Comparison Projection Method with Adomian's Decomposition Method for Solving System of Integral Equations

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Abstract. System of integral equations have been solved in many papers. In particular, systems of integral equations with degenerate kernels have been solved with Adomian’s decomposition method by some authors. In the present paper, we try to solve system of integral equations by using collocation method with Legendre polynomials which is more efficient and needs less computations than Adomian’s decomposition method.

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1. Introduction

Consider a system of Fredholm integral equations [18]

\[ \lambda u(x) = f(x) + \int_a^b k(x, t)u(t)dt, \]  

where \( \lambda \in \mathbb{R} \), and

\[ u(x) = [u_i(x)], \quad i = 1, ..., n, \]
\[ f(x) = [f_i(x)], \quad i = 1, ..., n, \]
\[ k(x, t) = [k_{i,j}(x, t)], \quad i, j = 1, ..., n. \]

This type of equations have been solved in many papers with different methods such as Taylor’s expansion [11, 13], operational matrices method [4, 15], homotopy perturbation method [1, 9], Sinc collocation method [16, 17] and Adomian’s decomposition method [5, 8, 12]. Some of these methods have restrictions such as \( k_{i,j}(x, t) \)
being degenerate and some of them are focused on system of integral equations of the second kind, with more computations leading to solutions with low accuracy. The aim of this paper is to solve system of integral equations by using collocation method with Legendre polynomials [2] as the basis for this projection method, and compare this method with Adomian decomposition method which has been used for solving this type of equations in [7]. Convergence of Legendre polynomials for solving Fredholm integral equation of the second kind has been discussed in [14], and we shall apply this discussion on convergence for system of integral equations.

2. Discretization of equations

In this section we apply collocation method to convert equation (1.1) to an algebraic system of linear equations $AX = b$. For this result, by using Legendre polynomials, we approximate $u_i(x)$’s, such that

$$u_i(x) \approx \sum_{k=1}^{m} c_{ik} L_{k-1}(x),$$

where $L_k(x)$ is $k$th Legendre polynomial and $c_{ik}$’s are unknown coefficients which are determined by solving an algebraic system of linear equations $AX = b$. By substituting relation (2.1) in (1.1) we have

$$\lambda \sum_{k=1}^{m} c_{1k} L_{k-1}(x) = f_1(x) + \sum_{i=1}^{n} \int_{a}^{b} k_{1i}(x, t) \sum_{k=1}^{m} c_{ik} L_{k-1}(t) dt,$$

$$\lambda \sum_{k=1}^{m} c_{2k} L_{k-1}(x) = f_2(x) + \sum_{i=1}^{n} \int_{a}^{b} k_{2i}(x, t) \sum_{k=1}^{m} c_{ik} L_{k-1}(t) dt,$$

$$\vdots$$

$$\lambda \sum_{k=1}^{m} c_{nk} L_{k-1}(x) = f_n(x) + \sum_{i=1}^{n} \int_{a}^{b} k_{ni}(x, t) \sum_{k=1}^{m} c_{ik} L_{k-1}(t) dt.$$

Now, we choose some collocation points such as

$$x_i = a + \frac{i(b-a)}{m}, \quad i = 1, 2, \ldots, m,$$

which are equidistant, and define a system of residual equations by

$$R_1(x) = \lambda \sum_{k=1}^{m} c_{1k} L_{k-1}(x) - f_1(x) - \sum_{i=1}^{n} \int_{a}^{b} k_{1i}(x, t) \sum_{k=1}^{m} c_{ik} L_{k-1}(t) dt,$$

$$R_2(x) = \lambda \sum_{k=1}^{m} c_{2k} L_{k-1}(x) - f_2(x) - \sum_{i=1}^{n} \int_{a}^{b} k_{2i}(x, t) \sum_{k=1}^{m} c_{ik} L_{k-1}(t) dt,$$

$$\vdots$$

$$R_n(x) = \lambda \sum_{k=1}^{m} c_{nk} L_{k-1}(x) - f_n(x) - \sum_{i=1}^{n} \int_{a}^{b} k_{ni}(x, t) \sum_{k=1}^{m} c_{ik} L_{k-1}(t) dt.$$

Then, by imposing the conditions

$$R_i(x_j) = 0, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m,$$
(where \(x_j\)'s are collocation points) to the system of residual equations, we deduce
an algebraic system of linear equations \(AX = b\) [3, 10].

