\bar{L}_0} (\bar{z} - \bar{d} - \frac{\bar{L}_0}{2}) \right] \right\}, & \text{in } \bar{d} \leq \bar{z} \leq \bar{d} + \bar{L}_0,
\end{cases}
\end{align*}

where $\bar{R}(\bar{z})$ and $\bar{R}_1$ are the radii of the stenosed artery with the peripheral region and core region respectively; $\bar{R}_0$ and $\beta \bar{R}_0$ are the radii of the normal artery and core region of the normal artery respectively; $\beta$ is the ratio of the central core radius to the normal artery radius; $\bar{L}_0$ is the length of the stenosis; $\bar{d}$ indicates the location of the stenosis; $\delta_P$ and $\delta_C$ are the maximum projections of the stenosis in the peripheral region and core region.
respectively such that $[\delta p/R_0] \ll 1$ and $[\delta C/R_0] \ll 1$. The boundary conditions are

$$\tau_C \text{ is finite and } \frac{\partial \tilde{u}_C}{\partial \tilde{r}} = 0 \text{ at } \tilde{r} = 0;$$

$$\tilde{u}_N = 0 \text{ at } r = \tilde{R};$$

$$\tau_C = \tau_N \text{ and } \tilde{u}_C = \tilde{u}_N \text{ at } \tilde{r} = 1.$$ 

Since the pressure gradient is a function of $\tilde{z}$ and $\tilde{r}$, we assume

$$-\frac{\partial \tilde{p}}{\partial \tilde{z}} = \tilde{q}(\tilde{z}) f(\tilde{r}),$$

where $\tilde{q}(\tilde{z}) = -(\partial \tilde{p}/\partial \tilde{z})(\tilde{z},0)$. Since, any periodic function can be expanded in a Fourier sine series, it is reasonable to choose $1 + A \sin \tilde{\omega} \tilde{t}$ as a good approximation for $f(\tilde{r})$, where $A$ and $\tilde{\omega}$ are the amplitude and angular frequency of the flow respectively. We introduce the following non-dimensional variables

$$z = \frac{\tilde{z}}{R_0}, \quad R(z) = \frac{\bar{R}(\tilde{z})}{R_0}, \quad R_1(z) = \frac{\bar{R}_1(\tilde{z})}{R_0}, \quad r = \frac{\tilde{r}}{R_0}, \quad \frac{\tilde{t}}{R_0}, \quad \frac{\partial}{\partial \tilde{r}} = \frac{\bar{d}}{R_0},$$

$$L_0 = \frac{R_0}{R_0}, \quad q(z) = \frac{\bar{q}(\tilde{z})}{q_0}, \quad \varepsilon_C = \frac{\alpha_C^2}{\mu_C}, \quad \varepsilon_N = \frac{\alpha_N^2}{\mu_N},$$

$$\frac{R_0}{R_0}, \quad \delta_p = \frac{\bar{R}_p}{R_0}, \quad \delta_C = \frac{\bar{R}_C}{R_0}, \quad \bar{u}_C = \frac{\tilde{u}_C}{(q_0 R_0^2/4 \mu_C)}, \quad \bar{u}_N = \frac{\tilde{u}_N}{(q_0 R_0^2/4 \mu_N)}.$$ 

$$\tau_C = \frac{\bar{t}_C}{(q_0 R_0/2)}, \quad \tau_N = \frac{\bar{t}_N}{(q_0 R_0/2)}, \quad \theta = \frac{\bar{\omega}}{(q_0 R_0/2)}, \quad t = \bar{\omega} \bar{t},$$

where $q_0$ is the negative of the pressure gradient in the normal artery, $\alpha_C$ and $\alpha_N$ are the pulsatile Reynolds numbers of the Casson fluid and Newtonian fluid, respectively. Using the non-dimensional variables, Equations (2.2)–(2.7) are simplified to

$$\frac{\varepsilon_C}{t} \frac{\partial u_C}{\partial t} = 4q(z)f(t) - \frac{2}{r} \frac{\bar{d}}{\partial \tilde{r}}(r \tau_C) \quad \text{if} \quad 0 \leq r \leq R_1(z)$$

