A Note on $[r,s,c,t]$-Colorings of Graphs

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Abstract. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset $S$ of $V(G)$ is called an independent set if no two vertices of $S$ are adjacent in $G$. The minimum number of independent sets which form a partition of $V(G)$ is called chromatic number of $G$, denoted by $\chi(G)$. A subset $S$ of $E(G)$ is called an edge cover of $G$ if the subgraph induced by $S$ is a spanning subgraph of $G$. The maximum number of edge covers which form a partition of $E(G)$ is called edge covering chromatic number of $G$, denoted by $\chi_c'(G)$. Given nonnegative integers $r$, $s$, $t$ and $c$, an $[r,s,c,t]$-coloring of $G$ is a mapping $f$ from $V(G) \cup E(G)$ to the color set $\{0, 1, \ldots, k-1\}$ such that the vertices with the same color form an independent set of $G$, the edges with the same color form an edge cover of $G$, and $|f(v_i) - f(v_j)| \geq r$ if $v_i$ and $v_j$ are adjacent, $|f(e_i) - f(e_j)| \geq s$ for every $e_i, e_j$ from different edge covers, $|f(v_i) - f(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively, and the number of edge covers formed by the coloring of edges is exactly $c$. The $[r,s,c,t]$-chromatic number $\chi_{r,s,c,t}(G)$ of $G$ is defined to be the minimum $k$ such that $G$ admits an $[r,s,c,t]$-coloring. In this paper, we present the exact value of $\chi_{r,s,c,t}(G)$ when $\delta(G) = 1$ or $G$ is an even cycle.

Keywords and phrases: $[r,s,c,t]$-coloring, edge covering coloring, chromatic number, $[r,s,t]$-coloring.

1. Introduction

In this paper, all graphs are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of graph $G$, respectively. In a proper vertex coloring of a graph $G$, $v_i$ and $v_j$ are colored differently if they are adjacent. In an edge covering coloring of a graph $G$, $E(G)$ is partitioned into edge covers and different edge covers has different colors. The minimum number of colors such that $G$ admits a proper vertex coloring is the chromatic number $\chi(G)$. The maximum number of colors such that $G$ admits an edge covering coloring is the edge covering coloring chromatic number.
\(\chi_c'(G)\). It is well known that \(\chi(G) \leq \Delta + 1\). For \(\chi_c'(G)\), R. P. Gupta first proved the following theorem.

**Theorem 1.1.** [3] Let \(G\) be a graph. Then

\[ \delta - 1 \leq \chi_c'(G) \leq \delta. \]

Kemnitz and Marangio introduced the \([r, s, t]\)-Colorings of graphs in [4]. Given nonnegative integers \(r, s\) and \(t\), an \([r, s, t]\)-coloring of a graph \(G\) is a function \(c\) from \(V(G) \cup E(G)\) to the color set \([0, 1, \ldots, k]\) such that \(|c(v_i) - c(v_j)| \geq r\) for every two adjacent vertices \(v_i, v_j \in V\), \(|c(e_i) - c(e_j)| \geq s\) for every two adjacent edges \(e_i, e_j \in E\), and \(|c(v_i) - c(e_j)| \geq t\) for every vertex \(v_i\) and its incident edges \(e_j\). The \([r, s, t]\)-chromatic number \(\chi_{r,s,t}(G)\) of \(G\) is the minimum \(k\) such that \(G\) admits an \([r, s, t]\)-coloring. In [4], the authors also gave exact values and some bounds of \(\chi_{r,s}(G)\) when at least one among the three parameters is fixed, for example if \(\min\{r, s, t\} = 0\) or if two of the three parameters are 1. Similarly, we give the definition of \([r, s, c, t]\)-colorings of graphs as follows. Given nonnegative integers \(r, s, t\) and \(c\), an \([r, s, c, t]\)-coloring of \(G\) is a mapping \(f\) from \(V(G) \cup E(G)\) to the color set \([0, 1, \ldots, k]\) such that the vertices with the same color form an independent set of \(G\), the edges with the same color form an edge cover of \(G\), and \(|f(v_i) - f(v_j)| \geq r\) if \(v_i\) and \(v_j\) are adjacent, \(|f(e_i) - f(e_j)| \geq s\) for every \(e_i, e_j\) from different edge covers, \(|f(v_i) - f(e_j)| \geq t\) for all pairs of incident vertices and edges, respectively, and the number of edge covers formed by the coloring of edges is exactly \(c\). The \([r, s, c, t]\)-chromatic number \(\chi_{r,s,c,t}(G)\) of \(G\) is defined to be the minimum \(k\) such that \(G\) admits an \([r, s, c, t]\)-coloring. It is obvious that we only consider the case that \(1 \leq c \leq \chi_c'(G)\), for otherwise either the edges can be colored arbitrarily (if \(c = 0\)) such that \(\chi_{r,s,c,t}(G) = \chi_{r,0,t}(G)\) which has been considered by Kemnitz and Marangio [4] or there is no \([r, s, c, t]\)-coloring of \(G\) if \(c > \chi_c'(G)\).