For example, for \(n = 2\) we have

\[
\begin{align*}
\lambda u_1(x) &= f_1(x) + \int_a^b k_{11}(x, t)u_1(t)dt + \int_a^b k_{12}(x, t)u_2(t)dt, \\
\lambda u_2(x) &= f_2(x) + \int_a^b k_{21}(x, t)u_1(t)dt + \int_a^b k_{22}(x, t)u_2(t)dt,
\end{align*}
\]

which after discretization, an algebraic system of linear equations \(AX = b\) is derived as follows

\[
A = (a_{ij}), \quad i, j = 1, 2, \ldots, 2m,
\]

\[
b^T = [f_1(x_1), f_1(x_2), \ldots, f_1(x_m), f_2(x_1), f_2(x_2), \ldots, f_2(x_m)],
\]

\[
X^T = [c_{11}, c_{12}, \ldots, c_{1m}, c_{21}, c_{22}, \ldots, c_{2m}],
\]

where

\[
a_{ij} = \begin{cases} 
\lambda L_{j-1}(x_i) - \int_a^b k_{11}(x_i, t)L_{j-1}(t)dt, & i = 1, 2, \ldots, m \\
- \int_a^b k_{12}(x_i, t)L_{j-m-1}(t)dt, & j = 1, 2, \ldots, m \\
- \int_a^b k_{21}(x_i-m, t)L_{j-1}(t)dt, & i = m + 1, \ldots, 2m \\
\lambda L_{j-m-1}(x_{i-m}) - \int_a^b k_{22}(x_{i-m}, t)L_{j-m-1}(t)dt, & j = m + 1, \ldots, 2m
\end{cases}
\]

\[
3. Convergence of method
\]

In this section by using the following Proposition we try to prove a convergence theorem which shows the error bound of the numerical method that we applied in the previous section.

**Proposition 3.1.** Let \(f(t) \in H^k(-1, 1)\) Sobolev space, \(P_m(f(t)) = \sum_{i=0}^m a_iL_i(t)\) be
the best approximation polynomial of \(f(t)\) in \(L_2\)-norm. Then

\[
\|f(t) - P_m(f(t))\|_{L_2[-1,1]} \leq C_0m^{-k}\|f(t)\|_{H^k(-1,1)},
\]

where \(C_0\) is a positive constant, which depends on the selected norm and is independent of \(f(t)\) and \(m\).

See [6], for the proof of Proposition 3.1.

For the above proposition we have defined the following norms:

\[
\|f(t)\|_{L_2[-1,1]} = \left(\int_{-1}^1 f^2(t)dt\right)^{1/2},
\]

\[
\|f(t)\|_{H^k(-1,1)} = \left(\sum_{i=0}^k \int_{-1}^1 |f^{(i)}(t)|^2dt\right)^{1/2}.
\]
Theorem 3.1. Assume $\kappa : H^k(-1,1) \to L_2[-1,1]$ is an operator defined by

$$\kappa(u(x)) = \int_{-1}^{1} k(x,t)u(t)dt,$$

where $k(x,t) \in L_2$ in square $[-1,1] \times [-1,1]$ which was introduced in equation (1.1), and $u_m(x)$ is the numerical solution of the equation (1.1). Then

$$\sup_{x \in [-1,1]} |u(x) - u_m(x)| \leq C_1 m^{-k} \|u(t)\|_{H^k(-1,1)},$$

where $C_1$ is a positive constant.

