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On a Graph Associated to Groups

[1]MOHAMMAD MASHKOURI AND [2]BIJAN TAERI
Department of Mathematical Sciences,
Isfahan University of Technology,
Isfahan 84156–83111, Iran
[1]mohammad@cc.iut.ac.ir, [2]b.taeri@cc.iut.ac.ir

Abstract. Let G be a group such that the subgroup $T^2(G) = \{x \in G \mid (gx)^2 = (xg)^2,\ \text{for all}\ \ g \in G\}$ is a proper subgroup. We define a graph Γ with the vertex set $G \setminus T^2(G)$ in which two vertices x, y are joined by an edge if $(xy)^2 \neq (yx)^2$. In this note we study this graph and find a characterization of some dihedral groups in terms of this graph.

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1. Introduction

Paul Erdös introduced the following graph: Let G be a group and consider a graph Γ whose vertex set is G and join two distinct elements if they do not commute. In response to a question of Erdös, on the cardinality of complete subgraphs of Γ, B. H. Neumann [7] proved that a group G is center-by-finite if and only if Γ has no infinite complete subgraph. In [2] this graph is called the non-commuting graph and, to avoid isolated vertices, the vertex set of this graph is taken as the elements of the group outside its center. There are many different ways of associating a graph to a group, for instance see [3, 4, 8]. The study of groups, using the properties of graphs, has been the object of such papers.

Let us introduce some necessary notation and definitions. Let $w(x, y)$ be a word and $\mathcal{W}$ be the variety of groups defined by the law $w(x, y) = 1$. Suppose that G is a group which is not in $\mathcal{W}$. Let $T(G) = \{x \in G \mid w(x, g) = w(g, x) = 1,\ \text{for all}\ g \in G\}$. We define a graph Γ as follows. The vertex set $V = V(\Gamma)$ of Γ is the set $G \setminus T(G)$ and two elements $x, y \in V$ are joined by an edge if $w(x, y) \neq 1$. Note that to avoid isolated vertices, the vertex set is taken as the elements of G outside $T(G)$. If $w(x, y) = [x, y] = x^{-1}y^{-1}xy$ the graph $\Gamma(G)$ is the non-commuting graph investigated in [2]. Let n be a positive integer. One can investigate the graph of a group using the word $u(x, y) = [x, y^n]$. The variety of groups defined by the word

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$[x, y^n]$ coincides with the variety of groups defined by the word $(xy)^n(yx)^{-n}$. So one can also investigate the graph of a group using the word $w(x, y) = (xy)^n(yx)^{-n}$. In this paper we consider a special word $w(x, y) = (xy)^2(yx)^{-2}$. This word has been considered in [1] for investigating the question posed by Erdös mentioned above.

Let g be an element of a group G. Recall that $C(g)$, the centralizer of g in G, is the subgroup of all elements of G commuting with g, i.e. $C(g) = \{x \in G \mid gx = xg\}$. Let n be a positive integer. Define the n-centralizer of an element g of G as $C^n(g) = \{x \in G \mid g^n x = xg^n\} = C(g^n)$. Then $C^n(g)$ is a subgroup of G and $\bigcap_{g\in G} C^n(g) = C(G^n)$, where $G^n = \{g^n \mid g \in G\}$. Now define $T^n(g) = \{x \in G \mid (gx)^n = (xg)^n\}$ and $T^n(G) = \bigcap_{g\in G} T^n(g)$. It is easy to see that $T^n(g)$ may not be a subgroup of G. But it can be seen easily that $T^n(G) = C(G^n)$, and so $T^n(G)$ is a normal subgroup of G. To prove $T^n(G) \subseteq C(G^n)$, let $g \in T^n(G)$. Then for all $x \in G$, $(gx)^n = (xg)^n$. Therefore $(g(g^{-1}x))^n = ((g^{-1}x)g)^n$ and so $x^n = g^{-1}x^n g$. Hence $gx^n = x^n g$ and $g \in C(G^n)$. To see $C(G^n) \subseteq T^n(G)$, let $g \in C(G^n)$. Then for all $x \in G$, $gx^n = x^n g$. Therefore $g(gx)^n = (gx)^n g$ and so $(gx)^n = g^{-1}(gx)^n g$. Hence $(gx)^n = (xg)^n$ and $g \in T^n(G)$.