2. The proofs of the main results

In this section, we give the \([r, s, c, t]\)-chromatic number \(\chi_{r,s,c,t}(G)\) if \(\delta(G) = 1\) or \(G\) is an even cycle. Firstly, we give some basic properties of \([r, s, c, t]\)-coloring of graphs.

**Lemma 2.1.** Let \(G\) be a graph. Given nonnegative integers \(r, r', s, s', t\) and \(t'\). If \(r' \leq r, s' \leq s, t' \leq t\), then \(\chi_{r',s',c,t'}(G) \leq \chi_{r,s,c,t}(G)\) holds for any fixed integer \(c\) with \(1 \leq c \leq \chi_c'(G)\).

**Proof.** An \([r, s, c, t]\)-coloring of \(G\) is by definition also an \([r', s', c, t']\)-coloring of \(G\) if \(r' \leq r, s' \leq s, t' \leq t\).

**Lemma 2.2.** If \(a \geq 0\) is an integer, then \(\chi_{ar,as,c,at}(G) = a(\chi_{r,s,c,t}(G) - 1) + 1\) holds for any fixed integer \(c\) with \(1 \leq c \leq \chi_c'(G)\).

**Proof.** If \(a = 0\) or 1, then the assertion is obvious. Let \(a \geq 2\) and \(f\) be an \([r, s, c, t]\)-coloring of a graph \(G\) with \(\chi_{r,s,c,t}(G)\) colors. If we multiply all assigned labels by \(a\), then we obtain a coloring \(f'\) such that \(f'(x) = af(x)\) for all elements \(x \in V(G) \cup E(G)\) and \(|f'(v_i) - f'(v_j)| = af(v_i) - f(v_j)| \geq ar\) if vertices \(v_i\) and \(v_j\) are adjacent, \(|f'(e_i) - f'(e_j)| = af(e_i) - f(e_j)| \geq as\) if edges \(e_i\) and \(e_j\) belong to different edge covers, and \(|f'(v_i) - f'(e_j)| = af(v_i) - f(e_j)| \geq at\) if \(v_i\) and \(e_j\) are incident, respectively. Furthermore, since \(f\) forms \(c\) edge covers of \(G\), \(f'\) also
forms $c$ edge covers of $G$. Therefore, $f'$ is an $[ar, as, c, at]$-coloring of $G$ with colors in $\{0, 1, \ldots, a(\chi_{r,s,c,t}(G) - 1)\}$ which implies that $a(\chi_{r,s,c,t}(G) - 1) + 1$ is an upper bound of $\chi_{as,as,c,at}(G)$.

On the other hand, assume that $G$ has an $[ar, as, c, at]$-coloring $f$ with $a(\chi_{r,s,c,t}(G) - 1)$ colors ($a \geq 2$). Define a coloring $f'$ by $f'(x) = \lfloor f(x)/a \rfloor$. If, for example, $x_i$ and $x_j$ are adjacent vertices, then $|\lfloor f(x_i)/a \rfloor - \lfloor f(x_j)/a \rfloor| \geq r$ by assumption which implies that $|f'(x_i) - f'(x_j)| = |\lfloor f(x_i)/a \rfloor - \lfloor f(x_j)/a \rfloor| \geq r$. Similar proof can be used when considering $s$ and $t$. Furthermore, since $f$ forms $c$ edge covers of $G$, $f'$ also forms $c$ edge covers of $G$. Therefore, $f'$ is an $[r, s, c, t]$-coloring of $G$ with $\chi_{r,s,c,t}(G) - 1$ colors which contradicts the definition of the $[r, s, c, t]$-chromatic number of $G$.

