Degree Conditions of Fractional ID-$k$-Factor-Critical Graphs

1 Renying Chang, 2 Guizhen Liu and 3 Yan Zhu
1 Department of Mathematics, Linyi Normal University, Linyi, Shandong, 276005 P. R. China
2 School of Mathematics and System Sciences, Shandong University, Jinan, Shandong, 250100 P. R. China
3 Department of Mathematics, East China University of Science and Technology, Shanghai, 200237 P. R. China
1 changrysd@163.com, 2 gzliu@sdu.edu.cn, 3 operationzy@163.com

Abstract. We say that a simple graph $G$ is fractional independent-set-deletable $k$-factor-critical, shortly, fractional ID-$k$-factor-critical, if $G - I$ has a fractional $k$-factor for every independent set $I$ of $G$. Some sufficient conditions for a graph to be fractional ID-$k$-factor-critical are studied in this paper. Furthermore, we show that the result is best possible in some sense.

2010 Mathematics Subject Classification: 05C70

Key words and phrases: fractional $k$-factor, independent set, fractional ID-$k$-factor-critical.

1. Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The minimum degree of $G$ is denoted by $\delta(G)$. For any vertex $x$ of $G$, the neighborhood of $x$ is denoted by $N_G(x)$, the degree of $x$ is denoted by $d_G(x)$, and we write $N_G[x]$ for $N_G(x) \cup \{x\}$. We use $G[S]$ and $G - S$ to denote the subgraph of $G$ induced by $S$ and $V(G) - S$, respectively, for $S \subseteq V(G)$. The join $G \vee H$ of disjoint graphs $G$ and $H$ is the graph obtained from $G + H$ by joining each vertex of $G$ to each vertex of $H$. Notations and definitions not given in this paper can be found in [1].

A subset $I$ of $V(G)$ is said to be independent if no two distinct vertices in $I$ are adjacent. A matching in a graph is a set of edges, no two of which meet a common vertex. A matching is perfect if it covers all vertices of the graph. A graph $G$ is factor-critical [5] if $G - v$ has a perfect matching for every vertex $v \in V(G)$. In [7], the concept of factor-critical graph was generalized to the ID-factor-critical graph. We say that $G$ is independent-set-deletable factor-critical (shortly, ID-factor-critical)
if for every independent set \( I \) of \( G \) which has the same parity with \(|V(G)|\), \( G - I \) has a perfect matching.

Let \( h : E(G) \rightarrow [0, 1] \) be a function, and let \( k \geq 1 \) be an integer. If \( \sum_{e \in x} h(e) = k \) holds for each vertex \( x \in V(G) \), we call \( G[F_h] \) a fractional \( k \)-factor of \( G \) with indicator function \( h \) where \( F_h = \{ e \in E(G) \mid h(e) > 0 \} \). A fractional 1-factor is also called a fractional perfect matching [6]. We say that \( G \) is fractional ID-\( k \)-factor-critical if for every independent set \( I \) of \( G \), \( G - I \) has a fractional \( k \)-factor. When \( k = 1 \), we say that \( G \) is fractional ID-factor-critical if for every independent set \( I \) of \( G \), \( G - I \) has a fractional perfect matching.

Liu and Zhang gave a necessary and sufficient condition for a graph to have fractional \((g,f)\)-factor and a \(k\)-factor in [4] and [8], respectively.

**Lemma 1.1.** Let \( G \) be a graph. Then \( G \) has a fractional \( k \)-factor if and only if for every subset \( S \) of \( V(G) \), \( \Phi_G(S;k) = k|S| - k|T| + d_{G-S}(T) \geq 0 \), where \( T = \{ x : x \in V(G) - S, d_{G-S}(x) \leq k - 1 \} \).

**Lemma 1.2.** Let \( G \) be a graph. Then \( G \) has a fractional \( k \)-factor if and only if for every subset \( S \) of \( V(G) \), \( k|S| - \sum_{i=0}^{k-1} (k-i)p_i(G - S) \geq 0 \), where \( p_i(G-S) = |\{ x : x \in V(G) - S, d_{G-S}(x) = i \}| \).

The degree condition of ID-factor-critical graphs was studied in [3].

**Lemma 1.3.** Let \( G \) be a graph with \( n \) vertices. Then \( G \) is ID-factor-critical if \( \delta(G) \geq (2n - 1)/3 \).

