Fuzzy Prime Ideals in $\Gamma$-rings

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Abstract. In this paper, we study fuzzy prime ideal of a $\Gamma$-ring via its operator rings. We obtain some characterisations of fuzzy prime ideal of a $\Gamma$-ring.

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1. Introduction

The notion of fuzzy ideal in $\Gamma$-ring was introduced by Jun and Lee [10]. They studied some preliminary properties of fuzzy ideals of $\Gamma$-rings. Later Hong and Jun [11] defined normalised fuzzy ideal and fuzzy maximal ideal in $\Gamma$-ring and studied them. Dutta and Chanda [2], studied the structures of the set of fuzzy ideals of a $\Gamma$-ring and characterise $\Gamma$-field, Noetherian $\Gamma$-ring, etc. with the help of fuzzy ideals via operator rings of $\Gamma$-ring. Jun [12] defined fuzzy prime ideal of a $\Gamma$-ring and obtained a number of characterisations for a fuzzy ideal to be a fuzzy prime ideal. In this paper, we prove a characterisation of a fuzzy prime ideal, already obtained by Jun in [12], in different way and also get few more new characterisations of fuzzy prime ideal. Lastly, we obtain a one-one correspondence between the set of all fuzzy prime ideals of a $\Gamma$-ring and the set of all fuzzy prime ideals of the operator rings of the $\Gamma$-ring.

2. Some basic definitions and examples

Definition 2.1. [1] Let $M$ and $\Gamma$ be two additive abelian groups. $M$ is called a $\Gamma$-ring if there exists a mapping $f : M \times \Gamma \times M \rightarrow M$, $f(a, \alpha, b)$ is denoted by $a\alpha b$, $a, b \in M$, $\alpha \in \Gamma$, satisfying the following conditions for all $a, b, c \in M$ and for all $\alpha, \beta, \gamma \in \Gamma$; $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$, and $a\alpha(b\beta c) = (a\alpha b)\beta c$.

Definition 2.2. [1] A subset $A$ of a $\Gamma$-ring $M$ is called a left (resp. right) ideal of $M$ if $A$ is an additive subgroup of $M$ and $maa \in A$ (resp. $aam \in A$) for all $m \in M$, $\alpha \in \Gamma$, $a \in A$. If $A$ is a left and a right ideal of $M$, then $A$ is called a two sided ideal of $M$ or simply an ideal of $M$.

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Definition 2.3. [9] Let $M$ be a $\Gamma$ ring and $F$ be the free abelian group generated by $\Gamma \times M$. Then $A = \sum_{i} n_{i} (\gamma_{i}, x_{i}) \in F : a \in M \Rightarrow \sum_{i} n_{i} a_{i} x_{i} = 0$ is a subgroup of $F$. Let $R = F/A$ be the factor group of $F$ by $A$. Let us denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified that $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ and $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$, for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication in $R$ by $\sum_{i}[\alpha_{i}, x_{i}] \sum_{j}[\beta_{j}, y_{j}] = \sum_{i,j}[\alpha_{i}, x_{i}\beta_{j}, y_{j}]$. Then $R$ forms a ring. This ring $R$ is called the right operator ring of the $\Gamma$ ring $M$. Similarly we can construct left operator ring $L$ of $M$. For the subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_{i}[\gamma_{i}, x_{i}] \in R$ where $\gamma_{i} \in \Phi$ and $x_{i} \in N$ and we denote by $[(\Phi, N)]$ the set of all elements $[\phi, x]$ in $R$, where $\phi \in \Phi$ and $x \in N$. Thus in particular, $R = [\Gamma, M]$ and $L = [M, \Gamma]$. If there exists an element $\sum_{i}[\delta_{i}, e_{i}] \in R$ such that $\sum_{i} x_{e_{i}} = x$ for every element $x \in M$ then it is called the right unity of $M$. It can be verified that $\sum_{i}[\delta_{i}, e_{i}]$ is the unity of $R$. Similarly we can define the left unity $\sum_{j}[f_{j}, \gamma_{j}]$ which is the unity of the left operator ring $L$.

Definition 2.4. [10] A nonempty fuzzy subset $\mu$ (i.e., $\mu(x) \neq 0$ for some $x \in M$) of a $\Gamma$ ring $M$ is called a fuzzy left (resp. right) ideal of $M$ if (i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, (ii) $\mu(xay) \geq \mu(y)$ (resp. $\mu(xay) \geq \mu(x)$) for all $x, y \in M$ and for all $\alpha \in \Gamma$.

A non-empty fuzzy subset $\mu$ of a $\Gamma$-ring $M$ is called a fuzzy ideal if it is a fuzzy left ideal and a fuzzy right ideal of $M$.