Now suppose that G is a group such that $T^n(G)$ is a proper subgroup. The vertex set $V = V(\Gamma)$ of the graph $\Gamma := \Gamma(G)$ is the set $G \setminus T^n(G)$ and two elements $x, y \in V$ are joined by an edge if $(xy)^n \neq (yx)^n$.

Now let us recall some notions of graphs. We consider simple graphs which are undirected, with no loops or multiple edges. For any graph Γ, we denote the sets of vertices and edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree of a vertex v in Γ is the number of edges incident to v. A graph Γ is regular if the degrees of all vertices of Γ are the same. A graph is complete if every pair of vertices are joined by an edge.

A path P is a sequence $v_0e_1v_1e_2\cdots e_kv_k$ whose terms are alternately distinct vertices and distinct edges, such that for any i, $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. In this case P is called a path between v_0 and v_k. The number k is called the length of P. If v_0 and v_k are adjacent in Γ by an edge e_{k+1}, then $P \cup \{e_{k+1}\}$ is called a cycle. The length of a cycle is defined by the number of its edges. The length of the shortest cycle in a graph Γ is called girth of Γ and denoted by girth(Γ). A Hamilton cycle of Γ is a cycle that contains every vertex of Γ. If v and w are vertices in Γ, then $d(v, w)$ denotes the length of the shortest path between v and w. The largest distance between all pairs of the vertices of Γ is called the diameter of Γ, and is denoted by diam(Γ). A graph Γ is connected if there is a path between each pair of the vertices of Γ. A planar graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex which both are incident.

2. Main results

Throughout this section we assume that G is a finite group, $n = 2$ and $\Gamma := \Gamma(G)$. So $V := V(\Gamma) = \{g \in G \mid (xg)^2 \neq (gx)^2, \text{ for some } x \in G\}$, with two elements $x, y \in V$ are joined by an edge if $(xy)^2 \neq (yx)^2$. First we classify groups with complete graphs.

Theorem 2.1. *The graph Γ is a complete graph if and only if $H := C(G^2)$ is abelian of odd order with index 2, $x^2 = 1$, $h^x = h^{-1}$, for all $x \in G \setminus H$ and all $h \in H$.*

Proof. ($\Longleftarrow$) Let x be any element of G outside H. Then $G = H \cup xH$ and $V(\Gamma) = G \setminus H = xH$. Let xh_1 and xh_2 be two distinct vertices of Γ, where $h_1, h_2 \in H$. Then $(xh_1xh_2)^2 \neq (xh_2xh_1)^2$. Otherwise $(h_1^x h_2)^2 = (h_2^x h_1)^2$ so $(h_1^{-1}h_2)^2 = (h_2^{-1}h_1)^2$ which implies that $(h_1^{-1}h_2)^4 = 1$, contradicting the hypothesis on the order of H. Therefore xh_1 and xh_2 are adjacent and Γ is a complete graph.

($\Longrightarrow$) Suppose that Γ is a complete graph. Suppose that $x \neq x^{-1}$ for some $x \in G \setminus H$. Then since $(x^{-1}x)^2 = (xx^{-1})^2$, x is not adjacent to x^{-1}, contradicting $\mathrm{diam}(\Gamma) = 1$. Hence $x^2 = 1$ for all $x \in G \setminus H$. For all $h \in H$ and for all $x \in G \setminus H$, we have $xh \in G \setminus H$ and so $(hx)^2 = 1$. Therefore $h^x = h^{-1}$.

To see that H is abelian, let $h_1, h_2 \in H$. Then

$$h_1^{-1}h_2^{-1} = h_1^x h_2^x = (h_1h_2)^x = (h_1h_2)^{-1},$$

for all $x \in G \setminus H$, and so $h_1h_2 = h_2h_1$. Hence H is abelian.

Now we prove that $|G/H| = 2$. Since G is not an elementary abelian 2-group, there exists $h \in H$ such that $h^2 \neq 1$. If $|G/H| > 2$, then there exist $x_1, x_2 \in G \setminus H$ such that $x_1x_2 = x_1^{-1}x_2 \notin H$. Hence $h^{x_1x_2} = h^{-1}$ and so

$$h^{-1} = h^{x_1x_2} = (h^{x_1})^{x_2} = (h^{-1})^{x_2} = h,$$

which is a contradiction. Hence $|G/H| = 2$.

