Generalized Fuzzy Compactness in \(L\)-Topological Spaces

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Abstract. In this paper, we shall introduce generalized fuzzy compactness in \(L\)-spaces where \(L\) is a complete de Morgan algebra. This definition does not rely on the structure of basis lattice \(L\) and no distributivity is required. The intersection of a generalized fuzzy compact \(L\)-set and a generalized closed \(L\)-set is a generalized fuzzy compact \(L\)-set. The generalized irresolute image of a generalized fuzzy compact \(L\)-set is a generalized fuzzy compact \(L\)-set.

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1. Introduction and preliminaries

In 1976, Lowen first introduced the concepts of fuzzy compactness in \([0, 1]\)-spaces in [6]. Subsequently its characterization was given by Wang in terms of \(\alpha\)-net in [11]. In 1988, it is again extended to \(L\)-spaces [12], where \(L\) is a completely distributive de Morgan algebra (i.e., a \(F\) lattice). However the above mentioned definitions of fuzzy compactness seriously depend on the structure of the basis lattice \(L\) and complete distributivity was required.

Kubiáčk also extended fuzzy compactness to \(L\)-spaces by means of closed \(L\)-sets and the way below relation in [4], where complete distributivity was not required. But his definition still depend on the structure of the basis lattice \(L\) and can’t be restated in terms of open \(L\)-sets by simply using quasi-complementation.

In [9, 10], a new definition of fuzzy compactness in presented in \(L\)-topological space by means of an inequality, which doesn’t depend on the structure of \(L\) and no distributivity is require in \(L\). When \(L\) is a completely distributive de Morgan algebra, it is equivalent to the notion of fuzzy compactness in [5, 7, 12].

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The notions of generalized open sets, generalized closed sets and generalized-
irresolute mapping were introduced by Balasubramanian and Sundaram in [1].

In this paper, following the lines of [9, 10], we shall introduce a concept of gen-
eralized compactness in \(L\)-topological spaces in terms of generalized open \(L\)-sets and
their inequality, where \(L\) is a complete de Morgan algebra. This definition doesn’t
rely on the structure of basis lattice \(L\) and no distributivity in \(L\) is required. It can
also be characterized by generalized closed \(L\)-sets and their inequality. When \(L\) is a
completely distributive de Morgan algebra, its many characterizations are presented.

Throughout this paper, \((L, \vee, \wedge, ')\) is a complete de Morgan algebra. 0 and 1
denote the smallest element and the largest element in \(L\), respectively.

A complete lattice \(L\) is a complete Heyting algebra if it satisfies the following
infinite distributive law: For all \(L\)

\[ a \wedge \bigvee B = \bigvee \{ a \wedge b \mid b \in B \}. \]

For a nonempty set \(X\), \(L^X\) denotes the set of all \(L\)-topological fuzzy sets (or
\(L\)-sets for short) on \(X\). 0 and 1 denote the smallest element and the largest element
in \(L^X\), respectively. An \(L\)-space (\(L\)-space for short) is a pair \((X, T)\), where \(T\) is a
subfamily \(L^X\) which contains 0, 1 and is closed for any suprema and finite infima.
\(T\) is called an \(L\)-topology on \(X\). Each member of \(T\) is called an open \(L\)-set and its
quasi-complementation is called a closed \(L\)-set. An element \(a\) in \(L\) is called a prime
element if \(b \wedge c \leq a\) implies \(b \leq a\) or \(c \leq a\). \(a\) in \(L\) is called co-prime element if
\(a'\) is a prime element. The set of all nonzero co-prime elements in \(L\) is denoted by
\(M(L)\). It is easy to see that \(M(L^X) = \{ x_\alpha \mid x \in X, \alpha \in M(L) \}\) is exactly the set of
all nonzero \(\vee\)-irreducible elements in \(L^X\).

According to [12], we know that \(L\) is completely distributive if and only if each
element \(a\) in \(L\) has the greatest minimal family (the greatest maximal family), de-
noted by \(\beta(a)(\alpha(a))\). Obviously \(\beta^*(a) = \beta(a) \cap M(L)\) is a minimal family of \(a\) and
\(\alpha^*(a) = \beta(a) \cap P(L)\) is a maximal family of \(a\).

For a subfamily \(\Phi \subset L^X\), \(2^\Phi\) denotes the set of all finite subfamily of \(\Phi\).