**Lemma 2.3.** Let $G$ be a graph. Then

\[
\max\{r(\chi(G) - 1) + 1, s(c - 1) + 1, t + 1\} \leq \chi_{r,s,c,t}(G) \leq r(\chi(G) - 1) + s(c - 1) + t + 1.
\]

**Proof.** By Lemma 2.1 and Lemma 2.2, $\chi_{r,s,c,t}(G) \geq \chi_{r,0,c,0}(G) = r(\chi(G) - 1) + 1$ as well as $\chi_{r,s,c,t}(G) \geq \chi_{0,s,c,0}(G) = s(c - 1) + 1$. Obviously, $\chi_{r,s,c,t}(G) \geq t + 1$.

If we color the vertices of $G$ with colors $0, r, \ldots, r(\chi(G) - 1)$ and the edges with colors $r(\chi(G) - 1) + t, r(\chi(G) - 1) + t + s, \ldots, r(\chi(G) - 1) + t + s(c - 1)$. (Note that $1 \leq c \leq \chi'_c(G)$), we obtain an $[r, s, c, t]$-coloring of $G$.

Next, we give the exact value of $\chi_{r,s,c,t}(G)$ where $\delta(G) = 1$ or $G$ is an even cycle. Given a graph $G$ with $\delta(G) = 1$, it is obvious that $\chi'_c(G) = \delta(G) = 1$. Since $1 \leq c \leq \chi'_c(G) = 1$, we can only let $c = 1$. Thus in order to get a $[r, s, c, t]$-coloring of $G$, we can only color all the edges of $G$ with the same color. Then we have the following theorem.

**Theorem 2.1.** Let $c = 1$ and $G$ be a graph with $\delta(G) = 1$. We have

\[
\chi_{r,s,c,t}(G) = \begin{cases} 
    r(\chi(G) - 1) + 1 & \text{if } r \geq 2t, \\
    r(\chi(G) - 2) + 2t + 1 & \text{if } t \leq r < 2t, \\
    r(\chi(G) - 1) + t + 1 & \text{if } r < t.
\end{cases}
\]

**Proof.** (a) By Lemma 2.2, $\chi_{r,s,c,t}(G) \geq r(\chi(G) - 1) + 1$. On the other hand, color the vertices of $G$ with colors $0, r, 2r, \ldots, r(\chi(G) - 1)$ to get a proper vertex coloring and all the edges of $G$ with color $t$. It is easy to see that it forms an $[r, s, c, t]$-coloring of $G$. The color set used by the coloring is $\{0, \ldots, r(\chi(G) - 1)\}$. By definition, $\chi_{r,s,c,t}(G) = r(\chi(G) - 1) + 1$.

(b) Firstly, we prove the lower bound. Given any $[r, s, c, t]$-coloring of $G$, suppose that the vertices of $G$ are colored with colors $f_1, f_2, \ldots, f_m$ where $m \geq \chi(G)$ and all the edges are colored with color $f_0$. If $f_i \leq f_0 \leq f_{i+1}$ holds for some $i \in \{1, 2, \ldots, m - 1\}$, we have $f_m - f_1 = f_m - f_{i+1} + f_{i+1} - f_i + f_i - f_1 \geq (m - 2)r + 2t \geq r(\chi(G) - 2) + 2t$. Otherwise, $f_0 \leq f_1$ or $f_0 \geq f_m$ holds. In this case, at least $(m - 1)r + t$ colors are needed which is greater than $r(\chi(G) - 2) + 2t$. Secondly, we color $G$ as follows. Color all vertices in $G$ with color $0, 2t, 2t + r, \ldots, 2t + r(\chi(G) - 2)$ and all edges of $G$ with color $t$. It is easy to see that it forms an $[r, s, c, t]$-coloring of $G$ with the color set $\{0, \ldots, r(\chi(G) - 2) + 2t\}$. By definition, $\chi_{r,s,c,t}(G) = r(\chi(G) - 2) + 2t$.

