Entire Functions That Share Fixed-Points

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Abstract. In this paper, we study the uniqueness problem on entire functions sharing fixed points with the same multiplicities. We generalize some previous results.

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1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the following standard notations of value distribution theory [9]: $T(r,f), m(r,f), N(r,f), N_1(r,f), \ldots$. We denote by $S(r,f)$ any function satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$ possibly outside a set $r$ of finite linear measure.

We say that two meromorphic functions $f$ and $g$ share a small function $a$ IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f$ and $g$ have the same zeros with the same multiplicities, then we say that $f$ and $g$ share a CM (counting multiplicities).

Let $p$ be a positive integer and $a \in \mathbb{C}$. We denote by $N_p(r, \frac{1}{f-a})$ the counting function of the zeros of $f - a$ where an $m$-fold zero is counted $m$ times if $m \leq p$ and $p$ times if $m > p$. We say that a finite value $z_0$ is a fixed point of $f$ if $f(z_0) = z_0$.

In answer to one famous question, Hayman [4], Fang and Hua [1], and Yang and Hua [8] obtained the following result.

Theorem 1.1. Let $f$ and $g$ be two non-constant entire functions, and let $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share $1$ CM, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where $c_1, c_2$ and $c$ are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant $t$ such that $t^{n+1} = 1$.

In [3], Fang also got the following results.
Theorem 1.2. Let \( f \) and \( g \) be two non-constant entire functions, and let \( n, k \) be two positive integers with \( n > 2k + 4 \). If \((f^n)^{(k)}(g^n)^{(k)}\) share 1 CM, then either \( f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \((-1)^k(c_1c_2)^n(nc)^{2k} = 1\) or \( f = tg \) for a constant \( t \) such that \( t^n = 1 \).

Theorem 1.3. Let \( f \) and \( g \) be two non-constant entire functions, and let \( n, k \) be two positive integers with \( n \geq 2k + 8 \). If \((f^n(f - 1))^{(k)}(g^n(g - 1))^{(k)}\) share 1 CM, then \( f = g \).

Recently, Zhang, Chen and Lin [11] proved the following result, which generalized some previous results.

Theorem 1.4. Let \( f(z) \) and \( g(z) \) be two entire functions; let \( n, m \) and \( k \) be three positive integers with \( n \geq 3m + 2k + 5 \), and let \( P(z) = a_m z^m + a_{m - 1} z^{m - 1} + \cdots + a_1 z + a_0 \) or \( P(z) = C \), where \( a_0 \neq 0 \), \( a_1, \ldots, a_{m - 1}, a_m \neq 0 \), \( C \neq 0 \) are complex constants. If \([f^n P(f)]^{(k)}\) and \([g^n P(g)]^{(k)}\) share 1 CM, then the following conclusions hold:

(i) If \( P(z) = a_m z^m + a_{m - 1} z^{m - 1} + \cdots + a_1 z + a_0 \), then \( f(z) = tg(z) \) for a constant \( t \) that satisfies \( t^d = 1 \), where \( d = (n + m, \ldots, n + m - i, \ldots, n) \), \( a_{m - i} \neq 0 \) for some \( i = 0, 1, \ldots, m \); or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(\omega_1, \omega_2) = \omega^n \left(a_m \omega_1^m + a_{m - 1} \omega_1^{m - 1} + \cdots + a_1 \omega_1 + a_0\right) - \omega_2^n(a_m \omega_2^m + a_{m - 1} \omega_2^{m - 1} + \cdots + a_1 \omega_2 + a_0) \);

(ii) If \( P(z) = C \), then \( f = tg \) for a constant \( t \) that satisfies \( t^n = 1 \), or \( f(z) = b_1/\sqrt{C} e^{b z}, g(z) = b_2/\sqrt{C} e^{-b z} \) for three constants \( b_1, b_2 \) and \( b \) that satisfy \((-1)^k(b_1b_2)^n(nb)^{2k} = -1\).

Corresponding to the above results, some authors considered the uniqueness problems of entire functions that have fixed points, see Fang and Qiu [2], Lin and Yi [7]. In the present paper, we consider the existence of fixed points of \((f^n P(f))^{(k)}\) and the corresponding uniqueness theorems, where \( n, k \) are positive integers and \( P(z) \) is a nonzero polynomial, and we obtain the following results which generalize the above theorems.

Theorem 1.5. Let \( f(z) \) be a transcendental entire function, \( n, k, m \) be three positive integers with \( n \geq k + 2 \), and let \( P(z) = a_m z^m + a_{m - 1} z^{m - 1} + \cdots + a_1 z + a_0 \) or \( P(z) = C \), where \( a_0, a_1, \ldots, a_{m - 1}, a_m \neq 0 \), \( C \neq 0 \) are complex constants. Then \([f^n P(f)]^{(k)}\) has infinitely many fixed points.