Let $M$ be a $\Gamma$-ring and $R$ and $L$ be the right operator ring and the left operator ring of $M$ respectively.

Definition 2.5. [2] For a fuzzy subset $\mu$ of $R$, we define a fuzzy subset $\mu^{*}$ of $M$ by $\mu^{*}(a) = \inf_{\gamma \in \Gamma} \mu([\gamma, a])$, where $a \in M$. For a fuzzy subset $\sigma$ of $M$ we define a fuzzy subset $\sigma^{*}$ of $R$ by $\sigma^{*}(\sum_{i}[\alpha_{i}, a_{i}]) = \inf_{m \in M} \sigma(\sum_{i}m\alpha_{i}a_{i})$, where $\sum_{i}[\alpha_{i}, a_{i}] \in R$. For a fuzzy subset $\delta$ of $L$, we define a fuzzy subset $\delta^{+}$ of $M$ by $\delta^{+}(a) = \inf_{\gamma \in \Gamma} \delta([a, \gamma])$, where $a \in M$. For a fuzzy subset $\eta$ of $M$ we define a fuzzy subset $\eta^{+}$ of $L$ by $\eta^{+}(\sum_{i}[a_{i}, \alpha_{i}]) = \inf_{m \in M} \eta(\sum_{i}a_{i}\alpha_{i}m)$, where $\sum_{i}[a_{i}, \alpha_{i}] \in L$.

Definition 2.6. [2] Let $\mu$, $\sigma$ be two fuzzy subsets of $M$. Then the sum $\mu \oplus \sigma$ and composition $\mu \circ \sigma$ of $\mu$ and $\sigma$ are defined as follows:

$$(\mu \oplus \sigma)(x) = \begin{cases} \sup_{x = u + v} \min[\mu(u), \sigma(v)], & u, v \in M \\ 0, & \text{otherwise.} \end{cases}$$

$$(\mu \circ \sigma)(x) = \begin{cases} \sup \min \min[\mu(u_{i}), \sigma(v_{i})], & 1 \leq i \leq n, x = \sum_{i=1}^{n} u_{i} \gamma_{i} v_{i}, \\ u_{i}, v_{i} \in M \text{ and } \gamma_{i} \in \Gamma, & \text{otherwise.} \end{cases}$$

Definition 2.7. [12] Let $\mu$, $\sigma$ be two fuzzy subsets of $M$. Then the product $\mu \Gamma \sigma$ of $\mu$ and $\sigma$ is defined by

$$(\mu \Gamma \sigma)(x) = \begin{cases} \sup_{x = u \gamma v} \min[\mu(u), \sigma(v)], & u, v \in M \text{ and } \gamma \in \Gamma \\ 0, & \text{otherwise.} \end{cases}$$
We denote the set of all fuzzy ideals of $M$, the set of all fuzzy ideals of $R$, the set of all fuzzy prime ideals of $M$, the set of all fuzzy prime ideals of $R$ by $FI(M)$, $FI(R)$, $FPI(M)$, $FPI(R)$ respectively.

**Definition 2.8.** [1] Let $M$ be a Γ-ring. A proper ideal $P$ of $M$ is called prime if for all pairs of ideals $S$ and $T$ of $M$, $S \cap T \subseteq P$ implies that $S \subseteq P$ or $T \subseteq P$.

**Remark 2.1.** [7] If $P$ is an ideal of a Γ-ring $M$, then the following conditions are equivalent:

(i) $P$ is a prime ideal of $M$;

(ii) If $a, b \in M$ and $a\Gamma b \subseteq P$ then $a \in P$ or $b \in P$.

**Definition 2.9.** [7] A fuzzy ideal $\mu$ of a ring $R$ is said to be prime if $\mu$ is a non-constant function and for any two fuzzy ideals $\sigma$ and $\delta$ of $R$, $\sigma \Gamma \delta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\delta \subseteq \mu$.

**Definition 2.10.** [8] Let $f$ be a mapping from a Γ-ring $M$ onto a Γ-ring $N$. Let $\mu \in FI(M)$. Now $\mu$ is said to be $f$-invariant if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$, for all $x, y \in M$.

**Definition 2.11.** [1] A function $f : M \to N$, where $M, N$ are Γ-rings is said to be a Γ-homomorphism if $f(a + b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$, for all $a, b \in M$, $\alpha \in \Gamma$.

**Definition 2.12.** [8] A fuzzy subset $\mu$ of a Γ-ring $M$ is called a fuzzy point if $\mu(x) \in [0, 1]$ for some $x \in M$ and $\mu(y) = 0$ for all $y \in M \setminus \{x\}$. If $\mu(x) = \beta$, then the fuzzy point $\mu$ is denoted by $x_\beta$.