$$\sqrt{\tau_C} = \sqrt{-\frac{1}{2} \frac{\partial u_C}{\partial r}} + \sqrt{\theta} \quad \text{if} \quad \tau_C \geq \theta \quad \text{and} \quad R_p \leq r \leq R_1(z),$$

$$\frac{\partial u_C}{\partial r} = 0 \quad \text{if} \quad \tau_C \leq \theta \quad \text{and} \quad 0 \leq r \leq R_p,$$

$$\varepsilon_N \frac{\partial u_N}{\partial t} = 4q(z)f(t) - \frac{2}{r} \frac{\bar{d}}{\partial \tilde{r}}(r \tau_N) \quad \text{if} \quad R_1(z) \leq r \leq R(z),$$

where

$$f(t) = 1 + A \sin t.$$ 

The boundary conditions (in the dimensionless form) are

$$\tau_C \text{ is finite and } \frac{\partial \tilde{u}_C}{\partial \tilde{r}} = 0 \text{ at } r = 0,$$

$$\tau_C = \tau_N \text{ and } u_C = u_N \text{ at } r = R_1,$$

$$\tau_N = 0 \text{ at } r = R.$$
The geometry of the stenosis in the peripheral region and core region (in the dimensionless form) are given by

\[ R(z) = \begin{cases} 1, & \text{in the normal artery region,} \\ 1 - \frac{\delta_P}{2} \left\{ 1 + \cos \left( \frac{2\pi}{\ell_0} \left( z - d - \frac{L_0}{2} \right) \right) \right\}, & \text{in } d \leq z \leq d + L_0, \end{cases} \]

(2.19)

\[ \tilde{R}_1(\bar{z}) = \begin{cases} \beta, & \text{in the normal artery region,} \\ \beta - \frac{\delta_C}{2} \left\{ 1 + \cos \left( \frac{2\pi}{\ell_0} \left( z - d - \frac{L_0}{2} \right) \right) \right\}, & \text{in } d \leq z \leq d + L_0. \end{cases} \]

(2.20)

The non-dimensional volume flow rate \( Q \) is given by

\[ Q = 4 \int_0^{R(z)} u(r,z,t) r \, dr, \]

(2.21)

where \( Q = \bar{Q} / (\pi \tilde{R}_0 \bar{q}_0 / 8 \bar{\mu}_0) \) and \( \bar{Q} \) is the volume flow rate.

3. Perturbation method of solution

Since it is not possible to find an exact solution to the system of nonlinear equations (2.13) – (2.16), the perturbation method is used to obtain the approximate solution to the unknowns \( u_C, u_N, \tau_C \), and \( \tau_N \). When we non-dimensionalize the momentum Equations (2.3) and (2.4), \( \varepsilon_C \) and \( \varepsilon_N \) occur naturally and hence, it is more appropriate to expand Equations (2.13)–(2.16) about \( \varepsilon_C \) and \( \varepsilon_N \). Let us expand the plug core velocity \( u_p \), and the velocity in the core region \( u_C \) in the perturbation series of \( \varepsilon_C \) as below (where \( \varepsilon_C \ll 1 \)):

\[ u_p(z,t) = u_{0p}(z,t) + \varepsilon_C u_{1p}(z,t) + \cdots, \]

(3.1)

\[ u_C(r,z,t) = u_{0C}(r,z,t) + \varepsilon_C u_{1C}(r,z,t) + \cdots. \]

(3.2)

Similarly, one can expand \( u_N, \tau_P, \tau_C, \tau_N \) and the plug core radius \( R_P \) in powers of \( \varepsilon_C \) and \( \varepsilon_N \), where \( \varepsilon_N \ll 1 \). Using the perturbation series in Equations (2.13) and (2.14) and then equating the constant terms and \( \varepsilon_C \) terms, the differential equations of the core region becomes

\[ \frac{\partial}{\partial r} (r \tau_{0C}) = 2q(z)f(t)r, \quad -\frac{\partial u_{0C}}{\partial r} = -\frac{2}{r} \frac{\partial}{\partial r} (r \tau_{1C}), \]

\[ \frac{\partial u_{0C}}{\partial t} = -2 \frac{\partial}{\partial r} (r \tau_{1C}), \quad \frac{\partial u_{1C}}{\partial r} = 2 \tau_{1C} \left( 1 - \sqrt{\theta / \tau_{0C}} \right). \]