Remark 1.1. It is easy to see that a polynomial \( Q(z) \) with degree \( n \geq 1 \) has exactly \( n \) fixed points (counting multiplicities), but a transcendental entire function may have no fixed points. For example, the function \( f = e^{\alpha(z)} + z \) has no any fixed points, where \( \alpha(z) \) is an entire function.

Here and forth, we define an integer \( m^* \), according to the nonzero polynomial \( P(z) \) in Theorem 1.6, by

\[
m^* = \begin{cases} m, & P(z) \neq C; \\
0, & P(z) = C.
\end{cases}
\]
Theorem 1.6. Suppose that $P(z)$ is given by Theorem 1.5. Let $f(z)$ and $g(z)$ be two transcendental entire functions, and let $n, m$ and $k$ be three positive integers with $n > 2k + m^* + 4$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $z$ CM, then the following conclusions hold:

(i) If $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ is not a monomial, then $f(z) = t g(z)$ for a constant $t$ that satisfies $t^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$; or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = \omega^1_1 (a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_1 w_1 + a_0) - \omega^2_2 (a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_1 w_2 + a_0)$.

(ii) If $P(z) = C$ or $P(z) = a_m z^m$, then $f = t g$ for a constant $t$ that satisfies $t^{n+m^*} = 1$, or $f(z) = b_1 e^{b_2 z^2}$, $g(z) = b_2 e^{-b_2 z^2}$ for three constants $b_1, b_2$ and $b$ that satisfy $4a^2(b_1 b_2)^{n+m^*}(n+m)b^2 = -1$, or $4C^2(b_1 b_2)^n(nb)^2 = -1$.

Remark 1.2. The condition of $n \geq 3m + 2k + 5$ in Theorem 1.4 is replaced by $n > 2k + 4 + m^*$ in Theorem 1.6.

2. Some lemmas

Lemma 2.1. [9] Let $f$ be a non-constant meromorphic function, and $a_0, a_1, a_2, \ldots, a_n$ be small functions of $f$ such that $a_n \neq 0$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = n T(r, f) + S(r, f).$$

Lemma 2.2. [6] Let $f$ be a non-constant meromorphic function, and $p, k$ be positive integers. Then

$$N_p \left( \frac{h'}{h} \right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f),$$

and

$$N_p \left( \frac{h'}{h} \right) \leq k N(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f).$$

Lemma 2.3. [10] Let

$$H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

where $F$ and $G$ are two non-constant meromorphic functions. If $F$ and $G$ share $1$ CM and $H \neq 0$, then

$$T(r, F) + T(r, G) \leq 2 N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + N_2 (r, F) + N_2 (r, G) + S(r, F) + S(r, G).$$

Lemma 2.4. [9] Let $f$ be a non-constant meromorphic function, and $a_1(z), a_2(z)$ and $a_3(z)$ be distinct small functions of $f$. Then

$$T(r, f) < \sum_{j=1}^{3} N \left( r, \frac{1}{f - a_j} \right) + S(r, f).$$
Lemma 2.5. [5] Suppose that $f$ is a non-constant meromorphic function, $k \geq 2$ is an integer. If

$$N(r, f) + N\left(r, \frac{1}{f}ight) + N\left(r, \frac{1}{f^{(k)}}\right) = S\left(r, \frac{f'}{f}\right),$$

then $f = e^{az^b}$, where $a \neq 0$, $b$ are constants.

Lemma 2.6. Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n$, $k$ be two positive integers with $n > k + 2$, and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ or $P(z) = C$, where $a_0, a_1, \ldots, a_{m-1}, a_m \neq 0$, $C \neq 0$ are complex constants. If $[f^n(z)P(f)]^{(k)}[g^n(z)P(g)]^{(k)} \equiv z^2$, then $P(z)$ is reduced to a nonzero monomial, that is, $P(z) = a_m z^m$ or $P(z) = C$.

Proof. If $P(z)$ is not reduced to a nonzero monomial, then $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_i$ is the last nonzero complex constant for $i = 0, 1, \ldots, m - 1$. Since

$$[f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f)]^{(k)}[g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_1 g)]^{(k)}$$

(2.5) $\equiv z^2$.

Suppose that $z_0$ is a $p$-fold zero of $f$, we know that $z_0$ must be a $(np + ip - k)$-fold zero of $[f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f)]^{(k)}$. Noting that $g$ is an entire function and $n > k + 2$, it follows from (2.5) that $z_0$ is a zero of $z^2$ with the order at least 3, which is impossible. Thus $f$ has no zeros. Let $f(z) = e^{\beta(z)}$, where $\beta(z)$ is a non-constant entire function. Then

$$\left(f^{m+n}\right)^{(k)} = (e^{(m+n)\beta})^{(k)} = P_m(\beta', \beta'', \ldots, \beta^{(k)}) e^{(m+n)\beta},$$

