On $K$-Starcompact Spaces

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Abstract. A space $X$ is $K$-starcompact if for every open cover $U$ of $X$, there exists a compact subset $K$ of $X$ such that $St(K, U) = X$, where $St(K, U) = \bigcup\{U \in U : U \cap K \neq \emptyset\}$. In this paper, we investigate the relations between $K$-starcompact spaces and other related spaces. We also study topological properties of $K$-starcompact spaces.

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1. Introduction

By a space, we mean a topological space. Let us recall that a space $X$ is countably compact if every countable open cover of $X$ has a finite subcover. Fleischman [1] defined a space $X$ to be starcompact if for every open cover $U$ of $X$, there exists a finite subset $F$ of $X$ such that $St(F, U) = X$, where $St(F, U) = \bigcup\{U \in U : U \cap F \neq \emptyset\}$, and he proved that every countably compact space is starcompact. Conversely, van Douwen et al. [2] proved that every Hausdorff starcompact space is countably compact, but this does not hold for $T_1$-space (see [3, Example 2.5]. As generalizations of starcompactness, the following classes of spaces were given.

Definition 1.1. (4) A space $X$ is $K$-starcompact if for every open cover $U$ of $X$, there exists a compact subset $K$ of $X$ such that $St(K, U) = X$.

Definition 1.2. (5) A space $X$ is $1\frac{1}{2}$-starcompact if for every open cover $U$ of $X$, there exists a finite subset $V$ of $U$ such that $St(\cup V, U) = X$.

In, a $1\frac{1}{2}$-starcompact space is called 1-starcompact. From the above definitions, it is not difficult to see that every starcompact space is $K$-starcompact and every $K$-starcompact space is $1\frac{1}{2}$-starcompact, but the converses do not hold (see Examples 2.1, 2.2 and 2.3 below).

Throughout the paper, the cardinality of a set $A$ is denoted by $|A|$. For a cardinal $\kappa$, $\kappa^+$ denotes the smallest cardinal greater than $\kappa$. Let $\omega$ be the first infinite cardinal, $\omega_1$ the first uncountable cardinal and $\mathfrak{c}$ the cardinality of continuum. As usual, a

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cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each ordinal \( \alpha, \beta \) with \( \alpha < \beta \), we write \( (\alpha, \beta) = \{ \gamma : \alpha < \gamma < \beta \} \). All unexplained concepts or symbols are standard as in Engelking [6].

2. \( \mathcal{K} \)-starcompact spaces and related spaces

Fleischman [1] proved that every countably compact space is starcompact. By the above definitions, it is clear that every starcompact space is \( \mathcal{K} \)-starcompact and every \( \mathcal{K} \)-starcompact space is \( 1_{\mathcal{K}} \)-starcompact. In [1], van Douwen et al. have proved that every \( 1_{\mathcal{K}} \)-starcompact space is pseudocompact. Since every normal pseudocompact space is countably compact, thus we have the following theorem.

**Theorem 2.1.** Let \( X \) be a normal space. Then, the following conditions are equivalent:

(a) \( X \) is countably compact;
(b) \( X \) is starcompact;
(c) \( X \) is \( \mathcal{K} \)-starcompact;
(d) \( X \) is \( 1_{\mathcal{K}} \)-starcompact;
(e) \( X \) is pseudocompact.

In the following, we show that Theorem 2.1 is not true for the classes of \( T_1 \) or Tychonoff spaces. van Douwen et al. [2] have proved that every starcompact Hausdorff space is countably compact, but this does not hold for \( T_1 \)-spaces (see [3, Example 2.5]). For a Tychonoff space \( X \), the symbol \( \beta(X) \) means the \( \check{\text{C}} \)ech-Stone compactification of \( X \).