In this paper, we discuss the degree conditions of fractional ID-\( k \)-factor-critical graphs. The main results will be given in the next section.

2. Main results

We begin our discussion with a well-known theorem of Dirac [2].

**Lemma 2.1.** Let \( G \) be a graph on \( n \geq 3 \) vertices with \( \delta(G) \geq n/2 \). Then \( G \) is hamiltonian.

The next result follows easily from Lemma 2.1.

**Lemma 2.2.** If \( G \) is a graph of order \( n \) and \( \delta(G) \geq 2n/3 \), then \( G \) is fractional ID-\( k \)-factor-critical when \( k = 1, 2 \).

*Proof.* Let \( I \) be an independent set of \( G \). It is easy to see that \( n - |I| \geq \delta(G) \). Hence

\[
2\delta(G) - |I| - n = 2\delta(G) + n - |I| - 2n \\
\geq 3\delta(G) - 2n \geq 0.
\]

It follows that \( \delta(G) - |I| \geq (n - |I|)/2 \).

Let \( H = G - I \). Then \( |V(H)| = n - |I| \), and \( \delta(H) \geq \delta(G) - |I| \geq |V(H)|/2 \). By Lemma 2.1, \( H \) has a hamiltonian cycle \( C \). \( C \) is also a fractional 2-factor and \( C \) also contains a fractional perfect matching. Thus Lemma 2.2 holds.

**Theorem 2.1.** Let \( k \) be a positive integer and \( G \) be a graph of order \( n \) with \( n \geq 6k - 8 \). If \( \delta(G) \geq 2n/3 \), then \( G \) is fractional ID-\( k \)-factor-critical.
Proof. Let $X$ be an independent set of $G$ and $H = G - X$. We have that $|V(H)| = n - |X|$ and $\delta(H) \geq |V(H)|/2$ by the same argument of Lemma 2.2. Clearly, Theorem 2.1 holds when $k = 1$ or $k = 2$. Therefore, we may assume $k \geq 3$.

We prove the theorem by contradiction. Suppose $H$ has no fractional $k$-factor. Then by Lemma 1.1, there exists some subset $S \subseteq V(H)$ such that $\Phi_H(S; k) = k|S| - k|T| + d_{H-S}(T) \leq -1$, where $T = \{x \mid x \in V(H) - S, d_{H-S}(x) \leq k - 1\}$. Set $\Psi_H(S; k) = \Phi_H(S; k) + 1$. It follows that $\Psi_H(S; k) \leq 0$.

Let $h_1 = \min\{d_{H-S}(x)\mid x \in T\}$. Choose $x_1 \in T$ such that $d_{H-S}(x_1) = h_1$. If $T - N_T[x_1] \neq \emptyset$, let $h_2 = \min\{d_{H-S}(x)\mid x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$.

Set $|S| = s$, $|T| = t$, and $|N_T[x_1]| = p$. We have $p \leq h_1 + 1$, $d_{H-S}(T) \geq h_1p + h_2(t-p)$, and

$$0 \geq \Psi_H(S; k) = ks - kt + d_{H-S}(T) + 1 \geq ks - kt + h_1p + h_2(t-p) + 1.$$ 

Set $|V(H)| = m$. Then $m = n - |X| \geq \delta(G) \geq 2n/3 \geq (12k - 16)/3 = 4k - 16/3$. Since $m$ is an integer, we have that $m \geq 4k - 5$.

We consider the following cases.

Case 1. $T = N_T[x_1]$.

In this case, we have $t = p \leq h_1 + 1$, $0 \leq h_1 \leq k - 1$, $h_2 = 0$. By $\delta(H) \geq (n - |X|)/2 \geq n/3 \geq (6k - 8)/3 \geq k$ ($k \geq 3$) and $d_H(x_1) \leq s + h_1$, we have $s \geq k - h_1$ and

$$\Psi_H(S; k) \geq ks - kt + h_1p + h_2(t-p) + 1 = ks + (h_1 - k)t + 1 \geq k(k - h_1) + (h_1 - k)t + 1 = (k - h_1)(k - t) + 1 \geq 1.$$ 

Then we get a contradiction.

Case 2. $T - N_T[x_1] \neq \emptyset$.

Subcase 2.1. $0 \leq h_1 \leq 2$.