(2.6)

$$\left(f^{n+i}\right)^{(k)} = (e^{(n+i)\beta})^{(k)} = P_i(\beta', \beta'', \ldots, \beta^{(k)}) e^{(n+i)\beta},$$

(2.7)

where $P_m$ and $P_i$ are differential polynomials in $\beta', \beta'', \ldots, \beta^{(k)}$. Obviously, $P_m \neq 0$, $P_i \neq 0$, $T(r, P_m) = S(r, f)$ and $T(r, P_i) = S(r, f)$. We obtain from (2.5) to (2.7) that

$$N\left(r, \frac{1}{a_m P_m e^{(m-i)\beta} + a_{m-1} P_{m-1} e^{(m-1-i)\beta} + \cdots + a_1 P_i}\right) = S(r, f).$$

By Lemma 2.4 and Lemma 2.1, we have

$$(m - i)T(r, f)$$

$$= T(r, a_m P_m e^{(m-i)\beta} + a_{m-1} P_{m-1} e^{(m-1-i)\beta} + \cdots + a_1 P_{i+1} e^{\beta}) + S(r, f)$$

$$\leq N\left(r, \frac{1}{a_m P_m e^{(m-i)\beta} + a_{m-1} P_{m-1} e^{(m-1-i)\beta} + \cdots + a_1 P_{i+1} e^{\beta}}\right)$$

$$+ N\left(r, \frac{1}{a_m P_m e^{(m-i)\beta} + a_{m-1} P_{m-1} e^{(m-1-i)\beta} + \cdots + a_1 P_{i+1} e^{\beta} + a_i P_i}\right)$$

$$+ S(r, f)$$

$$\leq N\left(r, \frac{1}{a_m P_m e^{(m-i)\beta} + a_{m-1} P_{m-1} e^{(m-2-i)\beta} + \cdots + a_{i+1} P_{i+1}}\right) + S(r, f)$$

$$\leq (m - i - 1)T(r, f) + S(r, f),$$

which is a contradiction. This completes the proof of Lemma 2.6. \[\blacksquare\]
Lemma 2.7. Assume that the assumptions of Lemma 2.6 hold, then \( f(z) = b_1 e^{b_2 z} \), \( g(z) = b_2 e^{-b_2 z} \) for three constants \( b_1, b_2 \) and \( b \) that satisfy \( 4a_m^2 (b_1 b_2)^m + 4m(b_2) = -1 \), or \( 4C^2 (b_1 b_2)^m + 4m = -1 \).

Proof. From Lemma 2.6, we get \( P(z) = a_m z^m \) or \( P(z) = C \), we distinguish two cases.

Case A. \( P(z) = a_m z^m \). In this case, we have \((a_m f^{m+n})^{(k)}(a_m g^{m+n})^{(k)} = z^2\).

If \( k = 1 \), then

\[
\begin{equation}
(2.8)
\end{equation}
\]

Since \( f \) and \( g \) are entire functions and \( n > k + 2 \), by using the similar arguments as in the proof of Lemma 2.6, we deduce from (2.8) that \( f \) and \( g \) have no zeros. Let \( f = e^{\alpha(z)} \), \( g = e^{\beta(z)} \), where \( \alpha(z) \), \( \beta(z) \) are non-constant entire functions. Set

\[
\begin{equation}
(2.9)
h(z) = \frac{1}{f(z)g(z)},
\end{equation}
\]

we know that \( h(z) = e^{\gamma(z)} \), where \( \gamma(z) \) is an entire function. We claim that \( \gamma(z) \) is a constant. In fact, suppose \( \gamma(z) \) is a non-constant entire function, then \( h(z) \) is a transcendental entire function. From (2.8), we get

\[
\begin{equation}
(2.10)
(m + n)^2 a_m^2 f^{n+m-1} f'(g^{n+m-1}) g' = z^2.
\end{equation}
\]

From (2.9) and (2.10), we have

\[
\begin{equation}
(2.11)
\left( \frac{g'}{g} + \frac{1}{2} \frac{h'}{h} \right)^2 = \frac{1}{4} \left( \frac{h'}{h} \right)^2 - \frac{z^2 h^{m+n}}{(m + n)^2 a_m^2}.
\end{equation}
\]

Let \( \xi = \frac{g'}{g} + \frac{1}{2} \frac{h'}{h} \), then (2.11) becomes

\[
\begin{equation}
(2.12)
\xi^2 = \frac{1}{4} \left( \frac{h'}{h} \right)^2 - \frac{z^2 h^{m+n}}{(m + n)^2 a_m^2}.
\end{equation}
\]

If \( \xi \equiv 0 \), from (2.12), we get

\[
\begin{equation}
(2.13)
h^{m+n} = \frac{(m + n)^2 a_m^2}{4z^2} \left( \frac{h'}{h} \right)^2.
\end{equation}
\]

