BULLETIN *of the*
Malaysian Mathematical
Sciences Society
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35***(1) (2012), 215–225*

On Joint Distributions of Order Statistics from *innid* Variables

M. Güngör
Department of Econometrics, University of Inonu, 44280 Malatya, Turkey
mgungor44@gmail.com

Abstract. In this study, the joint distributions of order statistics of *innid* random variables are expressed. Then, some results connecting distributions of order statistics of *innid* random variables to that of order statistics of *iid* random variables are given.

2010 Mathematics Subject Classification: 62G30, 62E15

Keywords and phrases: Order statistics, permanent, joint distribution, *iid* random variable, *innid* random variable.

1. Introduction

Several identities and recurrence relations for probability density function (*pdf*) and distribution function (*df*) of order statistics of independent and identically distributed (*iid*) random variables were established by numerous authors including Arnold *et al.* [1], Balasubramanian and Beg [3], David [13], and Reiss [18]. Furthermore, Arnold *et al.* [1], David [13], Gan and Bain [14], and Khatri [17] obtained the probability function and *df* of order statistics of *iid* random variables from a discrete parent. Corley [11] defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller [15]. Guilbaud [16] expressed the probability of the functions of independent but not necessarily identically distributed (*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West [9]. In addition, Vaughan and Venables [19] derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. Balakrishnan [2], and

Communicated by M. Ataharul Islam.
Received: October 20, 2009; *Revised:* April 7, 2010.

Bapat and Beg [7] obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents. Using multinomial arguments, the *pdf* of $X_{r:n+1}(1 \leq r \leq n+1)$ was obtained by Childs and Balakrishnan [10] by adding another independent random variable to the original n variables $X_1, X_2, ..., X_n$. Also, Balasubramanian *et al.* [6] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. In a paper published in 1991, Beg [8] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer *et al.* [12] derived the expressions for the distribution and density functions by Ryser's method and the distribution of maxima and minima based on permanents. In the first of two papers, Balasubramanian *et al.* [4] obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, ..., X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian *et al.* [5] generalized their previous results [4] to the case of the joint distribution function of several order statistics.

In general, the distribution theory for order statistics is complicated when the random variables are *innid*. In this study, the explicit expressions for the joint *df* and *pdf* of order statistics of *innid* random variables are obtained.

From now on, the subscripts and superscripts are defined in the first place in which they are used and these definitions will be valid unless they are redefined.

If $\mathrm{a}_1, \mathrm{a}_2, ...$ are defined as column vectors, then the matrix obtained by taking m_1 copies of a_1, m_2 copies of a_2,... can be denoted as

$$[\underset{m_1}{\mathrm{a}_1} \quad \underset{m_2}{\mathrm{a}_2} \ldots]$$

and per A denotes the permanent of a square matrix A, which is defined as similar to determinants except that all terms in the expansion have a positive sign.

Let $X_1, X_2, ..., X_n$ be *innid* continuous random variables and $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order statistics obtained by arranging the n X_i 's in increasing order of magnitude.

Let F_i and f_i be *df* and *pdf* of X_i $(i = 1, 2, \ldots, n)$, respectively. Moreover, $X^s_{1:n}, X^s_{2:n}, \ldots, X^s_{n:n}$ are order statistics of *iid* random variables with *df* F^s and *pdf* f^s, respectively, defined by

$$F^s = \frac{1}{n_s} \sum_{i \in s} F_i \tag{1.1}$$

and

$$f^s = \frac{1}{n_s} \sum_{i \in s} f_i \tag{1.2}$$

Here, s is a non-empty subset of the integers $\{1, 2, \ldots, n\}$ with $n_s \geq 1$ elements. $\mathrm{A}[s/.)$ is the matrix obtained from A by taking rows whose indices are in s.

The *df* and *pdf* of $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_d:n}$ $(1 \le r_1 < r_2 < \dots < r_d \le n,\ d = 1, 2, \dots, n)$ will then be given. For notational convenience we write

$$\sum\sum,\ \sum_{m_d,\dots,m_2,m_1}^{n,\dots,m_3,m_2},\ \sum_{t_d,\dots,t_2,t_1}^{n,\dots,m_3,m_2} \text{ and } \sum_{t_d,\dots,t_2,t_1}^{n,\dots,r_3-1,r_2-1}$$

instead of

$$\sum_{\kappa=1}^{n}(-1)^{n-\kappa}\frac{\kappa^n}{n!}\sum_{n_s=\kappa},\ \sum_{m_d=r_d}^{n}\dots\sum_{m_2=r_2}^{m_3}\sum_{m_1=r_1}^{m_2},\ \sum_{t_d=m_d}^{n}\dots\sum_{t_2=m_2}^{m_3}\sum_{t_1=m_1}^{m_2} \text{ and } \sum_{t_d=r_d}^{n}\dots\sum_{t_2=r_2}^{r_3-1}\sum_{t_1=r_1}^{r_2-1}$$

in the expressions below, respectively.

2. Theorems for distribution and probability density function

In this section, the theorems related to *df* and *pdf* of $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_d:n}$ are given. The theorems connect the *df* and *pdf* of order statistics of *innid* random variables to that of order statistics of *iid* random variables using (1.1) and (1.2).