In this case, we have $t > p$, $0 \leq h_1 \leq h_2$, $m/2 \leq d_H(x_1) \leq s + h_1$. Then $s \geq m/2 - h_1 \geq (4k - 5)/2 - h_1 = 2k - 5/2 - h_1$. Since $s$ is an integer and $m - s - t \geq 0$, we have $s \geq 2k - 2 - h_1$, $t \leq m - s \leq s + 2h_1$. Then we obtain that

$$0 \geq \Psi_H(S; k) \geq ks - kt + h_1p + h_2(t-p) + 1 \geq ks - kt + h_1t + 1 = ks + (h_1 - k)t + 1 \geq ks + (h_1 - k)(s + 2h_1) + 1 = h_1s + 2h_1^2 - 2h_1k + 1 \geq h_1(2k - 2 - h_1) + 2h_1^2 - 2h_1k + 1 = h_1^2 - 2h_1 + 1 = (h_1 - 1)^2.$$ 

\[\Box\]
When \( h_1 = 0 \) or \( h_1 = 2 \) (since \( 0 \leq h_1 \leq 2 \) and \( h_1 \) is an integer), we have
\[
0 \geq \Psi_H(S; k) \geq 1,
\]
a contradiction.

When \( h_1 = 1 \), we have \( \Psi_H(S; k) \geq 0 \) and we notice that \( \Psi_H(S; k) = 0 \) holds if and only if \( s = 2k - 2 - h_1 = 2k - 3 \) and \( t = s + 2h_1 = 2k - 1 \). Then \( m \leq 2s + 2h_1 = 4k - 4 \) and \( m \geq s + t = 4k - 4 \), so \( m = 4k - 4 = s + t \). Therefore \( H = G[S \cup T] \) and \( |N_T(x)| = n = h_1 + 1 = 2 \), \( |N_T(x)| = 1 \).

So for every vertex \( v \in T \), \( |N_T(v)| \geq |N_T(x)| \geq 1 \), and \( t = 2k - 1 \) is odd, it follows that there exists a vertex \( u \in T \) such that \( |N_T(u)| \geq 2 \).

\[
0 \geq \Psi_H(S; k) = ks - kt + d_{H-S}(T) + 1
\]
\[
\geq ks - kt + (t - 1) + 2 + 1
\]
\[
= k(2k - 3) - k(2k - 1) + (2k - 1 - 1) + 3
\]
\[
= 1,
\]
a contradiction, too.

**Subcase 2.2.** \( h_1 \geq 3 \).

In this case, \( 3 \leq h_1 \leq h_2 \leq k - 1 \). Then \( k - h_2 \geq 1 \) and \( m - s - t \geq 0 \). Thus \( (k - h_2)(m - s - t) \geq 0 \). So
\[
(k - h_2)(m - s - t) \geq \Psi_H(S; k)
\]
\[
\geq ks - kt + h_1p + h_2(t - p) + 1
\]
\[
= ks + (h_1 - k)p + (h_2 - k)(t - p) + 1.
\]

It follows that
\[
(2.1) \quad (k - h_2)(m - s) - ks \geq (h_1 - h_2)(h_1 + 1) + 1.
\]

Since \( m \geq 4k - 5 \), we have
\[
(2.2) \quad h_2m \geq h_2(4k - 5).
\]

Furthermore, since \( m/2 \leq d_H(x_1) \leq s + h_1 \) and \( m/2 \leq d_H(x_2) \leq s + h_2 \), we have \( 2s - m \geq -(h_1 + h_2) \). Then we can obtain that
\[
(2.3) \quad (2s - m)(2k - h_2) \geq -(h_1 + h_2)(2k - h_2).
\]

By (2.2) + (2.3) + 2 \times (2.1), we get
\[
0 \geq h_2(4k - 5) - (h_1 + h_2)(2k - h_2) + 2(h_1 - h_2)(h_1 + 1) + 2
\]
\[
= 2h_2(h_2 - h_1)k + h_2(h_2 + h_1 - 5) + 2(h_1 - h_2)(h_1 + 1) + 2.
\]

Set \( \Omega(k) = 2(h_2 - h_1)k + h_2(h_2 + h_1 - 5) + 2(h_1 - h_2)(h_1 + 1) + 2 \). Then we obtain
\[
(2.4) \quad 0 \geq \Omega(k).
\]

Since \( k \geq h_2 + 1 \) and \( \Omega(k) \) is a nondecreasing function for \( k (h_2 \geq h_1) \), then we obtain that \( \Omega(k) \geq \Omega(h_2 + 1) = 3h_2^2 - 3h_2 + 5h_2 + 2h_2^2 + 16 \). Set \( A = (3h_1 + 5)^2 - 12(2h_2^2 + 2) = -15(h_1 - 1)^2 + 16 \). And \( \Delta < 0 \) when \( h_1 \geq 3 \). It follows that \( \Omega(h_2 + 1) > 0 \) and \( \Omega(k) \geq \Omega(h_2 + 1) > 0 \), which contradicts (2.4).
The above arguments yield that $H$ has a fractional $k$-factor and $G$ is fractional ID-$k$-factor-critical. The proof is completed.