In [1], the notions of generalized open sets, generalized closed sets and generalized-
irresolute mapping were introduced in \([0,1]\)-fuzzy set theory by Balasubramanian and
Sundaram. They can easily be extended to \(L\)-sets as follows:

**Definition 1.1.** Let \((X, T)\) be an \(L\)-space and \(A \in L^X\). Then \(A\) is called generalized
closed \(L\)-set (or gl-closed for short) if \(cl(A) \leq U\) whenever \(A \leq U\) and \(U\) is open
\(L\)-set. \(A\) is called generalized open (gl-open for short) if \(A'\) is gl-closed.

\(\text{GLO}(X)\) and \(\text{GLC}(X)\) will always denote the family of all generalized open
\(L\)-sets and family of all generalized closed \(L\)-sets in \(X\), respectively.

**Definition 1.2.** Let \((X, T_1)\) and \((Y, T_2)\) be two \(L\)-spaces, \(f : X \to Y\) be a mapping
and \(f_L^- : L^X \to L^Y\) be the extension of \(f\). Then \(f\) called a generalized irresolute
mapping if \(f_L^-(B)\) is generalized open in \((X, T_1)\) for each generalized open \(L\)-set \(B\)
in \((Y, T_2)\).

**Definition 1.3.** [9, 10] Let \((X, T)\) be an \(L\)-space, \(G \in L^X\). Then \(G\) is called fuzzy
compact if for every family \(U \subset T\), it follows that

\[ \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in U} A(x) \right) \leq \bigvee_{\nu \in 2^{\nu(U)}} \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \nu} A(x) \right). \]
Lemma 1.1. [10] Let \((X, T_1)\) and \((Y, T_2)\) be two \(L\)-spaces, where \(L\) is a complete Heyting algebra, \(f : X \rightarrow Y\) be a mapping, \(f^L : L^X \rightarrow L^Y\) is the extension of \(f\). Then for any \(P \subset L^Y\), we have that

\[
\bigvee_{y \in Y} \left( f^L(G)(y) \land \bigwedge_{B \in P} B(y) \right) = \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in P} f^L(B)(x) \right).
\]

2. Generalized fuzzy compactness of \(L\)-subsets

Definition 2.1. Let \((X, T)\) be an \(L\)-space, \(G \in L^X\). Then \(G\) is called generalized fuzzy compact if for every family \(U \subset GLO(X)\), it follows that

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in U} A(x) \right) \leq \bigvee_{V \in \mathcal{P}(U)} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in V} A(x) \right).
\]

Now we consider characterizations of generalized fuzzy compactness. First we introduce the following concept.

Definition 2.2. Let \((X, T)\) be an \(L\)-space, \(a \in L \setminus \{1\}\) and \(G \in L^X\). A family \(U \subset GLO(X)\) is said to be a generalized open \(a\)-shading of \(G\) if for any \(x \in X\) with \(G(x) \geq a'\), there exists an \(A \in U\) such that \(A(x) \not\leq a\). \(U\) is said to be a generalized open strong \(a\)-shading of \(G\) if

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in U} A(x) \right) \not\leq a
\]

for any \(x \in X\).

Obviously, a generalized open strong \(a\)-shading of \(G\) is a generalized open \(a\)-shading of \(G\) and \(U\) is a generalized open \(a\)-shading of \(G\) if and only if

\[
G'(x) \lor \bigvee_{A \in U} A(x) \not\leq a.
\]

By Definition 2.1 and Definition 2.2 we obtain the following result.

Theorem 2.1. Let \((X, T)\) be an \(L\)-space and \(G \in L^X\). Then \(G\) is generalized fuzzy compact if and only if for any \(a \in L \setminus \{1\}\), each generalized open strong \(a\)-shading \(U\) of \(G\) has a finite subfamily \(V\) which is still a generalized open strong \(a\)-shading of \(G\).

Proof. Suppose that \(G\) is generalized fuzzy compact and for any \(a \in L \setminus \{1\}\), \(U\) is any generalized open strong \(a\)-shading of \(G\). Then

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in U} A(x) \right) \leq \bigvee_{V \in \mathcal{P}(U)} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in V} A(x) \right)
\]

and

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in U} A(x) \right) \not\leq a.
\]
So that
\[ \bigvee_{\mathcal{V} \in 2^{(U)}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \not\leq a, \]
hence there exists \( \mathcal{V} \in 2^{(U)} \) such that
\[ \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \not\leq a. \]
Thus \( \mathcal{V} \) is finite subfamily of \( \mathcal{U} \) and \( \mathcal{V} \) is a generalized open strong \( a \)-shading of \( G \).