Theorem 2.1.

$$F_{r_1,r_2,\dots,r_d:n}(x_1, x_2, \dots, x_d) = \sum_{m_d,\dots,m_2,m_1}^{n,\dots,m_3,m_2} C \sum_{t_d,\dots,t_2,t_1}^{n,\dots,m_3,m_2} (-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)} \left(\prod_{w=1}^{d}\binom{m_{w+1}-m_w}{t_w-m_w}\right)$$

(2.1)

$$\cdot \sum_{n_s=n-t_d+m_d}(t_d-m_d)! \sum_{n_{s_1},n_{s_2},\dots,n_{s_{d-1}}} \prod_{w=1}^{d} \operatorname{per}[\underset{m_{w+1}-m_{w-1}-t_w+t_{w-1}}{\mathrm{F}(x_w)}][s_w/.)\ ,$$

$$x_1 < x_2 < \dots < x_d,$$

where $\mathrm{F}(x_w) = (F_1(x_w), F_2(x_w), \dots, F_n(x_w))'$ *is column vector,* $x_w \in R$, $C = [\prod_{w=1}^{d+1} (m_w - m_{w-1})!]^{-1}$, $m_0 = 0$, $m_{d+1} = n$, $\sum_{n_{s_1},n_{s_2},\dots,n_{s_{d-1}}}$ *denotes the sum over* $\bigcup_{w=1}^{d-1} s_w$ *for which* $s_\upsilon \bigcap s_\nu = \phi$ *for* $\upsilon \ne \nu$, $s = \bigcup_{w=1}^{d} s_w$, $n_{s_w} = m_{w+1} - m_{w-1} - t_w + t_{w-1}$ *and* $t_0 = m_1$.

Proof. It can be written

$$F_{r_1,r_2,\dots,r_d:n}(x_1, x_2, \dots, x_d) = P\{X_{r_1:n} \le x_1, X_{r_2:n} \le x_2, \dots, X_{r_d:n} \le x_d\}\ . \tag{2.2}$$

(2.2) can be expressed as

$$F_{r_1,r_2,\dots,r_d:n}(x_1, x_2, \dots, x_d) = \sum_{m_d,\dots,m_2,m_1}^{n,\dots,m_3,m_2} C \operatorname{per} \mathrm{A}\ , \tag{2.3}$$

where

$$\mathrm{A} = \left[\underset{m_1}{\mathrm{F}(x_1)}\ \ \underset{m_2-m_1}{\mathrm{F}(x_2)-\mathrm{F}(x_1)}\ \ \dots\ \ \underset{n-m_d}{1-\mathrm{F}(x_d)}\right]$$

is matrix.

$\mathrm{F}(x_w) - \mathrm{F}(x_{w-1}) = (F_1(x_w) - F_1(x_{w-1}), F_2(x_w) - F_2(x_{w-1}), \dots, F_n(x_w) - F_n(x_{w-1}))'$ $(w = 1, 2, \dots, d+1)$, $F_i(x_0) = 0$ and $F_i(x_{d+1}) = 1$.

Using properties of permanent, we can write

$$
\begin{aligned}
\operatorname{per} \mathrm{A} &= \operatorname{per}[\underset{m_1}{\mathrm{F}(x_1)}\ \ \underset{m_2-m_1}{\mathrm{F}(x_2)-\mathrm{F}(x_1)}\ \ \underset{m_3-m_2}{\mathrm{F}(x_3)-\mathrm{F}(x_2)}\ \ \ldots\ \ \underset{m_d-m_{d-1}}{\mathrm{F}(x_d)-\mathrm{F}(x_{d-1})}\ \ \underset{n-m_d}{1-\mathrm{F}(x_d)}] \\
&= \sum_{t_d=0}^{n-m_d} (-1)^{n-m_d-t_d} \binom{n-m_d}{t_d} \ldots \sum_{t_2=0}^{m_3-m_2} (-1)^{m_3-m_2-t_2} \binom{m_3-m_2}{t_2} \\
&\quad \cdot \sum_{t_1=0}^{m_2-m_1} (-1)^{m_2-m_1-t_1} \binom{m_2-m_1}{t_1} \\
&\quad \cdot \operatorname{per}[\underset{m_2-t_1}{\mathrm{F}(x_1)}\ \ \underset{m_3-m_2-t_2+t_1}{\mathrm{F}(x_2)}\ \ \ldots\ \ \underset{t_d}{1}\ \ \underset{n-m_d-t_d+t_{d-1}}{\mathrm{F}(x_d)}] \\
&= \sum_{t_d=0}^{n-m_d} \ldots \sum_{t_2=0}^{m_3-m_2} \sum_{t_1=0}^{m_2-m_1} (-1)^{n-m_1-\sum_{w=1}^{d} t_w} \left(\prod_{w=1}^{d} \binom{m_{w+1}-m_w}{t_w} \right) \sum_{n_s=n-t_d} t_d! \\
&\quad \cdot \operatorname{per}[\underset{m_2-t_1}{\mathrm{F}(x_1)}\ \ \underset{m_3-m_2-t_2+t_1}{\mathrm{F}(x_2)}\ \ \ldots\ \ \underset{n-m_d-t_d+t_{d-1}}{\mathrm{F}(x_d)}\ ][s/.) \\
&= \sum_{t_d=m_d}^{n} \ldots \sum_{t_2=m_2}^{m_3} \sum_{t_1=m_1}^{m_2} (-1)^{\sum_{w=1}^{d} (m_{w+1}-t_w)} \left(\prod_{w=1}^{d} \binom{m_{w+1}-m_w}{t_w-m_w} \right) \\
&\quad \cdot \sum_{n_s=n-t_d+m_d} (t_d-m_d)!\ \operatorname{per}[\ \underset{m_2-t_1+m_1}{\mathrm{F}(x_1)}\ \ \underset{m_3-m_1-t_2+t_1}{\mathrm{F}(x_2)}\ \ \ldots\ \ \underset{n-m_d-t_d+t_{d-1}}{\mathrm{F}(x_d)}\ ][s/.) \\
&= \sum_{t_d=m_d}^{n} \ldots \sum_{t_2=m_2}^{m_3} \sum_{t_1=m_1}^{m_2} (-1)^{\sum_{w=1}^{d} (m_{w+1}-t_w)} \left(\prod_{w=1}^{d} \binom{m_{w+1}-m_w}{t_w-m_w} \right) \\
&\quad \cdot \sum_{n_s=n-t_d+m_d} (t_d-m_d)! \sum_{n_{s_1}, n_{s_2}, \ldots, n_{s_{d-1}}} \\
&\quad \cdot \operatorname{per}[\ \underset{m_2-t_1+m_1}{\mathrm{F}(x_1)}\ ][s_1/.)\ \operatorname{per}[\ \underset{m_3-m_1-t_2+t_1}{\mathrm{F}(x_2)}\ ][s_2/.)\ \ldots\ \operatorname{per}[\ \underset{n-m_{d-1}-t_d+t_{d-1}}{\mathrm{F}(x_d)}\ ][s_d/.) \\
&= \sum_{t_d, \ldots, t_2, t_1}^{n, \ldots, m_3, m_2} (-1)^{\sum_{w=1}^{d} (m_{w+1}-t_w)} \left(\prod_{w=1}^{d} \binom{m_{w+1}-m_w}{t_w-m_w} \right) \sum_{n_s=n-t_d+m_d} (t_d-m_d)!
\end{aligned}
$$