In [8] we have the following result about fractional $k$-factors.

**Theorem 2.2.** If $G$ has fractional $k$-factors, then $G$ has fractional $m$-factor for $1 \leq m \leq k$.

Theorem 2.2 implies immediately the following result.

**Theorem 2.3.** If a graph $G$ is fractional ID-$k$-factor-critical, then $G$ is fractional ID-$m$-factor-critical for $1 \leq m \leq k$.

3. **The sharpness of the bounds in Theorem 2.1**

In this section we show that the conditions in Theorem 2.1 are best possible.

Let $G = (2k - 4)K_1 \vee (2k - 3)K_1 \vee (k - 1)K_2$. Then we have $n = |V(G)| = 6k - 9$ and $\delta(G) = 4k - 6 \geq 2n/3$. Clearly, $A = (2k - 3)K_1$ is an independent set of $G$.

Let $H = G - A = (2k - 4)K_1 \vee (k - 1)K_2$. Choose $S = (2k - 4)K_1$. Then $
\sum_{i=0}^{k-1} (k-i)p_i(H-S) = (k-1)(2k-2) = k(2k-4) + 2 > k(2k-4) = k|S|.$

Therefore, by Lemma 1.2, $H$ has no fractional $k$-factor. Hence $G$ is not fractional ID-$k$-factor-critical. In this sense, the bound of $n$ is best possible.

This bound of $\delta(G)$ is sharp indeed. To see this, we construct a graph $G$ with $\delta(G) = \lceil 2n/3 \rceil - 1$ which is not fractional ID-$k$-factor-critical as follows.

**Case 1.** $n = 3m$.

In this case, let $G = (m-1)K_1 \vee mK_1 \vee (m+1)K_1$, $n = |V(G)| = 3m$ and $\delta(G) = 2m - 1 = \lceil 2n/3 \rceil - 1$.

Clearly, $A = (m-1)K_1$ is an independent set of $G$. Let $H = G - A = mK_1 \vee (m+1)K_1$. Choose $S = mK_1$. Then $\sum_{i=0}^{k-1} (k-i)p_i(H-S) = k(m+1) > km = k|S|.$

By Lemma 1.2, $H$ has no fractional $k$-factor. So $G$ is not fractional ID-$k$-factor-critical.

**Case 2.** $n = 3m + 1$.

In this case, let $G = mK_1 \vee mK_1 \vee (m+1)K_1$, $n = |V(G)| = 3m + 1$ and $\delta(G) = 2m + 1 = \lceil 2n/3 \rceil - 1$. Clearly, $A = mK_1$ is an independent set of $G$. Let $H = G - A = mK_1 \vee (m+1)K_1$. By the same argument as above, $H$ has no fractional $k$-factor. Thus $G$ is not fractional ID-$k$-factor-critical.

**Case 3.** $n = 3m + 2$.

In this case, let $G = (m+1)K_1 \vee mK_1 \vee (m+1)K_1$, $n = |V(G)| = 3m + 2$ and $\delta(G) = 2m + 1 = \lceil 2n/3 \rceil - 1$. Clearly, $A = (m+1)K_1$ is an independent set of $G$. We obtain that $G$ is not fractional ID-$k$-factor-critical by the same argument as above.

When $k = 1$, let $G$ be a graph and let $I$ be an arbitrary independent set of $G$. If $I$ has the same parity with $|V(G)|$, we have known that if $\delta(G) \geq (2n - 1)/3$, then $G$ is ID-factor-critical, that is, $G - I$ has a perfect matching [3]. Obviously, $G - I$ has a fractional perfect matching. If $I$ does not have the same parity with $|V(G)|$, we have known that if $\delta(G) \geq 2n/3$, $G$ is fractional ID-factor-critical, that is, $G - I$
has a fractional perfect matching. Hence the bound of $\delta(G)$ is sharp by the above argument.

References