**Example 2.1.** There exists a Tychonoff \( \mathcal{K} \)-starcompact space \( X \) which is not starcompact.

**Proof.** Let

\[
X = (\beta(\omega) \times (\omega_1 + 1)) \setminus ((\beta(\omega) \setminus \omega) \times \{\omega_1\}).
\]

To show that \( X \) is \( \mathcal{K} \)-starcompact. Let \( \mathcal{U} \) be an open cover of \( X \). Without loss of generality, we can assume that \( \mathcal{U} \) consists of basic open subsets of \( X \). For each \( n \in \omega \), there exist an \( \alpha_n < \omega_1 \) and \( U_n \in \mathcal{U} \) such that

\[
\{n\} \times (\alpha_n, \omega_1] \subseteq U_n.
\]

Let \( \alpha = \sup\{\alpha_n : n \in \omega\} \). Then, \( \alpha < \omega_1 \). If we put \( K_1 = \beta(\omega) \times \{\alpha + 1\} \), then \( K_1 \) is a compact subset of \( X \) and \( \omega \times \{\omega_1\} \subseteq St(K_1, \mathcal{U}) \), since \((n, \omega_1) \in U_n \in \mathcal{U} \) and \( U_n \cap K_1 \neq \emptyset \), for each \( n \in \omega \). On the other hand, since \( \beta(\omega) \times \omega_1 \) is countably compact, there is a finite subset \( F \) of \( \beta(\omega) \times \omega_1 \) such that

\[
\beta(\omega) \times \omega_1 \subseteq St(F, \mathcal{U}).
\]

If we put \( K = K_1 \cup F \), then \( K \) is a compact subset of \( X \) and \( X = St(K, \mathcal{U}) \). This shows that \( X \) is \( \mathcal{K} \)-starcompact.

However, \( X \) is not countably compact, since \( \{\{n, \omega_1\} : n \in \omega\} \) is a closed discrete infinite subset of \( X \). Thus, \( X \) is not starcompact, since every Hausdorff starcompact is countably compact.
Remark 2.1. Example 2.1 shows that the closed subset \( \{ \langle n, \omega_1 \rangle : n \in \omega \} \) of a Tychonoff \( K \)-starcompact space \( X \) is not \( K \)-starcompact, since it is a closed discrete infinite subset of \( X \).

Example 2.2. There exists a \( 1 \frac{1}{2} \)-starcompact \( T_1 \)-space \( X \) which is not \( K \)-starcompact.

Proof. Let \( X = \omega_1 \cup A \), where \( A = \{ a_\alpha \alpha \in \omega_1 \} \) is a set of cardinality \( \omega_1 \). We topologize \( X \) as follows: \( \omega_1 \) has the usual order topology and is an open subspace of \( X \); a basic neighborhood of a point \( a_\alpha \in A \) takes the form

\[
O_\beta(a_\alpha) = \{ a_\alpha \} \cup (\beta, \omega_1), \quad \text{where} \ \beta < \omega_1.
\]

Then, \( X \) is a \( T_1 \)-space. To show that \( X \) is \( 1 \frac{1}{2} \)-starcompact, let \( U \) be an open cover of \( X \). Without loss of generality, we can assume that \( U \) consists of basic open subsets of \( X \). Thus, it is suffices to show that there exists a finite subset \( V \) of \( U \) such that \( St(\cup \mathcal{V}, U) = X \). Since \( \omega_1 \) is countably compact, then it is \( 1 \frac{1}{2} \)-starcompact. Hence there is a finite subset \( \mathcal{V}_1 \) of \( U \) such that \( \omega_1 \subseteq St(\cup \mathcal{V}_1, U) \). On the other hand, if we pick \( \alpha_0 < \omega_1 \), then there exists a \( U_{\alpha_0} \in U \) such that \( a_{\alpha_0} \in U_{\alpha_0} \). For each \( \alpha < \omega_1 \), there is \( U_{\alpha} \in U \) such that \( a_\alpha \in U_{\alpha} \). Hence we have \( U_\alpha \cap U_\alpha_0 \neq \emptyset \), by the definition of the topology of \( X \). Therefore \( A \subseteq St(U_{\alpha_0}, U) \). If we put \( \mathcal{V} = \mathcal{V}_1 \cup \{ U_{\alpha_0} \} \), then \( \mathcal{V} \) is a finite subset of \( U \) and \( X = St(\cup \mathcal{V}, U) \).