Proof. Assume that the exact solution of equation (1.1) is $u(x)$, i.e.

$$u(x) = f(x) + \int_{-1}^{1} k(x,t)u(t)dt.$$

If we define the numerical solution of this equation by $u_m(x)$, then

$$u_m(x) = f(x) + \int_{-1}^{1} k(x,t)P_m(u(t))dt.$$

Hence

$$\sup_{x \in [-1,1]} |u(x) - u_m(x)| \leq \left| \int_{-1}^{1} k(x,t)u(t)dt - \int_{-1}^{1} k(x,t)P_m(u(t))dt \right|$$

$$\leq \left( \int_{-1}^{1} k^2(x,t)dt \right)^{1/2} \|u - P_m(u)\|.$$

Since $k(x,t) \in L_2$,

$$\max_{x \in [-1,1]} \left( \int_{-1}^{1} k^2(x,t)dt \right)^{1/2} \leq M.$$

By using Proposition 3.1, we have

$$\|u - P_m(u)\| \leq C_0 m^{-k} \|u(t)\|_{H^k(-1,1)},$$

and finally,

$$\sup_{x \in [-1,1]} |u(x) - u_m(x)| \leq MC_0 m^{-k} \|u(t)\|_{H^k(-1,1)}.$$

Letting $C_1 = MC_0$ completes proof of the theorem.

4. Numerical experiments

In this section, we compare Adomian’s decomposition method which has been discussed in [7] with the Legendre collocation method and present some examples that show the drawbacks of the Adomian’s decomposition method.

In [7], Adomian decomposition method was defined, and for solving system of integral equations (1.1), introduced the following successive process

$$u_{n+1} = G a^{(n)} = G B a^{(n-1)},$$

where $B$, $G$ and $a$ were defined in [7]. Then the solution of (1.1) is given by $u = \sum_{i=0}^{L} u_i$. In this process, if we increase the number of iterations, high powers of matrix $B$ are needed to compute. In addition, round-off errors in computing
powers of \( B \) and \( a^{(0)} \) are other weak points of this process that destroy the accuracy of solution. In Adomian’s decomposition method we need to generate matrix \( B \) which depends on the number of terms of degenerate kernels, for example if each \( k_{ij}; i,j = 1,2 \) has only one term then the rank of matrix \( B \) will be \((4 \times 4)\). By increasing the terms of kernels to two, the rank of matrix will be \((8 \times 8)\). So, the entries of matrix \( B \) will increase exponentially which lead to a huge amount of computation.

**Example 4.1.** In [7], the system of Fredholm integral equations
\[
\begin{align*}
  u_1(x) &= 2x^2 - \frac{1}{4} + \int_0^1 \left( \frac{1}{5} e^t u_1(t) + t^2 u_2(t) \right) dt, \\
  u_2(x) &= \frac{3}{2} x - x^2 + \int_0^1 \left( x^2 e^{-t} u_1(t) - x u_2(t) \right) dt,
\end{align*}
\]
has been solved by Adomian’s decomposition method, where the exact solutions are \( u_1(x) = e^x \) and \( u_2(x) = x \). After 30 steps of this method which needs a lot of computations, the following solutions are obtained
\[
\begin{align*}
  u_1(x) &= 1.0000002 e^x, \\
  u_2(x) &= 1.0000002 x - 0.0000002 x^2.
\end{align*}
\]
We have solved this system by using Legendre collocation method which was defined in previous section for \( m = 10 \). The numerical results are shown in Table 1. In this table \( E_{Adomian} \) and \( E_{Legendre} \) are the mean errors of Adomian’s decomposition method and Legendre collocation method for different values of \( x \), respectively.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( E_{Adomian} )</th>
<th>( E_{Legendre} )</th>
<th>( E_{Adomian} )</th>
<th>( E_{Legendre} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>( 2 \times 10^{-7} )</td>
<td>( 1.66392 \times 10^{-10} )</td>
<td>0</td>
<td>( 2.0212 \times 10^{-14} )</td>
</tr>
<tr>
<td>0.25</td>
<td>( 2.56805 \times 10^{-7} )</td>
<td>( 1.23479 \times 10^{-12} )</td>
<td>( 3.75 \times 10^{-8} )</td>
<td>( 3.53051 \times 10^{-14} )</td>
</tr>
<tr>
<td>0.5</td>
<td>( 3.29744 \times 10^{-7} )</td>
<td>( 1.02318 \times 10^{-12} )</td>
<td>( 5 \times 10^{-8} )</td>
<td>( 7.01772 \times 10^{-13} )</td>
</tr>
<tr>
<td>0.75</td>
<td>( 4.234 \times 10^{-7} )</td>
<td>( 1.0103 \times 10^{-12} )</td>
<td>( 3.75 \times 10^{-8} )</td>
<td>( 1.99962 \times 10^{-12} )</td>
</tr>
<tr>
<td>1</td>
<td>( 5.43656 \times 10^{-7} )</td>
<td>( 1.21414 \times 10^{-12} )</td>
<td>( 1.11022 \times 10^{-16} )</td>
<td>( 3.92941 \times 10^{-12} )</td>
</tr>
</tbody>
</table>