Conversely, suppose that for any \( a \in L \setminus \{1\} \), each generalized open strong \( a \)-shading \( \mathcal{U} \) of \( G \) has a finite subfamily \( \mathcal{V} \) which is still a generalized open strong \( a \)-shading of \( G \). Hence we have that
\[ \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a \implies \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \not\leq a, \]
therefore
\[ \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right). \]
Thus we obtain that
\[ \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(U)}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right). \]
Hence \( G \) is generalized fuzzy compact from Definition 2.1.

Moreover from Definition 2.1 we easily obtain the following theorem by simply using quasi-complementation.

**Theorem 2.2.** Let \( (X, T) \) be an \( L \)-space and \( G \in L^X \). Then \( G \) is generalized fuzzy compact if and only if for every subfamily \( \mathcal{P} \subset \text{GLC}(X) \), it follows that
\[ \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right). \]

**Definition 2.3.** Let \( (X, T) \) be an \( L \)-space, \( a \in L \setminus \{1\} \) and \( G \in L^X \). A family \( \mathcal{P} \subset \text{GLC}(X) \) is said to be a generalized closed \( a \)-remote family of \( G \) if for any \( x \in X \) with \( G(x) \geq a \), there exists a \( B \in \mathcal{P} \) such that \( B(x) \not\geq a \). \( \mathcal{P} \) is said to be a generalized closed strong \( a \)-remote family of \( G \) if
\[ \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a. \]
It is obvious that a generalized closed strong \( a \)-remote family of \( G \) is a generalized closed \( a \)-remote family of \( G \), \( \mathcal{P} \) is a generalized closed \( a \)-remote family of \( G \) if and only if
\[ G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \not\geq a \]
and \( \mathcal{P} \) is a generalized closed strong \( a \)-remote family of \( G \) if and only if \( \mathcal{P}' \) is a generalized open strong \( a \)-shading of \( G \).
From Theorem 2.2 we obtain the following result.

**Theorem 2.3.** Let \((X, T)\) be an \(L\)-space and \(G \in L^X\). Then \(G\) is generalized fuzzy compact if and only if for any \(a \in L \setminus \{0\}\), each generalized closed strong \(a\)-remote family \(P\) of \(G\) has a finite subfamily \(F\) which is still a generalized closed strong \(a\)-remote family of \(G\).

**Proof.** Analogous to the proof of Theorem 2.1. \(\square\)

**Theorem 2.4.** Let \(L\) be a complete Heyting algebra. If both \(G\) and \(H\) are generalized fuzzy compact, then \(G \vee H\) is generalized fuzzy compact.

**Proof.** For any family \(P \subset \text{GLC}(X)\), by Theorem 2.2 we have that

\[
\bigvee_{x \in X} \left( (G \vee H)(x) \land \bigwedge_{B \in P} B(x) \right) \\
= \left\{ \bigvee_{x \in X} (G(x) \land \bigwedge_{B \in P} B(x)) \right\} \lor \left\{ \bigvee_{x \in X} (H(x) \land \bigwedge_{B \in P} B(x)) \right\} \\
\geq \left\{ \bigwedge_{\mathcal{F} \in 2(P)} \bigvee_{x \in X} (G(x) \land \bigwedge_{B \in \mathcal{F} \cup \{H\}} B(x)) \right\} \lor \left\{ \bigwedge_{\mathcal{F} \in 2(P)} \bigvee_{x \in X} (H(x) \land \bigwedge_{B \in \mathcal{F}} B(x)) \right\} \\
= \bigwedge_{\mathcal{F} \in 2(P)} \bigvee_{x \in X} (G \vee H)(x) \land \bigwedge_{B \in \mathcal{F}} B(x).
\]

This shows that \(G \vee H\) is generalized fuzzy compact. \(\square\)

**Theorem 2.5.** If \(G\) is a generalized fuzzy compact \(L\)-set and \(H\) is a generalized closed \(L\)-set, then \(G \land H\) is a generalized fuzzy compact \(L\)-set.