**Definition 2.13.** [12] A non-constant fuzzy ideal $\mu$ of a Γ-ring $M$ is called a fuzzy prime ideal of $M$ if for any two fuzzy ideals $\sigma$ and $\theta$ of $M$, $\sigma \Gamma \theta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

3. Fuzzy Prime ideal in Γ-ring

**Theorem 3.1.** Let $\mu \in FI(M)$. Then $\mu$ is a fuzzy prime ideal of $M$ if and only if $\mu$ is non-constant and $\sigma \circ \theta \subseteq \mu$ where $\sigma, \theta \in FI(M)$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

**Proof.** The theorem follows since $\sigma \circ \theta \subseteq \mu$ if and only if $\sigma \Gamma \theta \subseteq \mu$ where $\sigma, \theta \in FI(M)$.

**Theorem 3.2.** Let $M$ be a commutative Γ-ring and $\mu \in FI(M)$. Then the following are equivalent:

(i) $x_r \Gamma y_t \subseteq \mu \Rightarrow x_r \subseteq \mu \lor y_t \subseteq \mu$ where $x_r$ and $y_t$ are two fuzzy points of $M$.

(ii) $\mu$ is a fuzzy prime ideal of $M$.

**Proof.** (i) $\Rightarrow$(ii) Let $\sigma, \theta \in FI(M)$ such that $\sigma \Gamma \theta \subseteq \mu$. Suppose $\sigma \not\subseteq \mu$. Then there exists $x \in M$ such that $\sigma(x) > \mu(x)$. Let $\sigma(x) = a$. Let $y \in M$ and $\theta(y) = b$. If $z = x\gamma y$ for some $\gamma \in \Gamma$, then $(x_a \Gamma y_b)(z) = \min\{a, b\}$. Hence $\mu(z) = \mu(x\gamma y) \geq (\sigma \Gamma \theta)(x\gamma y) \geq \min\{\sigma(x), \theta(y)\} = \min\{a, b\} = (x_a \Gamma y_b)(x\gamma y)$. If $(x_a \Gamma y_b)(z) = 0$ then $\mu(z) \geq (x_a \Gamma y_b)(z)$. Hence $x_a \Gamma y_b \subseteq \mu$. By (i) either $x_a \subseteq \mu$ or $y_b \subseteq \mu$. That is either $a \leq \mu(x)$ or $b \leq \mu(y)$. Since $a \not\leq \mu(x)$, $\theta(y) = b \leq \mu(y)$. So $\theta \subseteq \mu$. Thus $\mu$ is a fuzzy.
prime ideal of $M$.
(ii) $\Rightarrow$ (i) Suppose that $\mu$ is a fuzzy prime ideal of a commutative $\Gamma$ ring $M$. Suppose $x_r$ and $y_t$ be two fuzzy points of $M$ such that $x_r, \Gamma y_t \subseteq \mu$. Then

\[
(x_r, \Gamma y_t) (x \gamma y) = \min \{r, t\} \leq \mu(x \gamma y) \text{ for all } \gamma \in \Gamma.
\]

Let fuzzy subsets $\sigma, \theta$ be defined by

\[
\sigma(z) = \begin{cases} r, & \text{if } z \in \langle x \rangle, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \theta(z) = \begin{cases} t, & \text{if } z \in \langle y \rangle, \\ 0, & \text{otherwise}. \end{cases}
\]

Clearly $\sigma, \theta$ are fuzzy ideals of $M$. Now $(\sigma \Gamma \theta)(z) = \sup_{u=\gamma v} [\min(\sigma(u), \theta(v))] = \min \{r, t\}$, where $u \in \langle x \rangle$ and $v \in \langle y \rangle$. Hence $(\sigma \Gamma \theta)(z) = \min \{r, t\} \leq \mu(u \gamma v)$ from (a), when $z = u \gamma v$, where $u \in \langle x \rangle$, $v \in \langle y \rangle$. Otherwise $(\sigma \Gamma \theta)(z) = 0$. Hence $\sigma \Gamma \theta \subseteq \mu$. As $\mu$ is prime, $\sigma \subseteq \mu$ or $\theta \subseteq \mu$. Then $x_r \subseteq \sigma \subseteq \mu$ or $y_t \subseteq \theta \subseteq \mu$. Thus $x_r, \Gamma y_t \subseteq \mu$ implies that either $x_r \subseteq \mu$ or $y_t \subseteq \mu$.

**Theorem 3.3.** Let $I$ be an ideal of a $\Gamma$-ring $M$, $\alpha \in [0,1)$ and $\mu$ be a fuzzy subset of $M$ defined by

\[
\mu(x) = \begin{cases} 1, & \text{if } x \in I, \\ \alpha, & \text{if } x \notin I. \end{cases}
\]

Then $\mu$ is a fuzzy prime ideal of $M$ if and only if $I$ is a prime ideal of $M$.