(2.4)

$$
\cdot \sum_{n_{s_1}, n_{s_2}, \ldots, n_{s_{d-1}}} \prod_{w=1}^{d} \operatorname{per}\,[\ \underset{m_{w+1}-m_{w-1}-t_w+t_{w-1}}{\mathrm{F}(x_w)}\ ][s_w/.),
$$

where $1 = (\mathit{1,1,\ldots,1})'$. Using (2.4) in (2.3), (2.1) is obtained. ∎

Theorem 2.2.

$$
\begin{aligned}
&F_{r_1, r_2, \ldots, r_d : n}(x_1, x_2, \ldots, x_d) \\
&= \sum_{m_d, \ldots, m_2, m_1}^{n, \ldots, m_3, m_2} C \sum_{t_d, \ldots, t_2, t_1}^{n, \ldots, m_3, m_2} (-1)^{\sum_{w=1}^{d} (m_{w+1}-t_w)} \left(\prod_{w=1}^{d} \binom{m_{w+1}-m_w}{t_w-m_w} \right)
\end{aligned}
$$

(2.5)

$$
\cdot \sum_{n_s=n-t_d+m_d} (t_d-m_d)! \sum_{n_{s_1}, n_{s_2}, \ldots, n_{s_{d-1}}} \prod_{w=1}^{d} n_{s_w}! \prod_{l=1}^{n_{s_w}} F_{s_w^l}(x_w),
$$

where $s_w = \{s_w^1, s_w^2, ..., s_w^{n_{s_w}}\}$.

Proof. Omitted. ∎

Theorem 2.3.

$$F_{r_1,r_2,...,r_d:n}(x_1, x_2, ..., x_d)$$

$$= \sum\sum \sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} n!\ C$$

(2.6)

$$\cdot \sum_{t_d,...,t_2,t_1}^{n,...,m_3,m_2} (-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)} \prod_{w=1}^{d} \begin{pmatrix} m_{w+1} - m_w \\ t_w - m_w \end{pmatrix} [F^s(x_w)]^{m_{w+1}-m_{w-1}-t_w+t_{w-1}}.$$

Proof. (2.2) can be expressed as

$$F_{r_1,r_2,...,r_d:n}(x_1, x_2, ..., x_d)$$

$$= \sum\sum P\{X^s_{r_1:n} \le x_1,\ X^s_{r_2:n} \le x_2,\ ...\ , X^s_{r_d:n} \le x_d\} \tag{2.7}$$

is immediate from (2.1) and (2.7). Thus, (2.6) is obtained. ∎

Theorem 2.4.