**Proof.** Since \(G\) is a generalized fuzzy compact \(L\)-set, for any family \(P \subset \text{GLC}(X)\), by Theorem 2.2 we have that

\[
\bigvee_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in P} B(x) \right) \\
= \bigvee_{x \in X} (G(x) \land \bigwedge_{B \in P \cup \{H\}} B(x)) \geq \bigwedge_{\mathcal{F} \in 2(P \cup \{H\})} \bigvee_{x \in X} (G(x) \land \bigwedge_{B \in P} B(x)) \\
= \left\{ \bigwedge_{\mathcal{F} \in 2(P)} \bigvee_{x \in X} (G(x) \land \bigwedge_{B \in \mathcal{F}} B(x)) \right\} \\
\land \left\{ \bigwedge_{\mathcal{F} \in 2(P)} \bigvee_{x \in X} (G(x) \land (H(x) \land \bigwedge_{B \in \mathcal{F}} B(x))) \right\} \\
= \bigwedge_{\mathcal{F} \in 2(P)} \bigvee_{x \in X} (G \land H)(x) \land \bigwedge_{B \in \mathcal{F}} B(x).
\]

This shows that \(G \land H\) is a generalized fuzzy compact \(L\)-set. \(\square\)
Theorem 2.6. Let \((X, T_1)\) and \((Y, T_2)\) be two \(L\)-spaces, where \(L\) is a complete Heyting algebra, \(f : X \to Y\) be a generalized irresolute mapping. If \(G\) is generalized fuzzy compact in \((X, T_1)\), then so is \(f_L^{-1}(G)\) is in \((Y, T_2)\).

Proof. For any \(\mathcal{P} \subseteq \text{GLC}(X)\), by Lemma 1.1 and Theorem 2.2, we have that

\[
\bigvee_{y \in Y} \left( f_L^{-1}(G)(y) \land \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} f_L^{-1}(B)(x) \right)
\geq \bigwedge_{\mathcal{F} \in \mathfrak{P}(\mathcal{P})} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} f_L^{-1}(B)(x) \right)
= \bigwedge_{\mathcal{F} \in \mathfrak{P}(\mathcal{P})} \bigvee_{y \in Y} \left( f_L^{-1}(G)(y) \land \bigwedge_{B \in \mathcal{F}} B(y) \right).
\]

Therefore \(f_L^{-1}(G)\) is generalized fuzzy compact.

3. Some characterizations of generalized fuzzy compact

In this section, we assume that \(L\) is a completely distributive de Morgan algebra. We give many characterizations of generalized fuzzy compact.

Theorem 3.1. Let \((X, T)\) be an \(L\)-space and \(G \in L^X\). Then the following conditions are equivalent:

1. \(G\) is generalized fuzzy compact;
2. For any \(a \in L \setminus \{0\}\), each generalized closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily \(\mathcal{F}\) which is a generalized closed strong \(a\)-remote family of \(G\);
3. For any \(a \in L \setminus \{0\}\), each generalized closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily \(\mathcal{F}\) which is a generalized closed \(a\)-remote family of \(G\);
4. For any \(a \in L \setminus \{0\}\), each generalized closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily \(\mathcal{F}\) and \(b \in \beta(a)\) such that \(\mathcal{F}\) is a generalized closed strong \(b\)-remote family of \(G\);
5. For any \(a \in L \setminus \{0\}\), each generalized closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily \(\mathcal{F}\) of \(\mathcal{P}\) and \(b \in \beta(a)\) such that \(\mathcal{F}\) is a generalized closed \(b\)-remote family of \(G\);
6. For any \(a \in M(L)\), each generalized closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily \(\mathcal{F}\) which is a generalized closed strong \(a\)-remote family of \(G\);
7. For any \(a \in M(L)\), each generalized closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily \(\mathcal{F}\) which is a generalized closed \(a\)-remote family of \(G\);
8. For any \(a \in M(L)\), each generalized closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily \(\mathcal{F}\) of \(\mathcal{P}\) and \(b \in \beta^*(a)\) such that \(\mathcal{F}\) is a generalized closed strong \(b\)-remote family of \(G\);
9. For any \(a \in M(L)\), each generalized closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily \(\mathcal{F}\) of \(\mathcal{P}\) and \(b \in \beta^*(a)\) such that \(\mathcal{F}\) is a generalized closed \(b\)-remote family of \(G\).
Proof. By Theorem 2.3 we can obtain \((1)\iff(2)\). \((2)\implies(3)\) is obvious. Now to prove \((3)\implies(4)\), suppose that \(a \in L \setminus \{0\}\) and \(\mathcal{P}\) is a generalized closed strong \(a\)-remote family of \(G\), then we obtain that
\[
\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a,
\]
take \(c \in \beta(a)\) such that
\[
\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq c,
\]
Obviously \(\mathcal{P}\) is a strong generalized closed \(c\)-remote family of \(G\), by \((3)\) we know that \(\mathcal{P}\) has a finite subfamily \(\mathcal{F}\) which is a generalized closed \(c\)-remote family of \(G\). Take \(b \in \beta(c)\) such that \(\mathcal{F}\) is a generalized closed strong \(b\)-remote family of \(G\). \((4)\) is shown. \((4)\implies(5)\) is obvious, we prove \((5)\implies(2)\). For any \(a \in L \setminus \{0\}\), suppose that \(\mathcal{P}\) is any generalized closed strong \(a\)-remote family of \(G\), by \((5)\), \(\mathcal{P}\) has a finite subfamily \(\mathcal{F}\) and \(b \in \beta(a)\) such that \(\mathcal{F}\) is a generalized closed \(b\)-remote family of \(G\). So that for any
\[
x \in X, G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \not\geq b,
\]
we obtain
\[
\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \not\geq a,
\]
in fact, if
\[
\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \geq a,
\]
then by \(b \in \beta(a)\), there exists \(x_0 \in X\) such that
\[
G(x_0) \land \bigwedge_{B \in \mathcal{F}} B(x_0) \geq b,
\]
a contradiction. So that
\[
\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \not\geq a.
\]
This implies that \(\mathcal{F}\) is a generalized closed strong \(a\)-remote family of \(G\). Similarly we can prove that \((2)\implies(6)\implies(7)\implies(8)\implies(9)\implies(1)\).