**Proof.** Let $I$ be a prime ideal of $M$. Obviously $\mu$ is non-constant. If $\min \{\mu(a), \mu(b)\} = \alpha$, then $\mu(a - b) \geq \min \{\mu(a), \mu(b)\}$. If $\min \{\mu(a), \mu(b)\} = 1$, then $\mu(a) = \mu(b) = 1$. So $a, b \in I$ which implies that $a - b \in I$. So $\mu(a - b) = 1$. Hence for all $a, b \in M$, $\mu(a - b) \geq \min \{\mu(a), \mu(b)\}$. Similarly $\mu(a \gamma b) \geq \mu(a), \mu(b)$. Thus $\mu$ is a fuzzy ideal of $M$. Let $\sigma, \theta \in FI(M)$ be such that $\sigma \Gamma \theta \subseteq \mu$ and $\sigma \not\subseteq \mu, \theta \not\subseteq \mu$. Then there exist $x, y \in M$ such that $\sigma(x) > \mu(x)$, $\theta(y) > \mu(y)$. This implies that $\mu(x) = \mu(y) = \alpha$. Therefore $x, y \notin I$. Since $I$ is a prime ideal of $M$, $x \Gamma M \not\subseteq I$ [6]. Then there exist $m \in M, \gamma_1, \gamma_2 \in \Gamma$, such that $x \gamma_1 m \gamma_2 y \notin I$. Hence $\mu(x \gamma_1 m \gamma_2 y) = \alpha$. Now $(\sigma \Gamma \theta)(x \gamma_1 m \gamma_2 y) \geq \min \{\sigma(x), \theta(m \gamma_2 y)\} \geq \min \{\sigma(x), \theta(y)\} > \min \{\mu(x), \mu(y)\} = \alpha = \mu(x \gamma_1 m \gamma_2 y)$, a contradiction. Thus $\mu$ is prime.

Conversely let $\mu$ be a fuzzy prime ideal and $P, Q$ be two ideals of $M$ such that $P \Gamma Q \subseteq I$. Let $P \not\subseteq I$ and $Q \not\subseteq I$ and let $p \in P \setminus I$ and $q \in Q \setminus I$. We define fuzzy subsets $\sigma, \theta$ of $M$ as follows

\[
\sigma(x) = \begin{cases} 1, & \text{if } x \in P, \\ \alpha, & \text{if } x \notin P \end{cases} \quad \text{and} \quad \theta(x) = \begin{cases} 1, & \text{if } x \in Q, \\ \alpha, & \text{if } x \notin Q. \end{cases}
\]

Then $\sigma, \theta$ are fuzzy ideals of $M$. Since $\sigma(p) = 1 > \alpha = \mu(p)$, $\sigma \not\subseteq \mu$. Similarly $\theta \not\subseteq \mu$. But $\sigma \Gamma \theta \subseteq \mu$, a contradiction. So $I$ is a prime ideal of $M$.

**Corollary 3.1.** [12, Theorem 1, Corollary 1] Let $I$ be an ideal of a $\Gamma$-ring $M$. Then the characteristic function $\chi_I$ of $I$ is a fuzzy prime ideal of $M$ if $I$ is a prime ideal of $M$.

**Theorem 3.4.** [12, Theorem 2, Theorem 3] If $\mu$ is a fuzzy prime ideal of $M$ then the following conditions hold:

\[
\begin{align*}
& (i) \quad \mu(O_M) = 1, \\
& (ii) \quad \text{Im} \mu = \{1, \alpha\}, \quad \alpha \in [0,1),
\end{align*}
\]
(iii) \( \mu_0 = \{ x \in M : \mu(x) = \mu(O_M) \} \) is a prime ideal of \( M \).

Proof. (i) Let \( \mu \) be a fuzzy prime ideal of \( M \). Suppose \( \mu(O_M) < 1 \). Since \( \mu \) is non-constant, there exist \( a \in M \) such that \( \mu(a) < \mu(O_M) \). Let \( \sigma, \theta \in FI(M) \) be defined by

\[
\sigma(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ 0, & \text{if } x \notin \mu_0 \end{cases} \quad \text{and} \quad \theta(x) = \mu(O_M)
\]

for all \( x \in M \). Then \( \sigma \Gamma \theta \subseteq \mu \). Since \( \sigma(O_M) = 1 > \mu(O_M) \) and \( \theta(a) = \mu(O_M) > \mu(a) \), \( \sigma \not\subseteq \mu \) and \( \theta \not\subseteq \mu \). This contradicts the fact that \( \mu \) is a fuzzy prime ideal of \( M \). Hence \( \mu(O_M) = 1 \).