(3.3)

Similarly, using the perturbation series expansions in Equation (2.16) and then equating the constant terms and \( \varepsilon_N \) terms, the differential equations of the peripheral region reduced to

\[ \frac{\partial}{\partial r} (r \tau_{0N}) = 2q(z)f(t)r, \quad \frac{\partial u_{0N}}{\partial t} = -\frac{2}{r} \frac{\partial}{\partial r} (r \tau_{1N}), \]

\[ -\frac{\partial u_{0N}}{\partial r} = 2 \tau_{0N}, \quad -\frac{\partial u_{1N}}{\partial r} = 2 \tau_{1N}. \]

(3.4)
Substituting the perturbation series expansions in Equation (2.18) and then equating the constant terms and $\varepsilon_C$ and $\varepsilon_N$ terms, we get

$$\tau_{0P} \text{ and } \tau_{1P} \text{ are finite and } \frac{\partial u_{0P}}{\partial r} = 0, \frac{\partial u_{1P}}{\partial r} = 0 \text{ at } r = 0;$$

(3.5)

$$\tau_{0C} = \tau_{0N}, \tau_{1C} = \tau_{1N}, u_{0C} = u_{0N}, u_{1C} = u_{1N} \text{ at } r = R_1;$$

$$u_{0N} = u_{1N} = 0 \text{ at } r = R. $$

Solving the system of Equations (3.3) and (3.4) by using Equation (3.5) for the unknowns $u_{0C}, u_{1C}, \tau_{0C}, \tau_{1C}, u_{0N}, u_{1N}, \tau_{0N}$ and $\tau_{1N}$, one can obtain

(3.6) $\tau_{0P} = \nabla R_{0P}$, $\tau_{0C} = \nabla r$, $\tau_{0N} = \nabla r,$

(3.7) $u_{0N} = \nabla R^2 \left( 1 - \xi^2 \right),$ 

(3.8) $u_{0C} = \nabla R^2 \left( 1 - \Omega^2 + \Omega^2 \left[ 1 - \xi - \frac{8}{3} \sigma_1^{1/2} \left( 1 - \xi^{3/2} \right) + 2 \sigma_1 \left( 1 - \xi_1 \right) \right] \right),$ 

(3.9) $u_{0P} = \nabla R^2 \left\{ (1 - \Omega^2) + \Omega^2 \left[ (1 - \chi^2) - \frac{8}{3} \sigma_1^{1/2} \left( 1 - \chi^{3/2} \right) + 2 \sigma_1 \left( 1 - \chi \right) \right] \right\},$

(3.10) $\tau_{1P} = -\nabla B R^3 \left\{ \frac{1}{4} \sigma (1 - \Omega^2) + \Omega^2 \sigma_1 \left[ \frac{1}{4} - \frac{1}{3} \sigma_1^{1/2} + \frac{1}{12} \sigma_1^2 \right] \right\},$

$$\tau_{1N} = -\nabla B R^3 \left\{ \frac{1}{8} \xi (1 - \Omega^2) - \frac{1}{8} \Omega^2 \left[ 2 \xi_1 - \xi_3 - \sigma_1 \xi_1^{-1} \right] \right\},$$

(3.11) $$-\frac{8}{21} \sigma_1^{1/2} \left( 7 \xi_1 - 4 \xi_3^{5/2} - 3 \sigma_1^{7/2} \xi_1^{-1} \right) \right\},$$

(3.12) $u_{1N} = -\nabla B R^3 R_1 \left\{ \left[ \frac{1}{4} \xi_1 - \frac{1}{8} \Omega^2 \xi_1^{-1} - \frac{1}{8} \Omega^2 \xi_3^3 \right] + \xi_1^{-1} \Omega^2 \left[ \frac{1}{8} - \frac{1}{7} \sigma_1^{1/2} + \frac{1}{56} \sigma_1^4 \right] \right\},$