$$f_{r_1,r_2,...,r_d:n}(x_1, x_2, ..., x_d)$$

$$= D \sum_{t_d,...,t_2,t_1}^{n,...,r_3-1,r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \left(\prod_{w=1}^{d} \begin{pmatrix} r_{w+1} - r_w - 1 \\ t_w - r_w \end{pmatrix}\right)$$

$$\cdot \sum_{n_s=n+r_d-t_d} (t_d - r_d)! \sum_{n_{s_1},n_{s_2},...,n_{s_{d-1}}}$$

$$\cdot \prod_{w=1}^{d} \sum_{n_{\varsigma_w}} \mathrm{per}[\underset{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}{\mathrm{F}(x_w)}]\ [\varsigma_w/.)\ \ \mathrm{per}[\underset{1}{\mathrm{f}(x_w)}][\varsigma'_w/.), \tag{2.8}$$

where $\mathrm{f}(x_w) = (f_1(x_w), f_2(x_w), ..., f_n(x_w))'$, $D = [\prod_{w=1}^{d+1}(r_w - r_{w-1} - 1)!]^{-1}$, $r_0 = 0$, $r_{d+1} = n + 1$, $s = \bigcup_{w=1}^{d} s_w$, $s_\upsilon \bigcap s_\nu = \phi$ *for* $\upsilon \ne \nu$, $s_w = \varsigma_w \bigcup \varsigma'_w$, $\varsigma_w \bigcap \varsigma'_w = \phi$, $n_{s_w} = r_{w+1} - r_{w-1} - t_w + t_{w-1}$, $t_0 = r_1 - 1$, $n_{\varsigma w} = r_{w+1} - r_{w-1} - 1 - t_w + t_{w-1}$ *and* $n_{\varsigma'_w} = 1$.

Proof. Consider

(2.9)

$$P\{x_1 < X_{r_1:n} \le x_1 + \delta x_1, x_2 < X_{r_2:n} \le x_2 + \delta x_2,\ ...,\ x_d < X_{r_d:n} \le x_d + \delta x_d\}.$$

Dividing (2.9) by $\prod_{w=1}^{d} \delta x_w$ and then letting $\delta x_1, \delta x_2, ..., \delta x_d$ tend to zero, we obtain

$$f_{r_1,r_2,...,r_d:n}(x_1, x_2, ..., x_d) = D\,\mathrm{per}\,\mathrm{B}, \tag{2.10}$$

where

$$\mathrm{B} = [\underset{r_1-1}{\mathrm{F}(x_1)}\ \ \underset{1}{\mathrm{f}(x_1)}\ \ \underset{r_2-r_1-1}{\mathrm{F}(x_2) - \mathrm{F}(x_1)}\ \ \underset{1}{\mathrm{f}(x_2)}\ \ ...\ \ \underset{1}{\mathrm{f}(x_d)}\ \ \underset{n-r_d}{1 - \mathrm{F}(x_d)}]$$

is matrix. Using properties of permanent, we can write

$$\mathrm{per}\,\mathrm{B} = \mathrm{per}[\underset{r_1-1}{\mathrm{F}(x_1)}\ \ \underset{1}{\mathrm{f}(x_1)}\ \ \underset{r_2-r_1-1}{\mathrm{F}(x_2) - \mathrm{F}(x_1)}\ \ \underset{1}{\mathrm{f}(x_2)}\ \ ...\ \ \underset{1}{\mathrm{f}(x_d)}\ \ \underset{n-r_d}{1 - \mathrm{F}(x_d)}]$$

$$
\begin{aligned}
&= \sum_{t_d=0}^{n-r_d} (-1)^{n-r_d-t_d} \binom{n-r_d}{t_d} \dots \sum_{t_2=0}^{r_3-r_2-1} (-1)^{r_3-r_2-1-t_2} \binom{r_3-r_2-1}{t_2} \\
&\quad \cdot \sum_{t_1=0}^{r_2-r_1-1} (-1)^{r_2-r_1-1-t_1} \binom{r_2-r_1-1}{t_1} \\
&\quad \cdot \operatorname{per}[\, \underset{r_2-2-t_1}{\mathrm{F}(x_1)} \quad \underset{1}{\mathrm{f}(x_1)} \quad \underset{r_3-r_2-1-t_2+t_1}{\mathrm{F}(x_2)} \quad \underset{1}{\mathrm{f}(x_2)} \ \dots \ \underset{1}{\mathrm{f}(x_d)} \quad \underset{t_d}{1} \quad \underset{n-r_d-t_d+t_{d-1}}{\mathrm{F}(x_d)} \,] \\
&= \sum_{t_d=0}^{n-r_d} \dots \sum_{t_2=0}^{r_3-r_2-1} \sum_{t_1=0}^{r_2-r_1-1} (-1)^{n+1-r_1-d-\sum_{w=1}^{d} t_w} \left(\prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w} \right) \\
&\quad \cdot \sum_{n_s=n-t_d} t_d! \ \operatorname{per}[\, \underset{r_2-2-t_1}{\mathrm{F}(x_1)} \quad \underset{r_3-r_2-1-t_2+t_1}{\mathrm{F}(x_2)} \ \dots \ \underset{n-r_d-t_d+t_{d-1}}{\mathrm{F}(x_d)} \quad \underset{1}{\mathrm{f}(x_1)} \quad \underset{1}{\mathrm{f}(x_2)} \\
&\quad \dots \ \underset{1}{\mathrm{f}(x_d)}][s/.) \\
&= \sum_{t_d=r_d}^{n} \dots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^{d} (r_{w+1}-t_w)} \left(\prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w-r_w} \right) \\
&\quad \cdot \sum_{n_s=n+r_d-t_d} (t_d-r_d)! \ \operatorname{per}[\, \underset{r_2+r_1-2-t_1}{\mathrm{F}(x_1)} \quad \underset{r_3-r_1-1-t_2+t_1}{\mathrm{F}(x_2)} \ \dots \ \underset{n-r_{d-1}-t_d+t_{d-1}}{\mathrm{F}(x_d)} \\
&\quad \underset{1}{\mathrm{f}(x_1)} \quad \underset{1}{\mathrm{f}(x_2)} \ \dots \ \underset{1}{\mathrm{f}(x_d)}][s/.) \\
&= \sum_{t_d=r_d}^{n} \dots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^{d} (r_{w+1}-t_w)} \left(\prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w-r_w} \right) \\
&\quad \cdot \sum_{n_s=n+r_d-t_d} (t_d-r_d)! \\
&\quad \cdot \sum_{n_{s_1}, n_{s_2}, \dots, n_{s_{d-1}}} \operatorname{per}[\, \underset{r_2+r_1-2-t_1}{\mathrm{F}(x_1)} \quad \underset{1}{\mathrm{f}(x_1)}][s_1/.) \ \operatorname{per}[\, \underset{r_3-r_1-1-t_2+t_1}{\mathrm{F}(x_2)} \quad \underset{1}{\mathrm{f}(x_2)}][s_2/.) \dots \\
&\quad \cdot \operatorname{per}[\, \underset{n-r_{d-1}-t_d+t_{d-1}}{\mathrm{F}(x_d)} \quad \underset{1}{\mathrm{f}(x_d)}][s_d/.) \\
&= \sum_{t_d=r_d}^{n} \dots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^{d} (r_{w+1}-t_w)} \left(\prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w-r_w} \right) \\
&\quad \cdot \sum_{n_s=n+r_d-t_d} (t_d-r_d)! \\
&\quad \cdot \sum_{n_{s_1}, n_{s_2}, \dots, n_{s_{d-1}}} \prod_{w=1}^{d} \operatorname{per}[\, \underset{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}{\mathrm{F}(x_w)} \quad \underset{1}{\mathrm{f}(x_w)}][s_w/.) \\
&= \sum_{t_d=r_d}^{n} \dots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^{d} (r_{w+1}-t_w)} \left(\prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w-r_w} \right) \\
&\quad \cdot \sum_{n_s=n+r_d-t_d} (t_d-r_d)!
\end{aligned}
$$