(c) The lower bound has already been proved in (b). On the other hand, we color $G$ as follows. Color the vertices of $G$ with color $0, r, \ldots, r(\chi(G) - 1)$ and all edges of $G$ with color $r(\chi(G) - 1) + t$. It is easy to see that it forms an $[r, s, c, t]$-coloring of $G$ with the color set $\{0, \ldots, r(\chi(G) - 1) + t\}$. By definition, $\chi_{r,s,c,t}(G) = r(\chi(G) - 1) + t + 1$.\]
Given an even cycle $C_{2n}$, it is known that $\chi(C_{2n}) = 2, \chi'(C_{2n}) = 2$. So we must discuss two cases, $c = 1$ and $c = 2$. We have the following theorem.

**Theorem 2.2.** Let $c$ be an integer with $1 \leq c \leq 2$ and $C_{2n}$ be a cycle with $2n$ vertices. If $c = 1$, then

$$
\chi_{r,s,c,t}(C_{2n}) = \begin{cases} 
  r + 1 & \text{if } r \geq 2t, \\
  2t + 1 & \text{if } t \leq r < 2t, \\
  r + t + 1 & \text{if } r < t.
\end{cases}
$$

If $c = 2$, then

$$
\chi_{r,s,c,t}(C_{2n}) = \begin{cases} 
  r + 1 & \text{if } r \geq 2t + s, \\
  2t + s + 1 & \text{if } r \geq 2t, t + s \leq r < 2t + s, \\
  r + t + 1 & \text{if } r \geq 2t, s \leq r < t + s, \\
  s + t + 1 & \text{if } r \geq 2t, s - t \leq r < s, \\
  2t + r + 1 & \text{if } r \geq 2t, s - 2t \leq r < s - t, \\
  s + 1 & \text{if } r \leq s - 2t, \\
  2t + r + 1 & \text{if } t \leq r < 2t, t + r \leq s < 2t + r, \\
  s + t + 1 & \text{if } t \leq r < 2t, 2t s < t + r, \\
  3t + 1 & \text{if } t \leq r < 2t, t s < 2t, \\
  2t + s + 1 & \text{if } t \leq r < 2t, s < t, \\
  2t + r + 1 & \text{if } r < t, t s < 2t + r, \\
  r + t + s + 1 & \text{if } r < t, s < t.
\end{cases}
$$

**Proof.** If $c = 1$, the proof is the same as Theorem 2.1.

If $c = 2$, the edges of $C_{2n}$ are colored by two colors alternately. Suppose that $C_{2n} = x_1e_1y_1e'_1x_2e_2y_2e'_2 \ldots x_ie_iy_ie'_i \ldots x_ne_ny_ne'_n x_1$ and $f_1 = f(e_i) \leq f(e'_i) = f_2$, $i = 1, 2, \ldots, n$. It is obvious that $f_2 - f_1 \geq s$. Then we give the proof from (a) to (l).

(a) By Lemma 2.3, $\chi_{r,s,c,t}(C_{2n}) \geq r(\chi(C_{2n}) - 1) + 1 = r + 1$. On the other hand, let $f(x_i) = 0$, $f(y_i) = r$, $f_1 = t$, $f_2 = t + s$ for $i = 1, \ldots, n$. It is easy to see that $f$ forms an $[r, s, c, t]$-coloring of $C_{2n}$. Thus $\chi_{r,s,c,t}(C_{2n}) \leq r + 1$.

Then $\chi_{r,s,c,t}(C_{2n}) = r + 1$.