Since \( h(z) = e^{\gamma(z)} \), we obtain from (2.13) that

\[
\begin{align*}
(m + n)T(r, h) &= (m + n)m(r, h) + S(r, h) \\
&\leq m \left( r, \frac{1}{4z^2} \right) + 2m \left( r, \frac{h'}{h} \right) + S(r, h) = S(r, h).
\end{align*}
\]

Hence \( h \) is a constant, which is a contradiction. Therefore \( \xi \neq 0 \). Differentiating (2.12), we have

\[
\begin{equation}
(2.14)
2\xi' = \frac{1}{2} \frac{h'}{h} \left( \frac{h'}{h} \right)' - \frac{2z}{a_m^2 (m + n)^2} h^{m+n} - \frac{1}{a_m^2 (m + n)} z^2 h^{m+n-1} h'.
\end{equation}
\]

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From (2.12) and (2.14), we obtain
\[(2.15) \quad \frac{1}{a_m^2(m + n)^2} h^{m+n} \left( 2z + (m + n)z^2 \frac{h'}{h} - 2z^2 \frac{\xi'}{\xi} \right) = \frac{1}{2} \frac{h'}{h} \left( \left( \frac{h'}{h} \right)' - \frac{h'}{h} \frac{\xi'}{\xi} \right).\]

If \(2z + (m + n)z^2 \frac{h'}{h} - 2z^2 \frac{\xi'}{\xi} \equiv 0\), then, we deduce from (2.15) that either \(\frac{h'}{h} \equiv 0\) or \(\left( \frac{h'}{h} \right)' - \frac{h'}{h} \frac{\xi'}{\xi} \equiv 0\). If \(\frac{h'}{h} \equiv 0\), then \(h\) is a constant, which is a contradiction. If \(\left( \frac{h'}{h} \right)' - \frac{h'}{h} \frac{\xi'}{\xi} \equiv 0\), we have
\[(2.16) \quad \frac{h'}{h} = \frac{\xi}{d},\]
where \(d(\neq 0)\) is a constant. Thus we get from (2.12) and (2.16) that
\[(2.17) \quad \frac{z^2 h^{m+n}}{a_m^2(m + n)^2} = \left( \frac{1}{4} - d^2 \right) \left( \frac{h'}{h} \right)^2.\]

Hence, \((m + n)T(r, h) = S(r, h)\), which is also a contradiction.

Now we assume that \(2z + (m + n)z^2 \frac{h'}{h} - 2z^2 \frac{\xi'}{\xi} \neq 0\). Since \(h = e^{\gamma(z)}\) and \(\xi = \frac{g'}{g} + \frac{1}{2} \frac{h'}{h}\), from (2.12) and (2.15), we have
\[N \left( r, \frac{h'}{h} \right) = S(r, h), \quad N(r, \xi) = S(r, h),\]
and
\[(m + n)T(r, h) = (m + n)m(r, h) \leq m \left( r, \frac{1}{2z + (m + n)z^2 \frac{h'}{h} - 2z^2 \frac{\xi'}{\xi}} \right) \]
\[+ m \left( r, \frac{h'}{h} \left( \left( \frac{h'}{h} \right)' - \frac{h'}{h} \frac{\xi'}{\xi} \right) \right) + O(1) \]
\[\leq m \left( r, \frac{h'}{h} \left( \left( \frac{h'}{h} \right)' - \frac{h'}{h} \frac{\xi'}{\xi} \right) \right) + m \left( r, 2z + (m + n)z^2 \frac{h'}{h} - 2z^2 \frac{\xi'}{\xi} \right) \]
\[+ N \left( r, 2z + (m + n)z^2 \frac{h'}{h} - 2z^2 \frac{\xi'}{\xi} \right) \]
\[\leq N \left( r, \frac{\xi'}{\xi} \right) + S(r, h) + S(r, \xi)\]
\[(2.18) \quad \leq T(r, \xi) + S(r, h) + S(r, \xi).\]

Note that \(h = e^{\gamma(z)}\) is a transcendental entire function, we get from (2.12) that
\[2T(r, \xi) = T(r, \xi^2) + S(r, \xi) = T \left( r, \frac{1}{4} \left( \frac{h'}{h} \right)^2 - \frac{z^2 h^{m+n}}{a_m^2(m + n)^2} \right) + S(r, \xi) \]
\[= N \left( r, \frac{1}{4} \left( \frac{h'}{h} \right)^2 - \frac{z^2 h^{m+n}}{a_m^2(m + n)^2} \right).\]
\[ + m \left( r, \frac{1}{4} \left( \frac{h'}{h} \right)^2 - \frac{z^2 h^{m+n}}{a_m^2 (m+n)^2} \right) + S(r, \xi) \]
\[ \leq (m+n)m(r,h) + N \left( r, \left( \frac{h'}{h} \right)^2 \right) + S(r, h) + S(r, \xi) \]
\[ \leq (m+n)T(r, h) + S(r, h) + S(r, \xi). \]
(2.19)