(2.11)

$$\cdot \sum_{n_{s_1}, n_{s_2}, \dots, n_{s_{d-1}}} \prod_{w=1}^{d} \sum_{n_{\varsigma_w}} \text{per}[\underset{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}{\text{F}(x_w)}]\ [\varsigma_w/.)\ \ \text{per}[\text{f}(\underset{1}{x_w})][\varsigma'_w/.).$$

Using (2.11) in (2.10), (2.8) is obtained. ∎

Theorem 2.5.

$$f_{r_1,r_2,\dots,r_d:n}(x_1, x_2, \dots, x_d)$$

$$= D \sum_{t_d,\dots,t_2,t_1}^{n,\dots,r_3-1,r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \left(\prod_{w=1}^{d} \left(\begin{array}{c} r_{w+1}-r_w-1 \\ t_w - r_w \end{array} \right) \right)$$

(2.12)

$$\cdot \sum_{n_s=n+r_d-t_d} (t_d - r_d)! \sum_{n_{s_1}, n_{s_2}, \dots, n_{s_{d-1}}} \prod_{w=1}^{d} \sum_{n_{\varsigma_w}} n_{\varsigma_w}! \left(\prod_{l=1}^{n_{\varsigma_w}} F_{\varsigma_w^l}(x_w) \right) f_{\varsigma'^w_w}(x_w),$$

where $\varsigma_w = \{\varsigma_w^1, \varsigma_w^2, \dots, \varsigma_w^{n_{\varsigma_w}}\}$ *and* $\varsigma'_w = \{\varsigma'^w_w\}$.

Proof. Omitted. ∎

Theorem 2.6.

$$f_{r_1,r_2,\dots,r_d:n}(x_1, x_2, \dots, x_d)$$

$$= \sum\sum n!\ D \sum_{t_d,\dots,t_2,t_1}^{n,\dots,r_3-1,r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \prod_{w=1}^{d} \left(\begin{array}{c} r_{w+1}-r_w-1 \\ t_w - r_w \end{array} \right)$$

(2.13)

$$\cdot [F^s(x_w)]^{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}} f^s(x_w).$$

Proof. (2.9) can be expressed as

$$\sum\sum P\{x_1 < X^s_{r_1:n} \le x_1 + \delta x_1, x_2 < X^s_{r_2:n} \le x_2 + \delta x_2, \dots, x_d < X^s_{r_d:n} \le x_d + \delta x_d\}. \tag{2.14}$$

Dividing (2.14) by $\prod_{w=1}^{d} \delta x_w$ and then letting $\delta x_1, \delta x_2, \dots, \delta x_d$ tend to zero, (2.13) is obtained. ∎

3. Results for distribution and probability density function

In this section, the results related to *df* and *pdf* of $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_d:n}$ are given. The results connect the *df* and *pdf* of order statistics of *innid* random variables to that of order statistics of *iid* random variables.