**Example 4.2.** In this example which has been stated in [7], we have the following system
\[
\begin{align*}
  u_1(x) &= \frac{1}{18} x + \frac{17}{36} + \int_0^1 \left( \frac{2}{3} x + u_1(t) + u_2(t) \right) dt, \\
  u_2(x) &= x^2 - \frac{19}{12} x + 1 + \int_0^1 \left( x u_1(t) + u_2(t) \right) dt,
\end{align*}
\]
where the exact solutions are \( u_1(x) = 1 + x \) and \( u_2(x) = 1 + x^2 \). The results from Adomian’s decomposition method for 30 steps given in [7] are compared with Legendre collocation method for \( m = 10 \) in Table 2.
Example 4.3. In this example, we present a system of integral equations with degenerate kernels, but unfortunately the Adomian’s decomposition method [7] is not able to solve this system. In regard to the Adomian’s decomposition method for the system

\[
\begin{align*}
  u_1(x) &= -\frac{47}{30} + 2x + \frac{17}{12}x^2 + x^3 + \int_0^1 (2t^2 - x^2)(u_1(t) + u_2(t))dt, \\
  u_2(x) &= -\frac{1}{3}x - \frac{121}{60}x^2 + \int_0^1 3x^2t(u_1(t) + u_2(t))dt,
\end{align*}
\]

where the exact solutions are \( u_1(x) = x^3 + 2x \) and \( u_2(x) = x^2 - \frac{x}{3} \) we have the following process for solving it.

\[
\begin{align*}
  \mathbf{u}(x) &= [u_1(x), u_2(x)], \\
  \mathbf{f}(x) &= [f_1(x), f_2(x)] = \left[ -\frac{47}{30} + 2x + \frac{17}{12}x^2 + x^3, -\frac{1}{3}x - \frac{121}{60}x^2 \right], \quad g_{11}(x) = 1, \quad g_{12}(x) = x^2, \quad g_{13}(x) = 1, \quad g_{14}(x) = x^2, \quad g_{21}(x) = x^2, \quad g_{22}(x) = x^2, \quad h_{11}(t) = 2t^2, \quad h_{12}(t) = -1, \quad h_{13}(t) = 2t^2, \quad h_{14}(t) = -1, \quad h_{21}(t) = 3t, \quad h_{22}(t) = 3t,
\end{align*}
\]

\[
\mathbf{a}^{(0)} = \int_0^1 [h_{11}f_1, h_{12}f_1, h_{13}f_2, h_{14}f_2, h_{21}f_1, h_{22}f_2]dt = \begin{bmatrix}
  77 \\
  90' \\
  -7 \\
  45' \\
  -73 \\
  75' \\
  151 \\
  180' \\
  21 \\
  16' \\
  -443 \\
  240'
\end{bmatrix}
\]

and the matrices \( \mathbf{G}, \mathbf{B} \) are as follows

\[
\mathbf{G} = \begin{bmatrix}
  g_{11}(x) & g_{12}(x) & g_{13}(x) & g_{14}(x) & 0 & 0 \\
  0 & 0 & 0 & g_{21}(x) & 0 & g_{22}(x)
\end{bmatrix},
\]