Combining with (2.18), we have
\[ \frac{(m+n)}{2} T(r, h) = S(r, h), \]
which is a contradiction. Thus, \( \gamma(z) \) is a constant, and so \( h(z) = e^{\gamma(z)} \) is also a constant. From (2.9), we obtain
(2.20)
\[ f(z)g(z) = e^{\alpha(z)}e^{\beta(z)} = c_0, \]
where \( c_0(\neq 0) \) is a constant. So we have
(2.21)
\[ \beta(z) = -\alpha(z) + c_1, \]
for a constant \( c_1 \). Substituting \( f = e^{\alpha(z)} \), \( g = e^{\beta(z)} \) into (2.10), we get from (2.20) and (2.21) that
\[ f(z) = b_1e^{bz^2}, \quad g(z) = b_2e^{-bz^2}, \]
where \( b_1, b_2 \) and \( b \) are three constants that satisfy \( 4a_m^2(b_1b_2)^{n+m}((m+n)b)^2 = -1 \).
If \( k \geq 2 \), then
(2.22)
\[ a_m^2(f^{n+m}(k))g^{n+m}(k) = z^2. \]
Since \( f \) and \( g \) are entire functions and \( n > k + 2 \), by using the arguments similar to the proof of Lemma 2.6, we know from (2.8) that \( f \) and \( g \) have no zeros. Let
(2.23)
\[ f = e^{\alpha(z)}, \quad g = e^{\beta(z)}, \]
where \( \alpha(z), \beta(z) \) are non-constant entire functions. By (2.22), we have
(2.24)
\[ N \left( r, \frac{1}{f^{m+n}(k)} \right) \leq N \left( r, \frac{1}{z^2} \right) = O(\log r). \]
Combining with (2.23) and (2.24), we obtain
\[ N(r, f^{m+n}) + N \left( r, \frac{1}{f^{m+n}} \right) + N \left( r, \frac{1}{f^{m+n}(k)} \right) = O(\log r). \]
By (2.23), \( T(r, \frac{f^{m+n}}{f^{m+n}(k)}) = T(r, (m+n)a') \). If \( \alpha \) is transcendental, We know from Lemma 2.5 that \( f = e^\alpha = e^{az+b} \) for some constants \( a \neq 0 \) and \( b \), which is impossible. Hence \( \alpha \) must be a polynomial, and so \( \beta \) is also a polynomial. We suppose that \( \deg(\alpha) = p \) and \( \deg(\beta) = q \). If \( p = q = 1 \), we have
(2.25)
\[ f = e^{Az+B}, \quad g = e^{Cz+D}, \]
where \( A, B, C \) and \( D \) are constants that satisfy \( AC \neq 0 \). Substituting (2.25) into (2.22), we obtain
\[ a_m^2(m+n)^{2k}(AC)^k e^{(m+n)(A+C)z+(m+n)(B+D)} = z^2, \]
which is impossible. Thus \( \max\{p, q\} > 1 \). Without loss of generality, we suppose that \( p > 1 \). Then \( (f^{m+n})^{(k)} = Q_1(z)e^{(m+n)\alpha} \), where \( Q_1(z) \) is a polynomial of degree \( kp - k \geq k \geq 2 \). From (2.22), we have \( p = k = 2 \) and \( q = 1 \). Suppose that \( f^{m+n} = e^{(m+n)(A_1z^2 + B_1z + C_1)} \), \( g^{m+n} = e^{(m+n)(D_1z + E_1)} \),

where \( A_1, B_1, C_1, D_1, E_1 \) are constants such that \( A_1D_1 \neq 0 \). Then we have

\[
(f^{m+n})'' = (m + n)(4(m + n)A_1^2z^2 + 4(m + n)A_1B_1z + (m + n)B_1^2 + 2A_1)e^{(m+n)(A_1z^2 + B_1z + C_1)},
\]

where \( Q_2(z)e^{(m+n)(A_1z^2 + (B_1 + D_1)z + C_1 + E_1)} = z^2 \),

Substituting (2.26) and (2.27) into (2.22), we have

\[
Q_2(z)e^{(m+n)(A_1z^2 + (B_1 + D_1)z + C_1 + E_1)} = z^2,
\]

where \( Q_2(z) \) is a polynomial of degree 2. Since \( A_1 \neq 0 \), we get a contradiction.

**Case B.** \( P(z) = C \). In this case, by the similar arguments mentioned in the Case A, \( f \) and \( g \) must satisfy \( f(z) = b_1e^{b_2z}, g(z) = b_2e^{-b_2z} \), where \( b_1, b_2, b \) are constants that satisfy \( 4C^2(b_1b_2)^n(nb)^2 = -1 \). Lemma 2.7 follows.