Result 3.1.

$$F_{r:n}(x) = \sum_{m=r}^{n} \frac{1}{m!(n-m)!} \sum_{t=m}^{n} (-1)^{n-t} \left(\begin{array}{c} n-m \\ t-m \end{array} \right) \sum_{n_s=n-t+m} (t-m)!\ \text{per}[\underset{n-t+m}{\text{F}(x)}][s/.)$$

$$= \sum_{m=r}^{n} \frac{1}{m!(n-m)!} \sum_{t=m}^{n} (-1)^{n-t} \left(\begin{array}{c} n-m \\ t-m \end{array} \right) \sum_{n_s=n-t+m} (t-m)!\ (n-t+m)!$$

$$\cdot \prod_{l=1}^{n-t+m} F_{s^l}(x)$$

(3.1)

$$= \sum \sum \sum_{m=r}^{n} \binom{n}{m} \sum_{t=m}^{n} (-1)^{n-t} \binom{n-m}{t-m} [F^s(x)]^{n-t+m}.$$

Proof. If $d = 1$ in (2.1), (2.5) and (2.6), (3.1) is obtained. ∎

Result 3.2.

$$F_{1:n}(x) = 1 - \frac{1}{n!} \sum_{t=0}^{n} (-1)^{n-t} \binom{n}{t} \sum_{n_s=n-t} t! \ \operatorname{per}[\underset{n-t}{\mathrm{F}(x)}][s/.)$$

$$= 1 - \frac{1}{n!} \sum_{t=0}^{n} (-1)^{n-t} \binom{n}{t} \sum_{n_s=n-t} t! \ (n-t)! \prod_{l=1}^{n-t} F_{s^l}(x)$$

$$= \sum \sum \{1 - \sum_{t=0}^{n} (-1)^{n-t} \binom{n}{t} [F^s(x)]^{n-t}\}. \tag{3.2}$$

Proof. If $r = 1$ in (3.1), (3.2) is obtained. ∎

Result 3.3.

$$F_{n:n}(x) = \frac{1}{n!} \operatorname{per}[\underset{n}{\mathrm{F}(x)}] = \prod_{l=1}^{n} F_l(x) = \sum \sum [F^s(x)]^n. \tag{3.3}$$

Proof. In (3.1), if $r = n$, (3.3) is obtained. ∎

Result 3.4.

$$f_{r:n}(x)$$

$$= \frac{1}{(r-1)!(n-r)!} \ \operatorname{per}[\underset{r-1}{\mathrm{F}(x)} \ \ \underset{1}{\mathrm{f}(x)} \ \ \underset{n-r}{1-\mathrm{F}(x)}]$$

$$= \frac{1}{(r-1)!(n-r)!} \ \sum_{t=r}^{n} (-1)^{n-t} \binom{n-r}{t-r} \sum_{n_s=n+r-t} (t-r)! \ \operatorname{per}[\underset{n+r-1-t}{\mathrm{F}(x)} \ \ \underset{1}{\mathrm{f}(x)}][s/.)$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_{t=r}^{n} (-1)^{n-t} \binom{n-r}{t-r}$$

$$\cdot \sum_{n_s=n+r-t} (t-r)! \sum_{n_\varsigma=n+r-1-t} \operatorname{per}[\underset{n+r-1-t}{\mathrm{F}(x)}] [\varsigma/.) \ \operatorname{per}[\underset{1}{\mathrm{f}(x)}] \ [\varsigma'/.)$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_{t=r}^{n} (-1)^{n-t} \binom{n-r}{t-r} \sum_{n_s=n+r-t} (t-r)!$$

$$\cdot \sum_{n_\varsigma=n+r-1-t} (n+r-1-t)! \left(\prod_{l=1}^{n+r-1-t} F_{\varsigma^l}(x) \right) f_{\varsigma'}(x)$$

(3.4)

$$= \sum\sum r \begin{pmatrix} n \\ r \end{pmatrix} \sum_{t=r}^{n} (-1)^{n-t} \begin{pmatrix} n-r \\ t-r \end{pmatrix} [F^s(x)]^{n+r-1-t} f^s(x).$$

Proof. If $d = 1$ in (2.8), (2.12) and (2.13), (3.4) is obtained. ∎

Result 3.5.

$$\begin{aligned}
&f_{1:n}(x) \\
&= \frac{1}{(n-1)!} \operatorname{per}[\underset{1}{\mathrm{f}(x)} \quad \underset{n-1}{1-\mathrm{F}(x)}] \\
&= \frac{1}{(n-1)!} \sum_{t=1}^{n} (-1)^{n-t} \begin{pmatrix} n-1 \\ t-1 \end{pmatrix} \sum_{n_s=n+1-t} (t-1)! \operatorname{per}[\underset{n-t}{\mathrm{F}(x)} \quad \underset{1}{\mathrm{f}(x)}][s/.) \\
&= \frac{1}{(n-1)!} \sum_{t=1}^{n} (-1)^{n-t} \begin{pmatrix} n-1 \\ t-1 \end{pmatrix} \sum_{n_s=n+1-t} (t-1)! \\
&\quad \cdot \sum_{n_\varsigma=n-t} \operatorname{per}[\underset{n-t}{\mathrm{F}(x)}]\ [\varsigma/.)\ \operatorname{per}[\underset{1}{\mathrm{f}(x)}]\ [\varsigma'/.) \\
&= \frac{1}{(n-1)!} \sum_{t=1}^{n} (-1)^{n-t} \begin{pmatrix} n-1 \\ t-1 \end{pmatrix} \sum_{n_s=n-t+1} (t-1)! \\
&\quad \cdot \sum_{n_\varsigma=n-t} (n-t)! \left(\prod_{l=1}^{n-t} F_{\varsigma^l}(x) \right) f_{\varsigma'}(x)
\end{aligned}$$