Finally we show that $|H|$ is odd. Suppose there exists a non-identity $h \in H$ such that $h^2 = 1$ and let $x \in G \setminus H$. Then $(xxh)^2 = (x^2h)^2 = h^2 = 1$ and $(xhx)^2 = (x^2h^x)^2 = (h^{-1})^2 = 1$. Hence x and xh are not connected, contradicting to $\mathrm{diam}(\Gamma) = 1$. Therefore $|H|$ is odd. ∎

Now we prove some results when $|G : C(G^2)| = 2$.

Lemma 2.1. *Suppose that $H := C(G^2)$ has index 2 and let $x \in G \setminus H$ and $h \in H$ such that $h^2 \in C(x)$. Then $h^2 \in Z(G)$. Therefore $A := \{h \in H \mid h^2 \notin Z(G)\}$ is non-empty, and for all $h \in H$, x and hx (as well as x and xh) are adjacent if and only if $h \in A$. Also the graph Γ is $|A|$-regular.*

Proof. Let $g \in G$. It is clear that if $g \in H$, then $gh^2 = h^2g$. So suppose that $g = xh_1$, for some $h_1 \in H$. Thus

$$gh^2 = (xh_1)h^2 = x(h_1h^2) = x(h^2h_1) = h^2(xh_1) = h^2g$$

and therefore $h^2 \in Z(G)$, which proves the first assertion.

Now note that $(xhx)^2 = (hxx)^2$ if and only if $xhx^2hx = hx^2hx^2$ if and only if $x^3h^2x = x^2h^2x^2$ if and only if $xh^2 = h^2x$ if and only if $h^2 \in C(x)$. Thus x and hx are adjacent if and only if $h \in A$.

To see that A is nonempty, note that since $x \notin H$ and $H = C(G^2) = T^2(G)$, there exists $g \in G$ such that $(xg)^2 \neq (gx)^2$. Hence Γ has at least one edge. Suppose that (x, hx), for some $h \in H$ is an edge. Then by the above observation, $h^2 \notin Z(G)$ and so $A \neq \emptyset$. ∎

Now we can determine the girth of Γ. We thank the referee for pointing out that the conditions (i) and (ii), stated in Theorem 2.2, are equivalent to $\mathrm{girth}(\Gamma) = 3$.

Theorem 2.2. *Suppose that* $H := C(G^2)$ *has index* 2*. Then the graph* Γ *has girth* 3 *or* 4*. Also the girth is* 3 *if and only if either*

(i) *there exists* $h \in H$ *such that* $h^4 \notin Z(G)$ *or*

(ii) $\{h^4 \mid h \in H\} \subseteq Z(G)$ *and there exist* $h, k \in A$ *such that* $kh^{-1} \in A$*, where* $A = \{h \in H \mid h^2 \notin Z(G)\}$*, and* $k^x \neq h$*, for some* $x \in G \setminus H$*.*

Proof. Let x be any element of G outside H. First note that if $h \in H$, then xh and hx are not adjacent if and only if $h^4 \in Z(G)$. In fact if xh and hx are not adjacent, then $(xhhx)^2 = (hxxh)^2$. Since $hx^2 = x^2h$, we have $(xh^2x)^2 = (h^2x^2)^2$ and so $xh^2xxh^2x = h^2x^2h^2x^2$. Hence $xh^4x^3 = h^4x^4$ and $xh^4 = h^4x$. Since for all $u \in H$ we have $uh^4 = h^4u$, it follows that $h^4 \in Z(G)$. This argument also shows that if $h^4 \in Z(G)$, then xh and hx are not adjacent.

Suppose that the condition (i) holds, that is $h^4 \notin Z(G)$, for some $h \in H$. By Lemma 2.1, (x, xh) and (x, hx) are edges of Γ. Since $h^4 \notin Z(G)$, by above observation, xh and hx are adjacent. Thus $\{x, xh, hx\}$ form a triangle and so $\mathrm{girth}(\Gamma) = 3$.

Now suppose that the condition (ii) holds, that is $\{h^4 \mid h \in H\} \subseteq Z(G)$ and there exist $h, k \in H$ such that $\{h^2, k^2, (kh^{-1})^2\} \cap Z(G) = \emptyset$. By Lemma 2.1, (x, hx) and (x, kx) are edges of Γ. Also, by Lemma 2.1, $(hx, kh^{-1}(hx)) = (hx, kx)$ is an edge of Γ. Thus there is a triangle with vertices x, hx and kx. Hence $\mathrm{girth}(\Gamma) = 3$.