**Lemma 2.8.** Let \( f \) and \( g \) be two non-constant entire functions, \( n, m \) and \( k \) be three positive integers, and let \( F = (f^n(z)P(f))^{(k)}, G = (g^n(z)P(g))^{(k)} \), where \( P(z) \) is given by Theorem 1.5 and not a monomial. If there exist two nonzero constants \( a_1 \) and \( a_2 \) such that \( \overline{N}(r, \frac{1}{f - a_1}) = \overline{N}(r, \frac{1}{G}) \) and \( \overline{N}(r, \frac{1}{f - a_2}) = \overline{N}(r, \frac{1}{G}) \), then \( n \leq 2k + 2 + m \).

**Proof.** By the second fundamental theorem, we have

\[
T(r, F) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - a_1}\right) + S(r, F)
\]

\[
\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, F)
\]

(2.28)

\[
\leq N_1\left(r, \frac{1}{F}\right) + N_1\left(r, \frac{1}{G}\right) + S(r, F).
\]

From (2.28), Lemma 2.1 and Lemma 2.2, we obtain

\[
T(r, F) \leq T(r, F) - T(r, f^n(z)P(f)) + N_{k+1}\left(r, \frac{1}{f^n(z)P(f)}\right)
\]

\[
+ N_{k+1}\left(r, \frac{1}{g^n(z)P(g)}\right) + S(r, f) + S(r, g).
\]

Hence

\[
(n + m)T(r, f) \leq N_{k+1}\left(r, \frac{1}{f^n(z)P(f)}\right) + N_{k+1}\left(r, \frac{1}{g^n(z)P(g)}\right)
\]

\[
+ S(r, f) + S(r, g) \leq (k + 1)\left(\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right)\right)
\]

(2.29)

\[
+ m(T(r, f) + T(r, g)) + S(r, f) + S(r, g).
\]
By the similar reasoning, we have

\[(n + m)T(r, g) \leq (k + 1) \left( \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{g} \right) \right) + m(T(r, f) + T(r, g)) + S(r, f) + S(r, g).\]

From (2.29) and (2.30), we have

\[(n - 2k - 2 - m)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g),\]

which implies that \(n \leq 2k + 2 + m\). Lemma 2.8 is thus proved. \(\blacksquare\)

By the arguments much similar to the proof of Lemma 2.8, we have the following lemma.

**Lemma 2.9.** Let \(f\) and \(g\) be two non-constant entire functions, \(n, m\) and \(k\) be three positive integers, and let \(F = (f^n(z)P(f))^k\), \(G = (g^n(z)P(g))^k\), where \(P(z)\) is given by Theorem 1.5 and \(P(z) = a_mz^m\) or \(P(z) = C\). If there exist two nonzero constants \(a_1\) and \(a_2\) such that \(\mathcal{N}(r, \frac{1}{F-a_1}) = \mathcal{N}(r, \frac{1}{G})\) and \(\mathcal{N}(r, \frac{1}{G-a_2}) = \mathcal{N}(r, \frac{1}{F})\), then \(n \leq 2k + 2 - m^*\).

### 3. Proof of theorems

**Proof of Theorem 1.5.** Set \(F = f^n(z)P(f)\), by Lemma 2.4, we have

\[(3.1) \quad T(r, F^{(k)}) \leq \mathcal{N} \left( r, \frac{1}{F^{(k)}} \right) + \mathcal{N} \left( r, \frac{1}{F^{(k)} - z} \right) + S(r, f).\]

**Case 1.** \(P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0\), where \(a_m \neq 0\). By (3.1) and Lemma 2.2 with \(p = 1\), we obtain

\[(3.2) \quad T(r, F^{(k)}) \leq N_1 \left( r, \frac{1}{F^{(k)}} \right) + \mathcal{N} \left( r, \frac{1}{F^{(k)} - z} \right) + S(r, f),\]

and so

\[T(r, F) \leq N_{k+1} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{F^{(k)} - z} \right) + S(r, f)\]

\[\leq N_{k+1} \left( r, \frac{1}{F^n} \right) + N_{k+1} \left( r, \frac{1}{a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0} \right)\]

\[+ \mathcal{N} \left( r, \frac{1}{F^{(k)} - z} \right) + S(r, f)\]

\[\leq (k + 1 + m)T(r, f) + \mathcal{N} \left( r, \frac{1}{F^{(k)} - z} \right) + S(r, f).\]

Noting that \(T(r, F) = (m + n)T(r, f) + S(r, f)\) and \(n \geq k + 2\), we get \([f^n(z)P(f)]^{(k)}\) has infinitely many fixed points.
Case 2. $P(f) = C$, where $C \neq 0$. By using the same arguments as mentioned above, we have
\[
T(r, F) \leq N_{k+1} \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F^{(k)} - z} \right) + S(r, f)
\]
\[
\leq N_{k+1} \left( r, \frac{1}{Cf^n} \right) + N \left( r, \frac{1}{F^{(k)} - z} \right) + S(r, f)
\]
\[
\leq (k+1)T(r, f) + N \left( r, \frac{1}{F^{(k)} - z} \right) + S(r, f).
\]

Note that $T(r, F) = nT(r, f) + S(r, f)$ and $n \geq k + 2$, we obtain $[f^n(z)P(f)]^{(k)}$ has infinitely many fixed points. Theorem 1.5 follows.