(ii) Now we shall show that \( |Im\mu| = 2 \). Let \( x, y \in M \setminus \mu_0 \) and \( \mu(x) = c, c \neq 0 \). We define fuzzy ideal \( C(x) \) by

\[
C(x)(a) = \begin{cases} c, & \text{if } a \in \langle x \rangle \\ 0, & \text{if } a \notin \langle x \rangle \end{cases}
\]

For \( a \in \langle x \rangle \), \( C(x)(a) = c \leq \mu(a) \). For \( a \notin \langle x \rangle \), \( C(x)(a) = 0 \leq \mu(a) \). Hence \( C(x) \subseteq \mu \). Clearly \( 1_{\langle x \rangle} C \subseteq FI(M) \). Now \( 1_{\langle x \rangle} \not\subseteq \mu \) as \( 1_{\langle x \rangle}(x) = 1 > c = \mu(x) \).

Now \( 1_{\langle x \rangle} \Gamma C = 0 \) or \( c \) for any \( a \in M \). If \( 1_{\langle x \rangle} \Gamma C = 0 \), then clearly \( \mu(a) \geq (1_{\langle x \rangle}) \Gamma C \) and if \( 1_{\langle x \rangle} \Gamma C = c \), then \( a \in \langle x \rangle \); hence \( \mu(a) \geq c = (1_{\langle x \rangle}) \Gamma C \). Thus \( 1_{\langle x \rangle} \Gamma C \subseteq \mu \) and \( 1_{\langle x \rangle} \not\subseteq \mu \), \( C \subseteq \mu \). Now \( \mu(x) = c = C(y) \leq \mu(y) \). Hence \( \mu(x) \leq \mu(y) \). Similarly we can show that \( \mu(y) \leq \mu(x) \). Hence \( \mu(x) = \mu(y) \) for all \( x, y \in M \setminus \mu_0 \). This proves that \( |Im\mu| = 2 \). (iii) Clearly from (i) and (ii) it follows

\[
\mu(x) = \begin{cases} 1, & \text{for } x \in \mu_0, \\ \alpha, & \text{for } x \notin \mu_0 \end{cases}
\]

Then from Theorem 3.3, it follows that \( \mu_0 \) is a prime ideal of \( M \) as \( \mu \) is a fuzzy prime ideal of \( M \).

The converse of the above theorem is also true. We shall prove it later using operator rings of a \( \Gamma \)-ring.

**Lemma 3.1.** [2] If \( \mu \in FI(R) \) (resp. \( FLI(R), FRI(R) \)) then \( \mu^* \in FI(M) \) (resp. \( FLI(M), FRI(M) \)), where \( \mu^* \) is defined by \( \mu^*(m) = \inf_{\gamma \in \Gamma} \mu([\gamma, m]) \), \( m \in M \).

**Lemma 3.2.** [8, Theorem 1.2.48] \( \mu \) is a fuzzy prime ideal of a ring \( R \), if and only if \( \mu(O_R) = 1 \), \( \mu_0 \) is prime ideal of \( R \) and \( \mu(R) = \{1, \alpha\}, \alpha \in [0,1) \).

**Theorem 3.5.** If \( \mu \) be a fuzzy prime ideal of the right operator ring \( R \) of a \( \Gamma \)-ring \( M \), then \( \mu^* \) is a fuzzy prime ideal of \( M \).

Proof. Since \( \mu \) is a fuzzy prime ideal of \( R \), \( \mu(O_R) = 1 \), \( \mu_0 \) is prime ideal of \( R \) and \( \mu(R) = \{1, \alpha\}, \alpha \in [0,1) \) [7]. By definition of \( \mu^* \), it follows that \( |Im\mu^*| = 2 \), \( \mu^*(M) = \{1, \alpha\} \), \( \mu^*(O_M) = \inf_{\gamma \in \Gamma} \mu([\gamma, O_M]) = \mu(O_R) = 1 \). Now we shall prove \( (\mu^*)_0 = (\mu_0)^* \). Let \( x \in (\mu^*)_0 \). Now
Hence $x \in (\mu^*)_0 \iff \mu^*(x) = \mu^*(O_M) = 1$

$\iff \inf_{\gamma \in \Gamma} \mu[\gamma, x] = 1$

$\iff \mu[\gamma, x] = 1 = \mu(O_R)$, for all $\gamma \in \Gamma$

$\iff \gamma \in \Gamma$

$\iff x \in \Gamma$

Therefore $(\mu^*)_0 = (\mu_0)^*$. Since $\mu$ is a fuzzy prime ideal of $R$, $\mu_0$ is a prime ideal of $R$ and hence $(\mu^*)_0 = (\mu_0)^*$ is a prime ideal of $M$ [5]. Then from Theorem 3.3, it follows that $\mu^*$ is a fuzzy prime ideal of $M$.