\[
\mathbf{B} = \begin{bmatrix}
  \int_0^1 h_{11}g_{11}dt & \int_0^1 h_{11}g_{12}dt & \int_0^1 h_{11}g_{13}dt & \int_0^1 h_{11}g_{14}dt & 0 & 0 \\
  \int_0^1 h_{12}g_{11}dt & \int_0^1 h_{12}g_{12}dt & \int_0^1 h_{12}g_{13}dt & \int_0^1 h_{12}g_{14}dt & 0 & 0 \\
  \int_0^1 h_{21}g_{11}dt & \int_0^1 h_{21}g_{12}dt & \int_0^1 h_{21}g_{13}dt & \int_0^1 h_{21}g_{14}dt & 0 & 0 \\
  \int_0^1 h_{22}g_{11}dt & \int_0^1 h_{22}g_{12}dt & \int_0^1 h_{22}g_{13}dt & \int_0^1 h_{22}g_{14}dt & 0 & 0 \\
  \int_0^1 h_{13}g_{21}dt & \int_0^1 h_{13}g_{22}dt & \int_0^1 h_{13}g_{23}dt & \int_0^1 h_{13}g_{24}dt & 0 & 0 \\
  \int_0^1 h_{14}g_{21}dt & \int_0^1 h_{14}g_{22}dt & \int_0^1 h_{14}g_{23}dt & \int_0^1 h_{14}g_{24}dt & 0 & 0 \\
  \int_0^1 h_{23}g_{21}dt & \int_0^1 h_{23}g_{22}dt & \int_0^1 h_{23}g_{23}dt & \int_0^1 h_{23}g_{24}dt & 0 & 0 \\
  \int_0^1 h_{24}g_{21}dt & \int_0^1 h_{24}g_{22}dt & \int_0^1 h_{24}g_{23}dt & \int_0^1 h_{24}g_{24}dt & 0 & 0
\end{bmatrix}
\]
so that

\[
 B = \begin{bmatrix}
  \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\
 -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\
 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\
 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\
 \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 & 0 \\
 0 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{2} \\
 \end{bmatrix}
\]

Using the above information and the fact \( a^{(n)} = B a^{(n-1)} \) for 30 steps of [7], we obtain

\[
 u_{10} = -\frac{47}{30} + 2x + \frac{17}{12}x^2 + x^3, \\
 u_{11} = -\frac{53}{450} + \frac{41}{60}x^2, \\
 u_{12} = -\frac{1}{54} + \frac{61}{900}x^2, \\
 \vdots \\
 u_{1,30} = -0.0120628 + 0.0153814x^2, \\
 u_{20} = -\frac{1}{3}x - \frac{121}{60}x^2, \\
 u_{21} = -\frac{8}{15}x^2, \\
 u_{22} = -\frac{77}{1200}x^2, \\
 \vdots \\
 u_{2,30} = -0.0256153x^2.
\]

The approximated solution for some values of \( x \) after 30 steps of [7] and Legendre collocation method by \( m = 10 \) are given in Table 3 which shows the advantage of the Legendre collocation method.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( E_{Adomian}(u_1) )</th>
<th>( E_{Legendre} )</th>
<th>( E_{Adomian}(u_2) )</th>
<th>( E_{Legendre} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>2.01549</td>
<td>1.45472 \times 10^{-13}</td>
<td>0.0</td>
<td>2.24643 \times 10^{-14}</td>
</tr>
<tr>
<td>0.25</td>
<td>1.85487</td>
<td>7.70495 \times 10^{-14}</td>
<td>0.267494</td>
<td>1.10849 \times 10^{-14}</td>
</tr>
<tr>
<td>0.5</td>
<td>1.37299</td>
<td>5.66214 \times 10^{-14}</td>
<td>1.06998</td>
<td>4.41869 \times 10^{-14}</td>
</tr>
<tr>
<td>0.75</td>
<td>0.569873</td>
<td>2.28706 \times 10^{-14}</td>
<td>2.40745</td>
<td>9.94205 \times 10^{-14}</td>
</tr>
<tr>
<td>1</td>
<td>0.554498</td>
<td>2.44249 \times 10^{-14}</td>
<td>4.2799</td>
<td>1.7697 \times 10^{-13}</td>
</tr>
</tbody>
</table>

Increasing the number of iterations provide no improvement in the accuracy of the solutions. Numerical results for the above system by 60 steps for Adomian’s decomposition method are presented in Table 4.