Proof of Theorem 1.6. We consider the following two cases.
(i) $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ is not a monomial. Let
\[(3.3) \quad F = \frac{(f^n(z)P(f))^{(k)}}{z}, \quad G = \frac{(g^n(z)P(g))^{(k)}}{z}. \]
Then $F$ and $G$ are transcendental meromorphic functions that share 1 CM. Let $H$ be given by (2.3). If $H \neq 0$, by Lemma 2.3, we know that (2.4) holds. From Lemma 2.2, we have
\[(3.4) \quad N_2 \left( r, \frac{1}{F} \right) \leq N_{k+2} \left( r, \frac{1}{f^n(z)P(f)} \right) + S(r, f)
\]
\[(3.5) \quad N_2 \left( r, \frac{1}{G} \right) \leq T(r, G) - (m+n)T(r, f) + N_{k+2} \left( r, \frac{1}{g^n(z)P(g)} \right) + S(r, g).
\]
From (3.4) and (3.5), we obtain
\[(3.6) \quad N_2 \left( r, \frac{1}{F} \right) \leq N_{k+2} \left( r, \frac{1}{f^n(z)P(f)} \right) + S(r, f), \]
and
\[(3.7) \quad N_2 \left( r, \frac{1}{G} \right) \leq N_{k+2} \left( r, \frac{1}{g^n(z)P(g)} \right) + S(r, g).
\]
Again, from (3.4) and (3.5), we have
\[(m+n)(T(r, f) + T(r, g)) \leq T(r, F) + T(r, G) - N_2 \left( r, \frac{1}{F} \right) - N_2 \left( r, \frac{1}{G} \right)
\]
\[+ N_{k+2} \left( r, \frac{1}{f^n(z)P(f)} \right) + N_{k+2} \left( r, \frac{1}{g^n(z)P(g)} \right). \]
\[
+ N_{k+2} \left( r, \frac{1}{g^n(z)P(g)} \right) + S(r, f) + S(r, g).
\]

Combining with (3.6), (3.7) and Lemma 2.3, we get
\[
(m + n)(T(r, f) + T(r, g)) \leq 2N_{k+2} \left( r, \frac{1}{f^n(z)P(f)} \right) + 2N_{k+2} \left( r, \frac{1}{g^n(z)P(g)} \right) + S(r, f) + S(r, g)
\]
\[
\leq (2k + 4) \left( N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) \right) + 2N_{k+2} \left( r, \frac{1}{P(f)} \right) + S(r, f) + S(r, g).
\]

(3.8)

Thus, we deduce that
\[
(m + n - 2k - 4 - 2m)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g),
\]
which contradicts the assumption that \( n > 2k + 4 + m \). Therefore \( H \equiv 0 \). Integrating twice, we get from (2.3) that
\[
\frac{1}{F - 1} = \frac{A}{G - 1} + B,
\]
where \( A \neq 0 \) and \( B \) are constants. From (3.9), we have
\[
F = \frac{(B + 1)G + (A - B - 1)}{BG + (A - B)}, \quad G = \frac{(B - A)F + (A - B - 1)}{BF - (B + 1)}.
\]

We consider the following three cases.

**Case 1.** Suppose that \( B \neq 0, -1 \). From (3.10) we have \( N \left( r, \frac{1}{F - \frac{B + 1}{B}} \right) = N(r, G) \). From the second fundamental theorem, we have
\[
T(r, F) \leq N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - \frac{B + 1}{B}} \right) + S(r, F)
\]
\[
= N \left( r, \frac{1}{F} \right) + N(r, G) + S(r, F) \leq N \left( r, \frac{1}{F} \right) + S(r, F).
\]

(3.11)