Now suppose that the girth of Γ is 3. If $\{x, xh, hx\}$ is a triangle, then $h^4 \notin Z(G)$ and the condition (i) holds. So suppose that $\{h^4 \mid h \in H\} \subseteq Z(G)$. Given a triangle with vertices h_1x, h_2x and h_3x, set $z = h_1x$ and setting $k_2 = h_2h_1^{-1}$ and $k_3 = h_3h_1^{-1}$ the vertices of the triangle become z, k_2z and k_3z. So we may assume that any triangle has vertices x, hx and kx, where $x \notin H$, $h, k \in H$, and by Lemma 2.1, $h, k \in A$. Viewing the edge (hx, kx) as $(hx, kh^{-1}(hx))$, we also have $kh^{-1} \in A$. As, in this case, (hx, xh) is not an edge of Γ, we must have $kx \neq xh$ so $k^x \neq h$. Therefore the condition (ii) holds.

By Lemma 2.1, there exists a non-identity element h of H such that $h^2 \notin Z(G)$. Suppose that the conditions (i) and (ii) do not hold. Then $h^4 \in Z(G)$. Since $h^2 \notin Z(G)$, by Lemma 2.1, (x, hx), $(x, h^{-1}x)$, $(hx, h(hx)) = (hx, h^2x)$ are edges of Γ. Now if $h^3 = 1$, then since $h^4 \in Z(G)$ we conclude that $h \in Z(G)$, contradicting $h^2 \notin Z(G)$. Thus $h^3 \neq 1$ and, by Lemma 2.1, $(h^{-1}x, h^{-1}(h^3x)) = (h^{-1}x, h^2x)$ is an edge of Γ. Hence $\{x, hx, h^2x, h^{-1}x\}$ form a square. Therefore $\mathrm{girth}(\Gamma) = 4$. ∎

Theorem 2.3. *Suppose that* $H := C(G^2)$ *has index* 2*. Then the graph* Γ *is Hamiltonian and therefore is connected.*

Proof. Let $B = \{h \in H \mid h^2 \in Z(G)\}$. Then B is a normal subgroup of H and H is a disjoint union of A and B, where $A = \{h \in H \mid h^2 \notin Z(G)\}$. Since, by Lemma 2.1, $A \neq \emptyset$ we see that $B \neq H$ and so $|H/B| \geq 2$. Therefore $|B| \leq |H|/2$ and so $|A| \geq |H|/2$. Since Γ is $|A|$-regular, by Dirac's Theorem [5, p. 54], Γ is Hamiltonian. ∎

Now we classify groups G (such that $|G : C(G^2)| = 2$) with planar graph. Denote by $D_{2n} = \langle x, y \mid x^n = y^2 = 1, x^y = x^{-1} \rangle$ the dihedral group of order $2n$.

Theorem 2.4. *Suppose that* $H := C(G^2)$ *has index* 2*. The graph* Γ *is planar if and only if* $G \cong D_6$ *or* $G \cong D_{12}$ *or* $G = \langle x, y \mid x^3 = y^4 = 1, x^y = x^2 \rangle$*.*

Proof. Suppose that Γ is planar. Let $B = \{h \in H \mid h^2 \in Z(G)\}$ and $A = \{h \in H \mid h^2 \notin Z(G)\}$. Then by [5, Corollary 3.5.9], $|A| \leq 5$. Therefore $|H| = |A| + |B| \leq 5 + |B|$ and so $2 \leq |H/B| \leq \frac{5}{|B|} + 1$. Hence $|B| \leq 5$ and $|H| \leq 10$. Thus $|G| \leq 20$. Now one can prove directly that the non-abelian group of order 6 and two groups of order 12 have planar graph. These groups are D_6, D_{12}, and $G = \langle x, y \mid x^3 = y^4 = 1, x^y = x^2 \rangle$. Also we can see these facts using computer packages, GAP [6] and Mathematica [9]. The graph of D_{12} and G are isomorphic and shown in Figure 1.