**Lemma 3.3.** [2] If $\sigma \in FI(M)(\text{resp.} FLI(M), FRI(M))$, then $\sigma^x \in FI(R) (\text{resp.} FLI(R), FRI(R))$, where $\sigma^x$ is defined by $\sigma^x(\Sigma_i[\gamma_i, a_i]) = \inf_{m \in M} \sigma(\Sigma_i m\gamma_i a_i)$.

**Lemma 3.4.** [5] If $P$ is a prime ideal of a $\Gamma$-ring $M$, then $P^{x'}$ is a prime ideal of the right operator ring $R$ of the $\Gamma$-ring $M$.

**Theorem 3.6.** If $\sigma$ be a fuzzy prime ideal of $M$, then $\sigma^{x'}$ is a fuzzy prime ideal of $R$.

**Proof.** Since $\sigma$ is a fuzzy prime ideal of $M$, $\sigma(O_M) = 1$, $\sigma_0$ is prime ideal of $M$ and $\sigma(M) = \{1, \alpha\}$, $\alpha \in [0, 1)$. Now $\sigma^x(\Sigma_i[\gamma_i, a_i]) = \inf_{m \in M} \sigma(\Sigma_i m\gamma_i a_i)$. So $\sigma^x(O_R) = 1$, $\sigma^x(R) = \{1, \alpha\}$. We shall now show that $(\sigma^x)_0 = (\sigma_0)^{x'}$. Now $\Sigma_i[\gamma_i, a_i] \in (\sigma^x)_0$ if and only if $\sigma^x(\Sigma_i[\gamma_i, a_i]) = \sigma^x(O_R)$ if and only if $\inf_{m \in M} \sigma(\Sigma_i m\gamma_i a_i) = 1$ if and only if $\sigma(\Sigma_i m\gamma_i a_i) = 1 = \sigma(O_M)$ for all $m \in M$ if and only if $\Sigma_i m\gamma_i a_i \in \sigma_0$ for all $m \in M$ if and only if $\Sigma_i[\gamma_i, a_i] \in (\sigma_0)^{x'}$. Thus $(\sigma^x)_0 = (\sigma_0)^{x'}$. Since $\sigma_0$ is a prime ideal of $M$, $(\sigma^x)_0 = (\sigma_0)^{x'}$ is a prime ideal of $R$ by Lemma 3.4. Hence $\sigma^{x'}$ is a fuzzy prime ideal of $R$ by Lemma 3.2.

**Theorem 3.7.** The mapping $\mu \rightarrow \mu^*$ defines a one-one correspondence between the set of all fuzzy prime ideals of $R$ and the set of all fuzzy prime ideals of $M$, where $\mu$ is a fuzzy prime ideal of $R$.

**Proof.** Let $\mu$ be a fuzzy prime ideal of $R$. Then $\mu^*$ is a fuzzy prime ideal of $M$ by Theorem 3.5, and $(\mu^*)^{x'}$ is a fuzzy prime ideal of $R$ by Theorem 3.6. We shall show that $\mu = (\mu^*)^{x'}$. For this we first show that $\mu_0 = ((\mu^*)^{x'})_0$. Clearly by definition of $(\mu^*)^{x'}$, $\text{Im}\mu = \text{Im}(\mu^*)^{x'}$. Let $\Sigma_i[\gamma_i, a_i] \in \mu_0$. Now $(\mu^*)^{x'}(\Sigma_i[\gamma_i, a_i]) = \inf_{\gamma \in \Gamma} \inf_{m \in M} \mu(\gamma, \Sigma_i m\gamma_i a_i) = \inf_{\gamma \in \Gamma} \inf_{m \in M} \mu(\gamma, m) \Sigma_i[\gamma_i, a_i] = 1$.

Thus $\mu_0 \subseteq ((\mu^*)^{x'})_0$. Let $\Sigma_i[\gamma_i, a_i] \in ((\mu^*)^{x'})_0$. Then $(\mu^*)^{x'}(\Sigma_i[\gamma_i, a_i]) = 1 = \inf_{\gamma \in \Gamma} \inf_{m \in M} \mu(\gamma, m) \Sigma_i[\gamma_i, a_i])$.