<table>
<thead>
<tr>
<th>( x ) values</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{Adomian}(u_1) )</td>
<td>2.4132</td>
<td>2.22088</td>
<td>1.64392</td>
<td>0.682323</td>
<td>0.663914</td>
</tr>
<tr>
<td>( E_{Adomian}(u_2) )</td>
<td>0.0</td>
<td>0.320277</td>
<td>1.28111</td>
<td>2.8825</td>
<td>5.12444</td>
</tr>
</tbody>
</table>
Example 4.4. In this example, we try to solve a system of equations which has nonseparable kernels. It is mentioned in [7] that if the kernels in system of integral equations are nonseparable, then Taylor’s expansion method can degenerate them and the Adomian’s decomposition method [7] is able to solve it. This claim is not correct because of two reasons. Firstly, Taylor’s expansion has not enough authority to approximate a function with two variables and the second reason is that if we increase the terms of Taylor’s expansion, the rank of matrix $B$ will increase exponentially which leads to more computations with lots of round off errors.

Now, in equation (2.2) let

$$
\begin{align*}
    k_{11}(x, t) &= \sin(xt + t) \\
    k_{12}(x, t) &= e^{xt} \\
    k_{21}(x, t) &= x^2t - t^3 \\
    k_{22}(x, t) &= e^{xt^3} \\
    f_1(x) &= \frac{2 - e^x(2 - 2x + x^2)}{x^3} + \cos(4\pi x) + \frac{(1 + x)(\cos(1 + x) - 1)}{(1 + x)^2 - 16\pi^2} \\
    f_2(x) &= \frac{3}{16\pi^2} - \frac{2(e^x - 1)}{3x} + x^2
\end{align*}
$$

and the exact solutions are $u_1(x) = \cos(4\pi x)$ and $u_2(x) = x^2$. Numerical results for Legendre collocation method with $m = 15$ and $m = 20$ are shown in Tables 5 and 6.

Table 5. Numerical results for Example 4.4 for $m = 15$

<table>
<thead>
<tr>
<th>$x$ values</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{Legendre}(u_1)}$</td>
<td>$4.356 \times 10^{-5}$</td>
<td>$5.342 \times 10^{-5}$</td>
<td>$6.139 \times 10^{-5}$</td>
<td>$7.019 \times 10^{-5}$</td>
<td>$7.830 \times 10^{-5}$</td>
</tr>
<tr>
<td>$E_{\text{Legendre}(u_2)}$</td>
<td>$9.505 \times 10^{-6}$</td>
<td>$1.223 \times 10^{-5}$</td>
<td>$1.816 \times 10^{-5}$</td>
<td>$2.742 \times 10^{-5}$</td>
<td>$4.002 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table 6. Numerical results for Example 4.4 for $m = 20$

<table>
<thead>
<tr>
<th>$x$ values</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{Legendre}(u_1)}$</td>
<td>$5.417 \times 10^{-5}$</td>
<td>$2.445 \times 10^{-7}$</td>
<td>$2.827 \times 10^{-7}$</td>
<td>$3.178 \times 10^{-7}$</td>
<td>$3.612 \times 10^{-7}$</td>
</tr>
<tr>
<td>$E_{\text{Legendre}(u_2)}$</td>
<td>$7.026 \times 10^{-8}$</td>
<td>$6.684 \times 10^{-8}$</td>
<td>$9.293 \times 10^{-8}$</td>
<td>$1.334 \times 10^{-7}$</td>
<td>$1.883 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Now, if we degenerate the kernels of above system and then solve the new system by Legendre collocation method for $m = 20$, the following numerical results are obtained in Table 7 that show the weakness of Taylor’s expansion for degenerating of kernels in spite of the good accuracy that we obtained in Table 6.