(3.13) 

$$u_{1C} = -\nabla B R^3 R_1 \left\{ \left[ \frac{3}{16} \Omega^{-1} - \frac{1}{4} \Omega + \frac{1}{16} \Omega^3 + \frac{1}{4} \Omega^3 \log \xi \right] \right\},$$

$$-\Omega^3 \log \xi \left[ \frac{1}{4} - \frac{2}{7} \sigma_1^{1/2} + \frac{1}{28} \sigma_1^4 \right] \right\},$$

(3.14) 

$$u_{1C} = -\nabla B R^3 R_1 \left\{ \left[ \frac{3}{16} \Omega^{-1} - \frac{1}{4} \Omega + \frac{1}{16} \Omega^3 + \frac{1}{4} \Omega^3 \log \Omega \right] \right\},$$

$$-\Omega^3 \log \Omega \left[ \frac{1}{4} - \frac{2}{7} \sigma_1^{1/2} + \frac{1}{28} \sigma_1^4 \right]$$

$$+ \Omega \left( 1 - \Omega^2 \right) \left[ \frac{1}{4} \xi_1 - \frac{1}{3} \Omega + \frac{1}{16} \Omega^3 \log \xi \right]$$

$$+ \Omega \left( 1 - \Omega^2 \right) \left[ \frac{1}{4} \xi_1 - \frac{1}{3} \Omega + \frac{1}{16} \Omega^3 \log \xi \right]$$

$$+ \frac{53}{294} \sqrt{\sigma_1} \left( 1 - \sqrt{\xi_1} \right) - \frac{1}{3} \left( 1 - \xi_1^2 \right) + \frac{4}{9} \sigma_1 \left( 1 - \sqrt{\xi_1} \right)$$

$$- \frac{8}{63} \sigma_1 \left( 1 - \xi_1^3 \right) - \frac{1}{28} \sigma_1^4 \log \xi_1 + \frac{1}{14} \sqrt{\sigma_1^9} \left( 1 - \frac{1}{\sqrt{\xi_1}} \right) \right\}. $$
\[ u_{1P} = -\nabla BR^3 R_1 \left\{ \left[ \frac{3}{16} \Omega^{-1} - \frac{1}{4} \Omega + \frac{1}{16} \Omega^3 + \frac{1}{4} \Omega^3 \log \Omega \right] \\
- \Omega^3 \log \Omega \left[ \frac{1}{4} - \frac{2}{7} \sqrt{\sigma_1} + \frac{1}{28} \sigma_1^4 \right] \\
+ \Omega \left( 1 - \Omega^2 \right) \left[ \frac{1}{4} \left( 1 - \sigma_1^2 \right) - \frac{1}{3} \sqrt{\sigma_1} \left( 1 - \sqrt{\sigma_1^3} \right) \right] \\
+ \Omega^3 \left[ \frac{1}{4} \left( 1 - \sigma_1^2 \right) - \frac{1}{3} \sqrt{\sigma_1} \left( 1 - \sqrt{\sigma_1^3} \right) - \frac{1}{16} \left( 1 - \sigma_1^4 \right) \right] \\
- \frac{53}{294} \sqrt{\sigma_1} \left( 1 - \sqrt{\sigma_1^3} \right) - \frac{1}{3} \left( 1 - \sigma_1^2 \right) + \frac{4}{9} \sigma_1 \left( 1 - \sqrt{\sigma_1^3} \right) \\
+ \frac{16}{63} \sigma_1 \left( 1 - \sigma_1^3 \right) - \frac{1}{28} \sigma_1^4 \log \sigma_1 + \frac{1}{14} \sqrt{\sigma_1^9} \left( 1 - \frac{1}{\sqrt{\sigma_1}} \right) \right\} , \tag{3.15} \]

where \( \nabla = q(z)f(t) \), \( k^2 = r \mid_{R_0 = \theta} = R_{0P} = \theta /[q(z)f(t)] \), \( B = [1/f(t)](df(t)/dt) \), \( \xi = r/R \), \( \xi_1 = r/R_1 \), \( \Omega = R_1/R \), \( \sigma = k^2/R \), \( \sigma_1 = k^2/R_1 \) and \( \chi = R_{0P}/R_1 \). The wall shear stress \( \tau_w \) can be obtained as below:

\[ \tau_w = (\tau_{0N} + \epsilon_N \tau_{1N})_{r=R} = \tau_{0w} + \epsilon_N \tau_{1w} \tag{3.16} \]