Hence for all $\gamma \in \Gamma, m \in M$, $[\gamma, m] \Sigma_i[\gamma_i, a_i] \in \mu_0$. Now as $\mu_0$ is prime ideal of $R$, either $[\gamma, m] \subseteq \mu_0$ or $\Sigma_i[\gamma_i, a_i] \subseteq \mu_0$. Now as $\mu$ is non-constant, $[\gamma, m] \notin \mu_0$ for all $\gamma \in \Gamma, m \in M$. Hence $\Sigma_i[\gamma_i, a_i] \subseteq \mu_0$. Thus $((\mu^*)^{x'})_0 \subseteq \mu_0$. Hence $\mu_0 = ((\mu^*)^{x'})_0$. As $\mu$ is a fuzzy prime ideal of $R$, $\text{Im}\mu = \{1, \alpha\} = \text{Im}(\mu^*)^{x'}$ where $\alpha \in [0, 1)$. Hence $\mu = (\mu^*)^{x'}$. Now let $\sigma$ be a fuzzy prime ideal of $M$. We shall show that
σ = (σ*)*. Clearly Imσ = Im(σ*)* and both are fuzzy prime ideals of M. We shall first show that σ₀ = ((σ*)*)₀. Let a ∈ σ₀. Then σ(a) = 1. Now σ(mγa) ≥ σ(a), for all m ∈ M, for all γ ∈ Γ. So σ(mγa) = 1 for all m ∈ M, for all γ ∈ Γ. Now (σ*)*(a) = inf m∈M inf γ∈Γ σ(mγa) = 1. So a ∈ ((σ*)*)₀. Thus σ₀ ⊆ ((σ*)*)₀.

Now let a ∈ ((σ*)*)₀. Thus (σ*)* (a) = 1 = inf m∈M inf γ∈Γ σ(mγa). This implies that σ(mγa) = 1, for all m ∈ M, γ ∈ Γ. So mγa ∈ σ₀ i.e., m₁Mγa ⊆ σ₀ for all m₁ ∈ M. Now as σ₀ is a prime ideal of M, either m₁ ∈ σ₀ or a ∈ σ₀. Now m₁ ∉ σ₀ for all m₁ ∈ M as σ is non-constant. Thus a ∈ σ₀. Hence ((σ*)*)₀ ⊆ σ₀. Hence σ₀ = ((σ*)*)₀. As σ is a fuzzy prime ideal of M, σ* is also a fuzzy prime ideal of R. By Lemma 3.2, Imσ* = {1, α}, α ∈ [0, 1]. Since (σ*)*(m) = inf γ∈Γ σ*([γ, m]) where m ∈ M, Im(σ*)* = {1, α} = Imσ where α ∈ [0, 1]. This proves that σ = (σ*)*. Thus μ → μ* is a one-to-one correspondence between the set of all fuzzy prime ideals of R and the set of all fuzzy prime ideals of M.

Similar result holds for the Γ-ring and the left operator ring L of M. As a converse of Theorem 3.5, we have the following theorem.

**Theorem 3.8.** [12, Theorem 4] Let μ be a fuzzy ideal of M. Then μ is fuzzy prime ideal of M if the following conditions hold

(i) μ(O_M) = 1,

(ii) Imμ = {1, α} , α ∈ [0, 1),

(iii) μ₀ = {x ∈ M : μ(x) = μ(O_M)} is a prime ideal of M.

**Proof.** As μ is a fuzzy ideal of M, so μ* is a fuzzy ideal of R, where μ* is defined by μ*(∑ᵢ[γᵢ, αᵢ]) = μ(∑ᵢ mᵢγᵢαᵢ). Clearly if (i) μ(O_M) = 1 then μ*(O_R) = 1,

(ii) Imμ = {1, α} , α ∈ [0, 1) implies Imμ* = {1, α}, α ∈ [0, 1). From Theorem 3.6, (μ*)₀ = (μ₀)*. Now as μ₀ is prime ideal of M, (μ₀)* is a prime ideal of R by Lemma 3.4. Hence (μ*)₀ is prime ideal of R. Hence μ* is a fuzzy prime ideal of R by Lemma 3.2. So (μ*)* = μ is a fuzzy prime ideal of M.

**Lemma 3.5.** If f is a homomorphism of a Γ-ring M onto a Γ-ring N and μ be an f-invariant fuzzy ideal of M, then f(μ₀) = [f(μ)]₀.

**Proof.** Clearly

[f(μ)](O_N) = sup f(x)∈O_N μ(x) = sup f(x)∈f(O_M) μ(x) = sup f(x)∈f(O_M) μ(O_M) = μ(O_M),

since μ is f-invariant. Let y ∈ f(μ₀). Then y = f(x) for some x ∈ μ₀. Hence μ(x) = μ(O_M) = [f(μ)](O_N). Now

[f(μ)](y) = sup f(z)=y μ(z) = sup f(z)=f(x) μ(z) = μ(x) = μ(O_M) = f(μ)(O_N).

Hence y ∈ [f(μ)]₀. Again let f(x) ∈ [f(μ)]₀. Then

f(μ)(O_N) = [f(μ)](f(x)) = sup f(t)=f(x) μ(t) = μ(x). So μ(x) = [f(μ)](O_N) = μ(O_M).