In the following we present an example to show the efficiency of this numerical method for solving first kind system of integral equations.
In this example, we try to solve a system of equations

\[
\begin{align*}
    f_1(x) &= \int_{-1}^{1} \sin(x^2 + t)u_1(t)dt - \int_{-1}^{1} 3te^{x^2 t^2}u_2(t)dt, \\
    f_2(x) &= -\int_{-1}^{1} 3\cos(xt)u_1(t)dt + \int_{-1}^{1} txe^{x^3 t^2}u_2(t)dt,
\end{align*}
\]

where

\[
\begin{align*}
    f_1(x) &= -\frac{3e^{x^2 - 1}(-e^{e^x})}{2(1+x^2)} + \frac{\sin(x^2+3x)+\sin(x^2-3)}{6} + \frac{(\sin(x-x^2)+\sin(x^2+1))}{2}, \\
    f_2(x) &= \frac{e^{3x^3+2x^2-3x+1}}{2(e+3ex)} - \frac{6\cos 2x \cos x^2+3x \sin 6x \cos x \cos 2-x \sin x \sin x^2}{x^2-4},
\end{align*}
\]

and the exact solutions are \( u_1(x) = \sin 2x \) and \( u_2(x) = e^{x^2-1} \). Numerical results for Legendre collocation method with \( m = 10 \) and \( m = 15 \) are shown in Table 8.

### Table 8. Numerical results for Example 4.5

<table>
<thead>
<tr>
<th>( x )</th>
<th>( m = 10 )</th>
<th>( m = 15 )</th>
<th>( m = 10 )</th>
<th>( m = 15 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>( 1.7036 \times 10^{-4} )</td>
<td>( 4.5999 \times 10^{-10} )</td>
<td>( 1.0224 \times 10^{-5} )</td>
<td>( 5.4499 \times 10^{-8} )</td>
</tr>
<tr>
<td>-0.75</td>
<td>( 9.6772 \times 10^{-9} )</td>
<td>( 1.4551 \times 10^{-11} )</td>
<td>( 1.7953 \times 10^{-7} )</td>
<td>( 6.3424 \times 10^{-10} )</td>
</tr>
<tr>
<td>-0.5</td>
<td>( 5.2555 \times 10^{-9} )</td>
<td>( 1.4789 \times 10^{-11} )</td>
<td>( 6.9364 \times 10^{-8} )</td>
<td>( 1.7594 \times 10^{-10} )</td>
</tr>
<tr>
<td>-0.25</td>
<td>( 3.5251 \times 10^{-9} )</td>
<td>( 1.0557 \times 10^{-11} )</td>
<td>( 9.4624 \times 10^{-10} )</td>
<td>( 1.1806 \times 10^{-10} )</td>
</tr>
<tr>
<td>0</td>
<td>( 3.3230 \times 10^{-10} )</td>
<td>( 4.1108 \times 10^{-13} )</td>
<td>( 2.2215 \times 10^{-7} )</td>
<td>( 1.0512 \times 10^{-10} )</td>
</tr>
<tr>
<td>0.25</td>
<td>( 2.4917 \times 10^{-8} )</td>
<td>( 7.3872 \times 10^{-11} )</td>
<td>( 1.3333 \times 10^{-8} )</td>
<td>( 8.2249 \times 10^{-11} )</td>
</tr>
<tr>
<td>0.5</td>
<td>( 1.7392 \times 10^{-7} )</td>
<td>( 4.3160 \times 10^{-10} )</td>
<td>( 1.0306 \times 10^{-8} )</td>
<td>( 3.3478 \times 10^{-11} )</td>
</tr>
<tr>
<td>0.75</td>
<td>( 8.4662 \times 10^{-7} )</td>
<td>( 2.4831 \times 10^{-9} )</td>
<td>( 9.7882 \times 10^{-8} )</td>
<td>( 1.4628 \times 10^{-10} )</td>
</tr>
<tr>
<td>1</td>
<td>( 3.8222 \times 10^{-5} )</td>
<td>( 1.1492 \times 10^{-7} )</td>
<td>( 5.8332 \times 10^{-6} )</td>
<td>( 4.3467 \times 10^{-8} )</td>
</tr>
</tbody>
</table>

### Conclusion

In this paper, a projection method known as collocation method with Legendre polynomials was chosen to discretize the system of integral equations. This method has some advantages. It is easy to apply for first and second kind system of integral equations. It also requires less computations than other methods discussed in [7, 9, 11, 15, 16, 17]. For example when this method is applied to many systems of integral equations, by solving an algebraic system of linear equations with rank less than \( 10 \times 10 \), we can get good accuracy. In some methods the kernels of system are required to satisfy some conditions such as being separable but the method of this paper does not have such conditions.
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References