(b) Firstly we prove that $\chi_{r,s,c,t}(C_{2n}) \geq 2t + s + 1$. Let us pay attention to $x_i, y_i$ for some $i \in \{1, \ldots, n\}$. We might as well suppose $f(x_i) \leq f(y_i)$. If $f(x_i) \leq f(y_i) \leq f_1$, then $f_2 - f(x_i) = (f_2 - f_1) + (f_1 - f(y_i)) + (f(y_i) - f(x_i)) \geq r + t + s \geq 2t + s$. If $f(x_i) \leq f_1 \leq f(y_i) \leq f_2$, then $f_2 - f(x_i) = (f_2 - f(y_i) - f(x_i)) + (f(y_i) - f(x_i)) \geq t + r \geq 2t + s$. If $f(x_i) \leq f_1 \leq f_2 \leq f(y_i)$, then we have $f(y_i) - f(x_i) = (f(y_i) - f_2) + (f_2 - f_1) + (f_1 - f(x_i)) \geq t + s + t = 2t + s$. If $f_1 \leq f(x_i) \leq f(y_i) \leq f_2$, then $f_2 - f_1 = (f_2 - f(y_i) + (f(y_i) - f(x_i)) + (f(x_i) - f_1) \geq t + r \geq 2t + s$. The cases that $f_1 \leq f(x_i) \leq f_2 \leq f(y_i)$ or $f_1 \leq f_2 \leq f(x_i) \leq f(y_i)$ can be proved similarly.

From all the above we can see that $\chi_{r,s,c,t}(C_{2n}) \geq 2t + s + 1$. On the other hand, for $i = 1, \ldots, n$, let $f(x_i) = 0$, $f(y_i) = 2t + s$, $f_1 = t, f_2 = t + s$. It is obvious that $f$ is an $[r, s, c, t]$-coloring of $C_{2n}$. So $\chi_{r,s,c,t}(C_{2n}) = 2t + s + 1$. 
The same arguments can be used to prove the lower bounds from (c) to (l) except (f), so we only need to prove the upper bounds. We shall create an \([r, s, c, t]\)-coloring of \(G\) using given number of colors.

(c) Let \(f(x_i) = 0, f(y_i) = r, f_1 = t, f_2 = r + t\). So \(\chi_{r,s,c,t}(C_{2n}) \leq r + t + 1\).

(d) Let \(f(x_i) = 0, f(y_i) = r, f_1 = t, f_2 = t + s\). So \(\chi_{r,s,c,t}(C_{2n}) \leq s + t + 1\).

(e) Let \(f(x_i) = t, f(y_i) = t + r, f_1 = 0, f_2 = 2t + r\). So \(\chi_{r,s,c,t}(C_{2n}) \leq 2t + r + 1\).

(f) By Lemma 2.3, \(\chi_{r,s,c,t}(C_{2n}) \geq s(c - 1) + 1 = s + 1\). On the other hand, let \(f(x_i) = t, f(y_i) = t + r, f_1 = 0, f_2 = s\) for \(i = 1, \ldots, n\). It is easy to see that \(f\) forms an \([r, s, c, t]\)-coloring of \(C_{2n}\). Thus \(\chi_{r,s,c,t}(C_{2n}) \leq s + 1\). Then \(\chi_{r,s,c,t}(C_{2n}) = s + 1\).

(g) Let \(f(x_i) = t, f(y_i) = t + r, f_1 = 0, f_2 = 2t + r\). So \(\chi_{r,s,c,t}(C_{2n}) \leq 2t + r + 1\).

(h) Let \(f(x_i) = 0, f(y_i) = 2t, f_1 = t, f_2 = s + t\). So \(\chi_{r,s,c,t}(C_{2n}) \leq s + t + 1\).

(i) Let \(f(x_i) = 0, f(y_i) = 2t, f_1 = t, f_2 = 3t\). So \(\chi_{r,s,c,t}(C_{2n}) \leq 3t + 1\).

(j) Let \(f(x_i) = 0, f(y_i) = 2t + s, f_1 = t, f_2 = t + s\). So \(\chi_{r,s,c,t}(C_{2n}) \leq 2t + s + 1\).

(k) Let \(f(x_i) = t, f(y_i) = t + r, f_1 = 0, f_2 = 2t + r\). So \(\chi_{r,s,c,t}(C_{2n}) \leq 2t + r + 1\).

(l) Let \(f(x_i) = 0, f(y_i) = r, f_1 = r + t, f_2 = r + t + s\). So \(\chi_{r,s,c,t}(C_{2n}) \leq r + t + s + 1\).

Note that \([r, s, c, t]\)-coloring of \(C_{2n}\) is also an \([r, s, t]\)-coloring of \(C_{2n}\) when \(c = 2\), so it gives an upper bound of \(\chi_{r,s,c,t}(C_{2n})\).

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