By (3.11) and the same reasoning as in the proof of (3.4), we obtain
\[
T(r, F) \leq N_1 \left( r, \frac{1}{F} \right) + S(r, f)
\]
\[
\leq T(r, F) - (m + n)T(r, f) + N_{k+1} \left( r, \frac{1}{f^n(z)P(f)} \right) + S(r, f).
\]
Hence
\[
(m + n)T(r, f) \leq (k + 1)N \left( r, \frac{1}{f} \right) + N_{k+1} \left( r, \frac{1}{P(f)} \right) + S(r, f)
\]
\[
\leq (k + m + 1)T(r, f) + S(r, f),
\]
which contradicts \( n > 2k + 4 + m \).
Case 2. Suppose that $B = 0$. From (3.10) we have
\begin{equation}
F = \frac{G + (A - 1)}{A}, \quad G = AF - (A - 1).
\end{equation}
If $A \neq 1$, we get from (3.12) that \( \overline{\mathcal{N}} \left( r, \frac{1}{F - \frac{1}{A - 1}} \right) = \overline{\mathcal{N}} \left( r, \frac{1}{G} \right) \) and \( \overline{\mathcal{N}} \left( r, \frac{1}{F} \right) = \overline{\mathcal{N}}(r, \frac{1}{G + (A - 1)}) \). By Lemma 2.8, we have $n \leq 2k + 2 + m$. This contradicts the assumption that $n > 2k + 4 + m$. Thus $A = 1$ and $F = G$, that is,
\[
(f^n P(f))^{(k)} = (g^n P(g))^{(k)}.
\]
By integration, we have
\[
(f^n(z)P(f))^{(k-1)} = (g^n(z)P(g))^{(k-1)} + a_k^{(k-1)}.
\]
where $a_k^{(k-1)}$ is a constant. If $a_k^{(k-1)} \neq 0$, we get from Lemma 2.8 that $n \leq 2k + m$, which is a contradiction. Hence $a_k^{(k-1)} = 0$. Repeating the same process for $k - 1$ times, we obtain $f^n(z)P(f) = g^n(z)P(g)$, that is
\[
f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0) = g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_1 g + a_0).
\]
Let $h = \frac{f}{g}$. If $h$ is a constant, then substituting $f = gh$ into (3.13), we deduce
\[
a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \cdots + a_0 g^n(h^n - 1) = 0,
\]
which implies $h^d = 1$, where $d = (n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f(z) \equiv t g(z)$ for a constant $t$ such that $t^d = 1$. If $h$ is not a constant, then we know by (3.13) that $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \cdots + a_1 \omega_1 + a_0) - \omega_2^n(a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \cdots + a_1 \omega_2 + a_0)$.

Case 3. Suppose that $B = -1$. From (3.10) we obtain
\begin{equation}
F = \frac{A}{-G + (A + 1)}, \quad G = \frac{(A + 1)F - A}{F}.
\end{equation}
If $A \neq -1$, we obtain from (3.14) that \( \overline{\mathcal{N}} \left( r, \frac{1}{F - \frac{1}{A + 1}} \right) = \overline{\mathcal{N}} \left( r, \frac{1}{G} \right) \), \( \overline{\mathcal{N}}(r, F) = \overline{\mathcal{N}}(r, \frac{1}{G - A - 1}) \). By the same reasoning mentioned in Case 1 and Case 2, we get a contradiction. Hence $A = -1$. From (3.14), we have $FG = 1$, that is
\[
(f^n(z)P(f))^{(k)}(g^n(z)P(g))^{(k)} = z^2,
\]
by Lemma 2.6, this is impossible.

(ii) $P(z) = C$ or $P(z) = a_m z^m$, we distinguish two cases.

Case A. $P(z) = a_m z^m$. In this case, we have $F = (a_m f^{n+m}(z))^{(k)}$ and $G = (a_m g^{n+m}(z))^{(k)}$. Let
\[
F_1 = \frac{(a_m f^{n+m}(z))^{(k)}}{z}, \quad G_1 = \frac{(a_m g^{n+m}(z))^{(k)}}{z}.
\]
Then $F_1$ and $G_1$ share 1 CM. By the similar arguments mentioned in the proof of (i), we have $F_1 \equiv G_1$ or $F_1 G_1 \equiv 1$.

If $F_1 G_1 = 1$, we obtain from Lemma 2.7 that $f(z) = b_1 e^{b z^2}$, $g(z) = b_2 e^{-b z^2}$ for three constants $b_1$, $b_2$ and $b$ that satisfy $4a_m^2 (b_1 b_2)^n ((n+m)b)^2 = -1$.

If $F_1 \equiv G_1$, we get

$$(a_m f^{n+m})^{(k)} = (a_m g^{n+m})^{(k)}.$$ 

By integration, we have

$$(a_m f^{n+m})^{(k-1)} = (a_m g^{n+m})^{(k-1)} + a_{k-1}.$$ 

where $a_{k-1}$ is a constant. If $a_{k-1} \neq 0$, we get from Lemma 2.9 that $n \leq 2k + m$, which is a contradiction. Hence $a_{k-1} = 0$. Repeating the same process for $k-1$ times, we obtain $a_m f^{n+m} = a_m g^{n+m}$, we get that $f \equiv t g$, where $t$ is a constant that satisfies $t^{n+m} = 1$.

**Case B.** $P(z) = C$. In this case, by the similar arguments mentioned in the Case A, $f$ and $g$ must satisfy $f(z) = b_1 e^{b z^2}$, $g(z) = b_2 e^{-b z^2}$, where $b_1$, $b_2$ and $b$ are three constants satisfying $4C^2 (b_1 b_2)^n (nb)^2 = -1$ or $f = t g$ for a constant $t$ such that $t^n = 1$. This completes the proof of Theorem 1.6.

**References**