(3.5) $$= \sum\sum n \sum_{t=1}^{n} (-1)^{n-t} \begin{pmatrix} n-1 \\ t-1 \end{pmatrix} [F^s(x)]^{n-t} f^s(x)\,.$$

Proof. If $r = 1$ in (3.4), (3.5) is obtained. ∎

Result 3.6.

$$\begin{aligned}
f_{n:n}(x) &= \frac{1}{(n-1)!} \operatorname{per}[\underset{n-1}{\mathrm{F}(x)} \quad \underset{1}{\mathrm{f}(x)}] \\
&= \frac{1}{(n-1)!} \sum_{n_\varsigma=n-1} \operatorname{per}[\underset{n-1}{\mathrm{F}(x)}]\ [\varsigma/.)\ \operatorname{per}[\underset{1}{\mathrm{f}(x)}]\ [\varsigma'/.) \\
&= \sum_{n_\varsigma=n-1} \left(\prod_{l=1}^{n-1} F_{\varsigma^l}(x) \right) f_{\varsigma'}(x)
\end{aligned}$$

(3.6) $$= \sum\sum n [F^s(x)]^{n-1} f^s(x).$$

Proof. If $r = n$ in (3.4), (3.6) is obtained. ∎

Result 3.7.

$$\begin{aligned}
&f_{1,n:n}(x_1, x_2) \\
&= \frac{1}{(n-2)!} \operatorname{per}[\underset{1}{\mathrm{f}(x_1)} \quad \underset{n-2}{\mathrm{F}(x_2) - \mathrm{F}(x_1)} \quad \underset{1}{\mathrm{f}(x_2)}]
\end{aligned}$$

$$= \frac{1}{(n-2)!}\sum_{t=1}^{n-1}(-1)^{n-1-t}\left(\begin{array}{c} n-2 \\ t-1 \end{array}\right)\text{per}[\underset{n-1-t}{\text{F}(x_1)}\ \ \underset{t-1}{\text{F}(x_2)}\ \ \underset{1}{\text{f}(x_1)}\ \ \underset{1}{\text{f}(x_2)}]$$

$$= \frac{1}{(n-2)!}\sum_{t=1}^{n-1}(-1)^{n-1-t}\left(\begin{array}{c} n-2 \\ t-1 \end{array}\right)\sum_{n_{s_1}=n-t}\text{per}[\underset{n-1-t}{\text{F}(x_1)}\ \ \underset{1}{\text{f}(x_1)}]\ [s_1/.)$$

$$\cdot\ \text{per}[\underset{t-1}{\text{F}(x_2)}\ \ \underset{1}{\text{f}(x_2)}]\ [s_2/.)$$

$$= \frac{1}{(n-2)!}\sum_{t=1}^{n-1}(-1)^{n-1-t}\left(\begin{array}{c} n-2 \\ t-1 \end{array}\right)\sum_{n_{s_1}=n-t}\prod_{w=1}^{2}$$

$$\cdot\sum_{n_{\varsigma_w}}\text{per}[\underset{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}{\text{F}(x_w)}]\ [\varsigma_w/.)\ \ \text{per}[\underset{1}{\text{f}(x_w)}][\varsigma'{}_w/.)$$

$$= \frac{1}{(n-2)!}\sum_{t=1}^{n-1}(-1)^{n-1-t}\left(\begin{array}{c} n-2 \\ t-1 \end{array}\right)\sum_{n_{s_1}=n-t}\prod_{w=1}^{2}\sum_{n_{\varsigma_w}} n_{\varsigma_w}!$$

$$\cdot\left(\prod_{l=1}^{n_{\varsigma_w}} F_{\varsigma_w^l}(x_w)\right) f_{\varsigma'{}_w^w}(x_w)$$

$$= \sum\sum n(n-1)\sum_{t=1}^{n-1}(-1)^{n-1-t}\left(\begin{array}{c} n-2 \\ t-1 \end{array}\right)\ [F^s(x_1)]^{n-1-t}[F^s(x_2)]^{t-1}$$

$$\cdot\ f^s(x_1)f^s(x_2), x_1 < x_2. \tag{3.7}$$

Proof. In (2.8), (2.12) and (2.13), if $d = 2$ and $r_1 = 1$, $r_2 = n$, (3.7) is obtained. ∎

Result 3.8.

$$f_{1,2,\ldots,k:n}(x_1, x_2, \ldots, x_k) = \frac{1}{(n-k)!}\text{per}[\underset{1}{\text{f}(x_1)}\ \ \underset{1}{\text{f}(x_2)}\ \ \ldots\ \ \underset{1}{\text{f}(x_k)}\ \ \underset{n-k}{\text{1-F}(x_k)}]$$

$$= \frac{1}{(n-k)!}\sum_{t=k}^{n}(-1)^{n-t}\left(\begin{array}{c} n-k \\ t-k \end{array}\right)\sum_{n_s=n-t+k}(t-k)!$$

$$\cdot\ \text{per}[\underset{n-t}{\text{F}(x_k)}\ \ \underset{1}{\text{f}(x_1)}\ \ \underset{1}{\text{f}(x_2)}\ \ \ldots\ \ \underset{1}{\text{f}(x_k)}][s/.)$$