Now we present some characterizations of generalized fuzzy compactness by means of generalized open \(L\)-sets.

Theorem 3.2. Let \((X, T)\) be an \(L\)-space and \(G \in L^X\). Then the following conditions are equivalent:

1. \(G\) is generalized fuzzy compact;
2. For any \(a \in L \setminus \{1\}\), each generalized open strong \(a\)-shading \(U\) of \(G\) has a finite subfamily \(\mathcal{V}\) which is a generalized open strong \(a\)-shading of \(G\);
3. For any \(a \in L \setminus \{1\}\), each generalized open strong \(a\)-shading \(U\) of \(G\) has a finite subfamily \(\mathcal{V}\) which is a generalized open \(a\)-shading of \(G\);
(4) For any \(a \in L \setminus \{1\}\), each generalized open strong \(a\)-shading \(U\) of \(G\), there exists a finite subfamily \(V\) of \(U\) and \(b \in \alpha(a)\) such that \(V\) is a strong generalized open \(b\)-shading of \(G\);

(5) For any \(a \in L \setminus \{1\}\), each generalized open strong \(a\)-shading \(U\) of \(G\), there exists a finite subfamily \(V\) of \(U\) and \(b \in \alpha(a)\) such that \(V\) is a generalized open \(b\)-shading of \(G\);

(6) For any \(a \in P(L)\), each generalized open strong \(a\)-shading \(U\) of \(G\) has a finite subfamily \(V\) which is a generalized open \(a\)-shading of \(G\);

(7) For any \(a \in P(L)\), each generalized open strong \(a\)-shading \(U\) of \(G\) has a finite subfamily \(V\) which is a generalized open \(a\)-shading of \(G\);

(8) For any \(a \in P(L)\), each generalized open strong \(a\)-shading \(U\) of \(G\) has a finite subfamily \(V\) of \(U\) and \(b \in \alpha^*(a)\) such that \(V\) is a strong generalized open \(b\)-shading of \(G\);

(9) For any \(a \in P(L)\), each generalized open strong \(a\)-shading \(U\) of \(G\) has a finite subfamily \(V\) of \(U\) and \(b \in \alpha^*(a)\) such that \(V\) is a generalized open \(b\)-shading of \(G\).

Proof. By Theorem 2.1 we can obtain (1) \(\iff\) (2).

(2) \(\implies\) (3) is obvious.