Using Equations (3.7)–(3.9) and (3.13)–(3.15) in Equation (2.21), the volume flow rate is obtained as

\[ Q = \nabla R^4 \left\{ \left( 1 - \Omega^2 \right) \left( 1 + 3 \Omega^2 \right) + \Omega^4 \left[ 1 - \frac{16}{7} \sigma_1^{1/2} + \frac{4}{3} \sigma_1 - \frac{1}{21} \sigma_1^4 \right] \right\} \\
- \epsilon_c \nabla BR^3 R_1^3 \left\{ \left[ \frac{3}{8} \Omega^{-1} - \frac{1}{2} \Omega + \frac{1}{8} \Omega^3 + \frac{1}{2} \Omega^3 \log \Omega \right] \\
- \Omega^3 \log \Omega \left[ \frac{1}{2} - \frac{4}{7} \sqrt{\sigma_1} + \frac{1}{14} \sigma_1^4 \right] + \Omega \left( 1 - \Omega^2 \right) \left[ \frac{1}{4} - \frac{2}{7} \sqrt{\sigma_1} + \frac{1}{28} \sigma_1^4 \right] \\
+ \Omega^3 \left[ \frac{1}{6} - \frac{30}{77} \sqrt{\sigma_1} + \frac{8}{35} \sigma_1 - \frac{1}{3} \sigma_1^{5/2} + \frac{1}{14} \sigma_1^4 + \frac{5}{21} \sigma_1^{9/2} \right] \\
- \frac{41}{770} \sigma_1^6 - \frac{1}{14} \sigma_1^6 \log \sigma_1 + \frac{1}{14} \sigma_1^4 \left( 1 - \sigma_1^2 \right) \log k \right\} \right\} \\
- \epsilon_N \nabla BR^5 R_1 \left\{ \left[ \frac{1}{6} \Omega^{-1} - \frac{3}{8} \Omega + \frac{5}{24} \Omega^5 - \frac{1}{2} \Omega^3 \left( 1 - \Omega^2 \right) \log R_1 \right] \\
+ \Omega^4 \left( 1 - \Omega^2 \right) \left( 1 + 2 \log R_1 \right) \left[ \frac{1}{4} - \frac{2}{7} \sqrt{\sigma_1} + \frac{1}{28} \sigma_1^4 \right] \right\} . \tag{3.17} \]

The shear stress \( \tau_c = \tau_{0C} + \epsilon_c \tau_{1C} \) at \( r = R_P \) is given by

\[ |\tau_{0C} + \epsilon_c \tau_{1C}|_{r=R_P} = \theta. \tag{3.18} \]
Using the Taylor series of $\tau_{0C}$ and $\tau_{1C}$ about $R_{0p}$ and using $\tau_{0C}|_{r=R_{0p}} = \theta$, we get

\begin{equation}
R_{1p} = \frac{\tau_{1C}|_{r=R_{0p}}}{\nabla}.
\end{equation}

Using Equations (3.6), (3.11) and (3.19) in the two term approximated perturbation series of $R_p$, the expression for $R_p$ can be obtained as

\begin{equation}
R_p = k^2 - \frac{B\epsilon R^3}{4} \left[ \sigma^2 (1 - \Omega^2) + \Omega^3 \left( \sigma_1 - \frac{4}{3} \sigma_1^{3/2} + \frac{1}{3} \sigma_1^3 \right) \right].
\end{equation}

The resistance to flow is given by

\begin{equation}
\Lambda = \frac{\Delta p f(t)}{Q},
\end{equation}

where $\Delta p$ is the pressure drop. When $R_1 = R$, the present model reduces to the single fluid Casson model and in such case, the expressions obtained in the present model for velocity $u_C$, shear stress $\tau_C$, wall shear stress $\tau_w$, flow rate $Q$ and plug core radius $R_p$ are identical with those of Chaturani and Ponnalagar Samy [3].