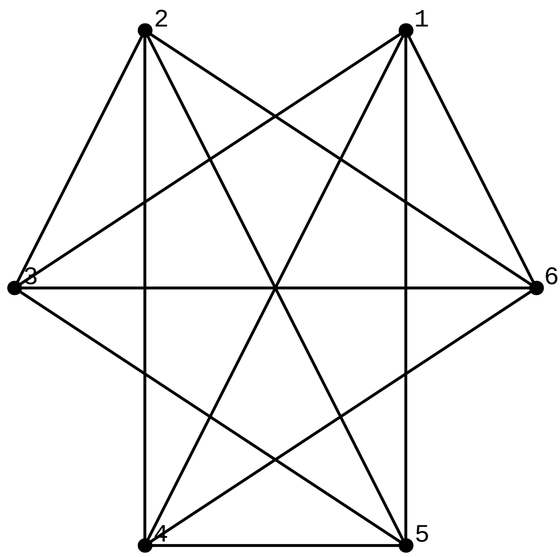

Figure 1. The graph of D_{12}.

For this purpose we use some commands of GAP to producing non-abelian groups for order ≤ 20, and generate the vertices and edges of the graph of each group. We use this information in Mathematica and determine the planarity of the graphs. Also we can see the figure of the graphs in Mathematica. Note that the GRAPE package of GAP can produce a graph and determine many properties of the graph, but can not determine the planarity of a graph. The following GAP program computes the vertex set and edge set of the graph of a given group. We use this program to find edges of the graph associated to groups of order ≤ 20.

```
################### generating the set of vertices of graph of a group
ver:=G->Difference(G,Centralizer(G,Group(Set(List(G,x->x^2)))));
################### generating the set of edges of graph of a group
edge:=function(G)
 local vertices,edges,r,temp,y,x,edgeslabel;
 vertices:=ver(G);  edgeslabel:=[];  edges:=[];
 for x in vertices do
     temp:=Difference(vertices,[x]);
```

```
     r:=Filtered(temp,y->(x*y)^2<>(y*x)^2);
     for y in r do
         if not([x,y] in edges) and not([y,x] in edges) then
             Add(edgeslabel,[Position(vertices,x),Position(vertices,y)]);
         fi;
     od;
 od;
return edgeslabel;
end;
################### generating the set of non-abelian groups of order <= 20
L:=Flat(List([6..20],x->AllSmallGroups(x,IsAbelian,false)));
L:=Filtered(L,x->Size(ver(x))>=1);
```

Now by the above program we find the edges of 15 non-abelian groups of order ≤ 20 with non-trivial graphs. Using this information and the following Mathematica program we see that 3 of these are planar. We can find the presentations of these groups by GAP and see that these are the claimed groups.

```
<< DiscreteMath`Combinatorica`
gengraph[edg_]:=
      Module[{vert=Union[Flatten[edg]],t=Length[edg],e={},n,z},
         n=Length[vert];
         v=Table[{{N[Cos[2Pi*t/n],2],N[Sin[2Pi*t/n],2]},
                         VertexLabel->vert[[t]]},{t,1,n}];
         For[i=1,i<=t,
            z = edg[[i]];
            AppendTo[e,{Flatten[{Position[vert,z[[1]]],
                               Position[vert,z[[2]]]}]}];
        i++ ];
 Return[Graph[e,v]]      ]
res=Table[gengraph[g[[i]]],{i,1,Length[g]}]; Map[PlanarQ, res]
```

∎

In what follows we characterize some dihedral groups in terms of their graphs. Note that if G and K are two groups, then $\Gamma(G) \cong \Gamma(K)$ if and only if there exists a bijective map $\varphi : V(\Gamma(G)) \to V(\Gamma(K))$ such that for every two distinct elements $x, y \in V(\Gamma(G))$, we have $(xy)^2 = (yx)^2$ if and only if $(\varphi(x)\varphi(y))^2 = (\varphi(y)\varphi(x))^2$.

Recall that the dihedral group of order $2n$ is

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, x^y = x^{-1} \rangle = \{1, x, x^2, \ldots, x^{n-1}, y, yx, yx^2, \ldots, yx^{n-1}\}.$$

The dihedral group has a normal cyclic subgroup $\langle x \rangle$ of index 2. Every element of the set $\{y, yx, yx^2, \ldots, yx^{n-1}\}$ has order 2. Let $H_1 = C(D_{2n}^2)$, then $H_1 = \langle x \rangle$ and so $V(D_{2n}) = G - H_1 = \{y, yx, yx^2, \ldots, yx^{n-1}\}$.

We want to find the adjacency of vertices of $\Gamma(D_{2n})$. Let $u = yx^i$ and $v = yx^j$, where $0 \leq i < j \leq n-1$, be two vertices of $\Gamma(D_{2n})$. Then u and v are not adjacent if and only if $n|4(j-i)$. It follows that if n is odd, $u = yx^i$ and $v = yx^j$, for all $0 \leq i < j \leq n-1$, are adjacent. Therefore $\Gamma(D_{2n})$ is complete if and only if n is odd. Note that we can see this result using Theorem 2.1.