(3) \(\implies\) (4). Suppose that \(a \in L \setminus \{1\}\) and \(U\) is a generalized open strong \(a\)-shading of \(G\), then

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{B \in U} B(x) \right) \not\leq a.
\]

Take \(c \in \alpha(a)\) such that

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{B \in U} B(x) \right) \not\leq c,
\]

obviously \(U\) is a generalized open strong \(c\)-shading of \(G\) and by (3) we know that \(U\) has a finite subfamily \(V\) which is a generalized open \(c\)-shading of \(G\). Take \(b \in \alpha(a)\) such that \(c \in \alpha(b)\), then \(V\) is a generalized open strong \(b\)-shading of \(G\), (4) is shown.

(4) \(\implies\) (5) is obvious.

(5) \(\implies\) (2). For any \(a \in L \setminus \{1\}\), suppose that \(U\) is any generalized open strong \(a\)-shading of \(G\), by (5), \(U\) has a finite subfamily \(V\) and \(b \in \alpha(a)\) such that \(V\) is a generalized open \(b\)-shading of \(G\). So that for any \(x \in X\),

\[
G'(x) \lor \bigvee_{B \in V} B(x) \not\leq b,
\]

we obtain

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{B \in V} B(x) \right) \not\leq a,
\]

in fact, if

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{B \in V} B(x) \right) \leq a,
\]
then by \( b \in \alpha(a) \), there exists \( x_0 \in X \) such that 
\[
G(x_0) \lor \bigvee_{B \in V} B(x_0) \leq b,
\]
a contradiction. So that

\[
\bigwedge_{x \in X} \left(G'(x) \lor \bigvee_{B \in V} B(x)\right) \not\leq a.
\]

This implies that \( V \) is a generalized open strong \( a \)-shading of \( G \).
Similarly we can prove that (2)\( \Rightarrow \)(6)\( \Rightarrow \)(7)\( \Rightarrow \)(9)\( \Rightarrow \)(1). 

**Definition 3.1.** Let \( (X, T) \) be an \( L \)-space, \( a \in L \setminus \{0\} \) and \( G \in L^X \). A family \( U \subset \text{GLO}(X) \) is said to be a generalized open \( \beta_a \)-cover of \( G \) if for any \( x \in X \) with \( a \notin \beta(G'(x)) \), there exists \( A \in U \) such that \( a \in \beta(A(x)) \). \( U \) is said to be a generalized open strong \( \beta_a \)-cover of \( G \) if

\[
a \in \beta\left(\bigwedge_{x \in X} \left(G'(x) \lor \bigvee_{A \in U} A(x)\right)\right).
\]

It is obvious that a generalized open strong \( \beta_a \)-cover of \( G \) is generalized open \( \beta_a \)-cover \( G \) and \( U \) is a generalized open \( \beta_a \)-cover of \( G \) if and only if for any \( x \in X \),

\[
a \in \beta\left(\bigwedge_{x \in X} \left(G'(x) \lor \bigvee_{A \in U} A(x)\right)\right).
\]

**Theorem 3.3.** Let \( (X, T) \) be an \( L \)-space and \( G \in L^X \). Then the following conditions are equivalent:

1. \( G \) is generalized fuzzy compact;
2. For any \( a \in L \setminus \{0\} \), each generalized open strong \( \beta_a \)-cover \( U \) of \( G \) has a finite subfamily \( V \) which is a generalized open strong \( \beta_a \)-cover of \( G \);
3. For any \( a \in L \setminus \{0\} \), each generalized open strong \( \beta_a \)-cover \( U \) of \( G \) has a finite subfamily \( V \) which is a generalized open \( \beta_a \)-cover of \( G \);
4. For any \( a \in L \setminus \{0\} \), any generalized open strong \( \beta_a \)-cover \( U \) of \( G \), there exists a finite subfamily \( V \) of \( U \) and \( b \in L \) with \( a \in \beta(b) \) such that \( V \) is a generalized open strong \( \beta_a \)-cover of \( G \);
5. For any \( a \in L \setminus \{0\} \), any generalized open strong \( \beta_a \)-cover \( U \) of \( G \), there exists a finite subfamily \( V \) of \( U \) and \( b \in L \) with \( a \in \beta(b) \) such that \( V \) is a generalized open \( \beta_a \)-cover of \( G \);
6. For any \( a \in M(L) \), each generalized open strong \( \beta_a \)-cover \( U \) of \( G \) has a finite subfamily \( V \) which is a generalized open strong \( \beta_a \)-cover of \( G \);
7. For any \( a \in M(L) \), each generalized open strong \( \beta_a \)-cover \( U \) of \( G \) has a finite subfamily \( V \) which is a generalized open \( \beta_a \)-cover of \( G \);
8. For any \( a \in M(L) \) and any generalized open strong \( \beta_a \)-cover \( U \) of \( G \), there exists a finite subfamily \( V \) of \( U \) and \( b \in M(L) \) with \( a \in \beta^*(b) \) such that \( V \) is a generalized open strong \( \beta_a \)-cover of \( G \);
9. For any \( a \in M(L) \) and any generalized open strong \( \beta_a \)-cover \( U \) of \( G \), there exists a finite subfamily \( V \) of \( U \) and \( b \in M(L) \) with \( a \in \beta^*(b) \) such that \( V \) is a generalized open \( \beta_a \)-cover of \( G \).
Proof. We only prove (1) \iff (2).