4. Results and discussion

The objective of the present model is to analyze the effects of the pulsatility, non-Newtonian nature, peripheral layer and stenosis on various flow quantities in a blood flow through a stenosed artery when blood is modeled by a two-fluid model with a core region of suspension of red cells represented by the Casson fluid and a peripheral layer of plasma treated as the Newtonian fluid. In this study, we have used the range 0–0.15 for the yield stress $\theta$. Though the amplitude $A$ varies from 0 to 1, the range 0.2–0.5 is used to pronounce its effect. The ratio $\alpha$ ($= \alpha_C/\alpha_N$) between the pulsatile Reynolds numbers of the Newtonian fluid and Casson fluid is called the pulsatile Reynolds number ratio. Though, the ratio $\alpha$ ranges from 0 to 1, the range 0.2 to 0.5 is used [11]. The same range is used for the pulsatile Reynolds number $\alpha_C$ [15]. Given the values of $\alpha$ and $\alpha_C$, the value of $\alpha_N$ is obtained from $\alpha = \alpha_N/\alpha_C$. The value of $\beta$ is generally taken as 0.95 and 0.985 [15]. To deduce the present model to a single-fluid Casson model, $\beta$ is assigned the value 1. The relations $R_1 = \beta R$ and $\delta_C = \beta \delta_p$ are used to estimate $R_1$ and $\delta_C$ [15]. The range 0.1 to 0.15 is used for $\delta_p$ [14]. But, to compare the present model with the earlier results, the value 0.2 is used for $\delta_C$.

It is observed that in Equation (3.17), $f(t)$, $R$ and $\theta$ are known and, $Q$ and $q(z)$ are the unknowns to be determined. A careful observation of Equation (3.17) reveals the fact that $q(z)$ is the pressure gradient of the steady flow. Thus, if steady flow is assumed, then Equation (3.17) can be solved for $q(z)$ [10, 3]. For steady flow, Equation (3.17) reduces to

\begin{equation}
0 = (R^4 - 4R^2R_1^2 + 3R_1^4) y^4 - Q_s y^3
+ \left[ (R_1 y)^4 - \frac{16}{7} \theta \left( \sqrt{R_1 y} \right)^7 + \frac{4}{3} \theta (R_1 y)^3 - \frac{1}{21} \theta^4 \right],
\end{equation}

where $y = q(z)$ and $Q_s$ is the steady state flow rate. Equation (4.1) is solved numerically for $y$ by using Newton-Raphson method with variation in the axial direction and yield stress with $\beta = 0.95$ and $\delta_p = 0.1$. Throughout the analysis, the steady flow rate $Q_s$ value is taken as 1.0. Only that root which gives the realistic value for plug core radius has been considered.
4.1. Pressure drop

The variation of pressure drop $\Delta p$ (across the stenosis) in a time cycle for different values of $A$, $\theta$ and $\delta_P$ with $\beta = 0.95$ is depicted in Figure 2. It is clear that the pressure drop increases as time $t$ increases from $0^\circ$ to $90^\circ$ and then decreases from $90^\circ$ to $270^\circ$ and again it increases from $270^\circ$ to $360^\circ$. The pressure drop is maximum at $90^\circ$ and minimum at $270^\circ$. It is observed that for a given value of $A$, the pressure drop increases with the increase of $\delta_P$ or yield stress $\theta$ when the other parameters held constant. It is found that as the amplitude $A$ increases, the pressure drop increases when $t$ lies between $0^\circ$ and $180^\circ$ and decreases when $t$ lies between $180^\circ$ and $360^\circ$ while $\theta$ and $\delta_P$ are held fixed. The pressure drop increases with the increase of the width of the peripheral layer thickness.

![Figure 2. Variation of pressure drop in a time cycle for different values of $A$, $\theta$, $\delta_P$ and $\beta$](image)

4.2. Plug core radius

The variation of plug core radius with axial distance for different values of $A$ and $\delta_P$ with $\beta = 0.95$, $\alpha_C = 0.5$, $\theta = 0.1$ and $t = 60^\circ$ is shown in Figure 3. It is noted that the plug core radius decreases as $z$ increases from 4 to 5 and it increases as $z$ increases further from 5 to 6. It is observed that for a given value of $\delta_P$, the plug core radius decreases with the increase of $A$ and the same behavior is noted as $\delta_P$ increases for a given value of $A$. It is also noticed that the plug core radius increases with the increase of the yield stress of the fluid in the core region. It is of interest to note that the plot of the single-fluid Casson model is in good agreement with Figure 4 of Chaturani and Ponnalagar Samy [3].