So x ∈ μ₀. Hence f(x) ∈ f(μ₀). Thus f(μ₀) = [f(μ)]₀.

**Lemma 3.6.** [13] Let f be a homomorphism of a Γ-ring M onto a Γ-ring N. If μ is an f-invariant fuzzy ideal of M, then f(μ) is a fuzzy ideal of N.
Theorem 3.9. Let \( f \) be a homomorphism of a \( \Gamma \)-ring \( M \) onto a \( \Gamma \)-ring \( N \). If \( \mu \) is an \( f \)-invariant fuzzy prime ideal of \( M \), then \( f(\mu) \) is a fuzzy prime ideal of \( N \).

Proof. Let \( \mu \) be an \( f \)-invariant fuzzy prime ideal of \( M \). Then \( f(\mu) \) is a fuzzy ideal of \( N \) by Lemma 3.15. Since \( \mu \) is fuzzy prime:

(i) \( \mu(O_M) = 1 \),
(ii) \( \mu(M) = \{1, \alpha\} \), \( \alpha \in [0, 1) \),
(iii) \( \mu_0 = \{x \in M : \mu(x) = \mu(O_M)\} \) is a prime ideal of \( M \).

From the proof of the Lemma 3.5, \( [f(\mu)](O_N) = \mu(O_M) = 1 \). Also by Lemma 3.5, \( [f(\mu)]_0 = f(\mu_0) \) is a prime ideal of \( N \). Now we prove \( [f(\mu)](N) = \{1, \alpha\} \), \( \alpha \in [0, 1) \).

Let \( x \in M \) be such that \( \mu(x) = \alpha \). Then \( [f(\mu)](f(x)) = \sup \mu(z) = \mu(x) = \alpha \), \( f(z) = f(x) \) as \( \mu \) is \( f \)-invariant. Also \( (f(\mu))(O_N) = 1 \). So \( (f(\mu))(N) = \{1, \alpha\} \). By Theorem 3.8, it follows that \( f(\mu) \) is a fuzzy prime ideal of \( N \). \( \blacksquare \)

Lemma 3.7. Let \( f \) be a homomorphism of a \( \Gamma \)-ring \( M \) to a \( \Gamma \)-ring \( N \). If \( \eta \in FI(N) \), then \( f^{-1}(\eta_0) = [f^{-1}(\eta)]_0 \).

Proof. Let \( x \in M \). Now

\[
x \in f^{-1}(\eta_0) \iff f(x) \in \eta_0 \\
\iff \eta(f(x)) = \eta(O_N) = \eta(f(O_M)) \\
\iff f^{-1}(\eta)(x) = f^{-1}(\eta)(O_M) \\
\iff x \in [f^{-1}(\eta)]_0.
\]

Hence \( f^{-1}(\eta_0) = [f^{-1}(\eta)]_0 \). \( \blacksquare \)

Lemma 3.8. [10] Let \( f \) be a homomorphism of a \( \Gamma \)-ring \( M \) onto a \( \Gamma \)-ring \( N \) and \( \eta \in FI(N) \). If \( \eta \) is a fuzzy ideal of \( N \), then \( f^{-1}(\eta) \) is a fuzzy ideal of \( M \).

Theorem 3.10. Let \( f \) be a homomorphism of a \( \Gamma \)-ring \( M \) onto a \( \Gamma \)-ring \( N \) and \( \eta \in FI(N) \). If \( \eta \) is a fuzzy prime ideal of \( N \), then \( f^{-1}(\eta) \) is a fuzzy prime ideal of \( M \).

Proof. By Lemma 3.8, \( f^{-1}(\eta) \) is a fuzzy ideal of \( M \). \( f^{-1}(\eta)(O_M) = \eta(f(O_M)) = \eta(O_N) = 1 \) as \( \eta \) is a fuzzy prime ideal of \( N \). Now \( \eta(N) = \{1, \alpha\} \), where \( \alpha \in [0, 1) \).

Let \( y \in N \) be such that \( \eta(y) = \alpha \), then there exists \( x \in M \) such that \( f(x) = y \) as \( f \) is onto. Now \( f^{-1}(\eta)(x) = \eta(f(x)) = \alpha \). Thus \( f^{-1}(\eta)(M) = \{1, \alpha\} \), \( \alpha \in [0, 1) \). Hence by Lemma 3.5

(i) \( f^{-1}(\eta)(O_M) = 1 \),
(ii) \( |f^{-1}(\eta)(M)| = 2 \),
(iii) \( [f^{-1}(\eta)]_0 \) is a prime ideal of \( M \).

Hence from (i), (ii), (iii) it follows from Theorem 3.8 that \( f^{-1}(\eta) \) is a fuzzy prime ideal of \( M \). \( \blacksquare \)

References