(1) \implies (2). Suppose that $G$ is generalized fuzzy compact and for any $a \in L \setminus \{0\}$, $\mathcal{U}$ is any generalized open strong $\beta_a$-cover of $G$. Then

$$\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \right).$$

So

$$\beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \right) \leq \beta \left( \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \right) \right).$$

By

$$a \in \beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \right),$$

we obtain

$$a \in \beta \left( \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \right) \right),$$

therefore

$$a \in \bigcup_{\mathcal{V} \in 2^{(\mathcal{U})}} \beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \right),$$

hence there exists a $\mathcal{V} \in 2^{(\mathcal{U})}$ such that

$$a \in \beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \right).$$

Thus $\mathcal{V}$ is a generalized open strong $\beta_a$-cover of $G$.

(2) \implies (1). Suppose that for any $a \in L \setminus \{0\}$, each generalized open strong $\beta_a$-cover $\mathcal{U}$ of $G$ has a finite subfamily $\mathcal{V}$ which is a generalized open strong $\beta_a$-cover of $G$, then we know that

$$a \in \beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \right) \ implies \ that \ a \in \beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \right)$$

where $\mathcal{V} \in 2^{(\mathcal{U})}$. Hence

$$\beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \right) \leq \beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \right).$$

Thus

$$\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} A(x) \right) \right).$$

This prove that $G$ is generalized fuzzy compact.
Definition 3.2. Let \((X, T)\) be an \(L\)-space, \(a \in L \setminus \{0\}\) and \(G \in L^X\). A family \(U \subset \text{GLO}(X)\) is said to be a generalized open \(Q_a\)-cover of \(G\) if for any \(x \in X\) it follows that
\[
G' \lor \bigvee_{A \in U} A(x) \geq a.
\]

It is obvious that a generalized open \(\beta_a\)-cover of \(G\) is a generalized open \(Q_a\)-cover of \(G\). Moreover from Definition 2.1 we also can obtain the following result.

Theorem 3.4. Let \((X, T)\) be an \(L\)-space and \(G \in L^X\). Then the following conditions are equivalent:

1. \(G\) is generalized fuzzy compact;
2. For any \(a \in L \setminus \{0\}\) and any \(b \in \beta(a) \setminus \{0\}\), each generalized open \(Q_a\)-cover of \(G\), has a finite subfamily which is a generalized open \(Q_b\)-cover of \(G\);
3. For any \(a \in L \setminus \{0\}\) and any \(b \in \beta(a) \setminus \{0\}\), each generalized open \(Q_a\)-cover of \(G\), has a finite subfamily which is a generalized open \(\beta_a\)-cover of \(G\);
4. For any \(a \in L \setminus \{0\}\) and any \(b \in \beta(a) \setminus \{0\}\), each generalized open \(Q_a\)-cover of \(G\), has a finite subfamily which is a generalized open strong \(\beta_a\)-cover of \(G\);
5. For any \(a \in M(L)\) and any \(b \in \beta^*(a)\), each generalized open \(Q_a\)-cover of \(G\), has a finite subfamily which is a generalized open \(Q_b\)-cover of \(G\);
6. For any \(a \in M(L)\) and any \(b \in \beta^*(a)\), each generalized open \(Q_a\)-cover of \(G\), has a finite subfamily which is a generalized open \(\beta_b\)-cover of \(G\);
7. For any \(a \in M(L)\) and any \(b \in \beta^*(a)\), each generalized open \(Q_a\)-cover of \(G\), has a finite subfamily which is a generalized open strong \(\beta_b\)-cover of \(G\).

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References