![Figure 3. Variation of plug core radius with axial distance for different values of $A$, $\delta_P$ and $\theta$ with $\beta = 0.95$, $\alpha_C = 0.95$ and $t = 60^\circ$.](image)
4.3. Velocity distribution

The velocity distributions for different values of $A$, $\alpha$, $\alpha_C$ and $\beta$ with $z = 5$, $\theta = \delta_p = 0.1$ and $t = 45^\circ$ are shown in Figure 4. One can notice the plug flow around $r = 0$. It is found that for the given values of $\alpha$, $\alpha_H$ and $\beta$, the velocity increases as $A$ increases. Further, it is observed that for given values of $A$, $\alpha$ and $\alpha_C$, the velocity decreases considerably near $r = 0$ as $\beta$ increases. The velocity decreases with the increase of the yield stress when all the other parameters are kept as constant. For given values of $A$ and $\beta$, the same behavior is noted for increasing values of $\alpha$ and $\alpha_C$, but the decrease is only a slight.

Figure 4. Velocity distribution for different values of $A$, $\alpha$, $\alpha_C$, $\beta$ and $\theta$ with $t = 45^\circ$, $\delta_p = 0.1$ and $z = 5$.

4.4. Wall shear stress

The variation of wall shear stress with axial distance for different values of $\theta$ and $\alpha_N$ with $t = 45^\circ$, $A = 0.5$, $\beta = 0.95$ and $\delta_p = 0.1$ is shown in Figure 5. One can notice that the wall shear stress increases as $z$ increases from 4 to 5 and then it decreases symmetrically as $z$ increases further from 5 to 6. It is found that the wall shear stress increases with the increase of the amplitude of the flow while $\theta$ and $\alpha_H$ are kept as constants. Also, it is noticed that for a given value of $\theta$ and increasing values of $\alpha_N$, the wall shear stress decreases slightly while the other parameters were kept as constants.

Figure 5. Variation of wall shear stress with axial distance for different values of $A$, $\theta$ and $\alpha_N$ with $t = 45^\circ$, $\beta = 0.95$ and $\delta_p = 0.2$. 
4.5. Resistance to flow

Figure 6 depicts the variation of resistance to flow in a time cycle for different values of $A$, $\theta$, $\beta$, $\alpha$ and $\alpha_H$ with $\delta_p = 0.1$. It is clear that the resistance to flow decreases as time $t$ increases from $0^\circ$ to $90^\circ$ and then increases as $t$ increases from $90^\circ$ to $270^\circ$ and again decreases as $t$ increases further from $270^\circ$ to $360^\circ$. The resistance to flow is minimum at $90^\circ$ and maximum at $270^\circ$. It is observed that for fixed values of $\beta$, $\alpha$ and $\alpha_H$ and increasing values of $A$, the resistance to flow decreases when $t$ lies between $0^\circ$ and $180^\circ$ and increases when $t$ lies between $180^\circ$ and $360^\circ$. The resistance to flow increases with increasing values of $\theta$ and $\delta_p$ while all other parameters are held fixed. It is noticed that the resistance to flow decreases as $\beta$ increases when the other parameters held fixed.

Figure 6. Variation of resistance to flow in a time cycle for different values of $A$, $\theta$, $\beta$, $\alpha$, $\alpha_C$ and $\delta_p$.

5. Conclusions

The present mathematical analysis brings out many interesting fluid mechanical phenomena due to the presence of the peripheral layer. It is observed that the pressure drop, plug core radius, wall shear stress and resistance to flow increase as the yield stress $\theta$ or stenosis size $\delta_p$ increases while all other parameters are held constant. It is also found that the velocity increases and the plug core radius decreases as $A$ increases. Thus, the results demonstrate that the present model is capable of predicting the hemodynamic features most interesting to physiologists. Thus, the presence of the peripheral layer helps in the functioning of the diseased arterial system. The extension of the present study to the blood flow through arteries with elastic walls may have more applications in the medical field which would be done in the near future.

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