Now let n be even. Suppose that $u = yx^i$ and $v = yx^j$, where $0 \le i < j \le n-1$, are not adjacent. Then since $n|4(j-i)$, we have to consider two cases:

Case 1. $\frac{n}{2}$ **is odd:** Since $\frac{n}{2}|2(j-i)$ and $\frac{n}{2}$ is odd, we have $\frac{n}{2}|(j-i)$. Since $0 \le i < j \le n-1$, we have $j-i = \frac{n}{2}$. Therefore in $\Gamma(D_{2n})$, for all $0 \le i < j \le n-1$, $j-i \ne \frac{n}{2}$, yx^i and yx^j are adjacent, that is the only nonadjacent vertices are $\frac{n}{2}$ pairs $(y, yx^{\frac{n}{2}})$, $(yx, yx^{\frac{n}{2}+1})$, $(yx^2, yx^{\frac{n}{2}+2})$, $\cdots$, $(yx^{\frac{n}{2}-1}, yx^{n-1})$. Note that in this case the vertices $y, yx^{\frac{n}{2}-1}, yx^{\frac{n}{2}+1}$ form a triangle, so girth$(\Gamma(D_{2n}) = 3$.

Case 2. $\frac{n}{2}$ **is even:** Since $\frac{n}{2}$ is even, we have $\frac{n}{4}|(j-i)$, where $0 \le i < j \le n-1$. So $j-i \in \{\frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\}$. Therefore in $\Gamma(D_{2n})$, for all $0 \le i < j \le n-1$, with $j-i \notin \{\frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\}$, yx^i and yx^j are adjacent. Thus we have $\frac{3n}{4}$ pairs $(y, yx^{\frac{n}{4}})$, $(yx, yx^{\frac{n}{4}+1})$, $(yx^2, yx^{\frac{n}{4}+2})$, $\cdots$, $(yx^{\frac{3n}{4}-1}, yx^{n-1})$, and $\frac{n}{4}$ pairs $(y, yx^{\frac{3n}{4}})$, $(yx, yx^{\frac{3n}{4}+1})$, $(yx^2, yx^{\frac{3n}{4}+2})$, $\cdots$, $(yx^{\frac{n}{4}-1}, yx^{n-1})$, and $\frac{n}{2}$ pairs $(y, yx^{\frac{n}{2}})$, $(yx, yx^{\frac{n}{2}+1})$, $(yx^2, yx^{\frac{n}{2}+2})$, $\cdots$, $(yx^{\frac{n}{2}-1}, yx^{n-1})$, of nonadjacent vertices. It follows that there are $\frac{3n}{2}$ nonadjacent pairs of vertices and so there are $\binom{n}{2} - \frac{3n}{2} = \frac{n^2}{2} - 2n$ edges.

Note that $yx^{\frac{n}{2}-1}$ and $yx^{\frac{n}{2}+1}$ are adjacent if and only if $n \ne 8$. So if $n = 8$, then the vertices y, yx^3, yx^4, yx^7 form a square and $\Gamma(D_{16})$ has no triangle. Thus girth$(\Gamma(D_{16})) = 4$. Also if $n > 8$, then the vertices $y, yx^{\frac{n}{2}-1}, yx^{\frac{n}{2}+1}$ form a triangle. Thus girth$(\Gamma(D_{2n})) = 3$.

Theorem 2.5. *Let n be a positive odd integer. Suppose that G is a finite non-abelian group. If $\Gamma(D_{2n}) \cong \Gamma(G)$, then H is abelian of index 2 and $|G| = 2n$. In particular if H is cyclic, then $G \cong D_{2n}$.*

Proof. Since $\Gamma(D_{2n})$ is a complete graph, so is $\Gamma(G)$. Thus by Theorem 2.1, H is abelian of odd order and index 2 such that $x^2 = 1$ and $h^x = h^{-1}$, for all $x \in G \setminus H$. Since $2|H| = |G| = |H| + |V(\Gamma(G))| = |H| + n$, we have $|H| = n$ and so $|G| = 2n$. If $H = \langle h \rangle$, then $G = \langle x, h \rangle$, where $x \in G \setminus H$. It follows that $G \cong D_{2n}$. ∎

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