$$= \frac{1}{(n-k)!}\sum_{t=k}^{n}(-1)^{n-t}\left(\begin{array}{c} n-k \\ t-k \end{array}\right)\sum_{n_s=n-t+k}(t-k)!\sum_{n_{s_1},n_{s_2},\ldots,n_{s_{k-1}}}$$

$$\cdot\prod_{w=1}^{k-1}\text{per}[\underset{1}{\text{f}(x_w)}][s_w/.)\,\text{per}[\underset{n-t}{\text{F}(x_k)}\ \ \underset{1}{\text{f}(x_k)}]\ [s_k/.)$$

$$= \frac{1}{(n-k)!}\sum_{t=k}^{n}(-1)^{n-t}\left(\begin{array}{c} n-k \\ t-k \end{array}\right)\sum_{n_s=n-t+k}(t-k)!\sum_{n_{s_1},n_{s_2},\ldots,n_{s_{k-1}}}$$

$$\cdot\prod_{w=1}^{k}\sum_{n_{\varsigma_w}}\text{per}[\underset{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}{\text{F}(x_w)}]\ [\varsigma_w/.)\ \ \text{per}[\underset{1}{\text{f}(x_w)}][\varsigma'_w/.)$$

$$
\begin{aligned}
&= \frac{1}{(n-k)!} \sum_{t=k}^{n} (-1)^{n-t} \binom{n-k}{t-k} \\
&\quad \cdot \sum_{n_s = n-t+k} (t-k)! \sum_{n_{s_1}, n_{s_2}, \ldots, n_{s_{k-1}}} \prod_{w=1}^{k} \sum_{n_{\varsigma_w}} n_{\varsigma_w}! \left(\prod_{l=1}^{n_{\varsigma_w}} F_{\varsigma_w^l}(x_w) \right) f_{\varsigma'^w_w}(x_w)
\end{aligned}
$$

$$
(3.8) \qquad = \sum \sum \frac{n!}{(n-k)!} \sum_{t=k}^{n} (-1)^{n-t} \binom{n-k}{t-k} [F^s(x_k)]^{n-t} \prod_{w=1}^{k} f^s(x_w).
$$

Proof. If $d = k$ and $r_1 = 1$, $r_2 = 2$, $\ldots$, $r_k = k$ in (2.8), (2.12) and (2.13), (3.8) is obtained. ∎

References

[1] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, *A First Course in Order Statistics*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, Wiley, New York, 1992.

[2] N. Balakrishnan, Permanents, order statistics, outliers, and robustness, *Rev. Mat. Complut.* **20** (2007), no. 1, 7–107.

[3] K. Balasubramanian and M. I. Beg, On special linear identities for order statistics, *Statistics* **37** (2003), no. 4, 335–339.

[4] K. Balasubramanian, M. I. Beg and R. B. Bapat, On families of distributions closed under extrema, *Sankhyā Ser. A* **53** (1991), no. 3, 375–388.

[5] K. Balasubramanian, M. I. Beg and R. B. Bapat, An identity for the joint distribution of order statistics and its applications, *J. Statist. Plann. Inference* **55** (1996), no. 1, 13–21.

[6] K. Balasubramanian, N. Balakrishnan and H. J. Malik, Identities for order statistics from non-independent non-identical variables, *Sankhyā Ser. B* **56** (1994), no. 1, 67–75.

[7] R. B. Bapat and M. I. Beg, Order statistics for nonidentically distributed variables and permanents, *Sankhyā Ser. A* **51** (1989), no. 1, 79–93.

[8] M. I. Beg, Recurrence relations and identities for product moments of order statistics corresponding to non-identically distributed variables, *Sankhyā Ser. A* **53** (1991), no. 3, 365–374.

[9] G. Cao and M. West, Computing distributions of order statistics, *Comm. Statist. Theory Methods* **26** (1997), no. 3, 755–764.

[10] A. Childs and N. Balakrishnan, Relations for order statistics from non-identical logistic random variables and assessment of the effect of multiple outliers on the bias of linear estimators, *J. Statist. Plann. Inference* **136** (2006), no. 7, 2227–2253.

[11] H. W. Corley, Multivariate order statistics, *Comm. Statist. A—Theory Methods* **13** (1984), no. 10, 1299–1304.

[12] E. Cramer, K. Herle and N. Balakrishnan, Permanent expansions and distributions of order statistics in the INID case, *Comm. Statist. Theory Methods* **38** (2009), no. 11–12, 2078–2088.

[13] H. A. David, *Order Statistics*, second edition, Wiley, New York, 1981.

[14] G. Gan and L. J. Bain, Distribution of order statistics for discrete parents with applications to censored sampling, *J. Statist. Plann. Inference* **44** (1995), no. 1, 37–46.

[15] C. M. Goldie and R. A. Maller, Generalized densities of order statistics, *Statist. Neerlandica* **53** (1999), no. 2, 222–246.

[16] O. Guilbaud, Functions of non-iid random vectors expressed as functions of iid random vectors, *Scand. J. Statist.* **9** (1982), no. 4, 229–233.

[17] C. G. Khatri, Distributions of order statistics for discrete case, *Ann. Inst. Statist. Math.* **14** (1962), 167–171.

[18] R.-D. Reiss, *Approximate Distributions of Order Statistics*, Springer Series in Statistics, Springer, New York, 1989.

[19] R. J. Vaughan and W. N. Venables, Permanent expressions for order statistic densities, *J. Roy. Statist. Soc. Ser. B* **34** (1972), 308–310.