Next, we show that \( X \) is not \( K \)-starcompact. Let us consider the open cover

\[
\mathcal{V} = \{ \omega_1 \} \cup \{ O_\alpha(a_\alpha) : \alpha < \omega_1 \}.
\]

Let \( K \) be a compact subset of \( X \). Since \( A \) is discrete closed in \( X \), there exists an \( \alpha_1 < \omega_1 \) such that

\[
K \cap \{ a_\alpha : \alpha > \alpha_1 \} = \emptyset;
\]

On the other hand, since \( \omega_1 \) is a countably compact space, there exists \( \alpha_2 < \omega_1 \) such that \( K \cap (\alpha_2, \omega_1) = \emptyset \), since \( K \cap \omega_1 \) is compact in \( \omega_1 \). Choose \( \beta > \max\{ \alpha_1, \alpha_2 \} \). Then \( a_\beta \not\in St(K, \mathcal{V}) \), since \( O_\beta(a_\beta) \) is the unique element of \( \mathcal{V} \) containing \( a_\beta \) and \( O_\beta \cap K = \emptyset \). This shows that \( X \) is not \( K \)-starcompact.

Question 1. Is there a \( 1 \frac{1}{2} \)-starcompact Hausdorff (or Tychonoff) space which is not \( K \)-starcompact?

Example 2.3. [2, Example 2.2.5] There exists a pseudocompact Tychonoff space which is not \( 1 \frac{1}{2} \)-starcompact.

3. Topological properties of \( K \)-starcompact spaces

In this section, we study topological properties of \( K \)-starcompact spaces. Example 2.2 shows that a closed subset of a \( K \)-starcompact space \( X \) need not be \( K \)-starcompact. Now, we give an example showing that a regular closed subset of a Tychonoff \( K \)-starcompact space need not be \( K \)-starcompact. Here, a subset \( A \) of a space \( X \) is said to be regular closed in \( X \) if \( A = \text{cl}_X \text{int}_X A \).

Example 3.1. There exists a \( K \)-starcompact Tychonoff space \( X \) having a regular-closed subset which is not \( K \)-starcompact.
Proof. Let $D$ be a discrete space of cardinality $\mathfrak{c}$. Let

$$X = (\beta(D) \times (\mathfrak{c}^+ + 1)) \setminus ((\beta(D) \setminus D) \times \{\mathfrak{c}^+\}).$$

To show that $X$ is $\mathcal{K}$-starcompact. For this end, let $U$ be an open cover of $X$. Since $\beta(D) \times \mathfrak{c}^+$ is countably compact, then it is star-compact. Hence there exists a finite subset $F$ of $\beta(D) \times \mathfrak{c}^+$ such that

$$\beta(D) \times \mathfrak{c}^+ \subseteq \text{St}(F, U).$$

It remains to find a compact subset $K_1$ such that $D \times \{\mathfrak{c}^+\} \subseteq \text{St}(K_1, U)$. For each $d \in D$, there exists a $\alpha_d < \mathfrak{c}^+$ such that $\{d\} \times (\alpha_d, \mathfrak{c}^+]$ is included in some member of $U$. Let $\alpha_0 = \sup\{\alpha_d : d \in D\}$. Then, $\alpha_0 < \mathfrak{c}^+$, since $|D| = \mathfrak{c}$. Thus, if we put $K_1 = \beta(D) \times \{\alpha_0 + 1\}$, then $K_1$ is a compact subset of $X$ and $D \times \{\mathfrak{c}^+\} \subseteq \text{St}(K_1, U)$. If we put $K = F \cup K_1$, then $K$ is a compact subset of $X$ and $X = \text{St}(K, U)$, which completes the proof. Let $\Psi = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space, where $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of $\omega$ with $|\mathcal{R}| = \mathfrak{c}$ (see Mrówka [7]). Then, the space $\Psi$ is not $1_2$-starcompact [2, Example 2.25], and hence, it is not $\mathcal{K}$-starcompact, since every $\mathcal{K}$-starcompact space is $1_2$-starcompact.

Assume $X \cap \Psi = \emptyset$. Define a bijection $f : D \times \{\mathfrak{c}^+\} \to \mathcal{R}$. Let $Y$ be the quotient space obtained from the topological sum $X \oplus \Psi$ by identifying $p$ with $f(p)$ for every $p \in D \times \{\mathfrak{c}^+\}$ and let $\varphi : X \oplus \Psi \to Y$ be the quotient map. Then, $\varphi(\Psi)$ is a regular-closed subspace of $Y$ which is not $\mathcal{K}$-starcompact, since $\varphi(\Psi)$ is homomorphic to $\Psi$.

Next, we show that $Y$ is $\mathcal{K}$-starcompact. For this end, let $U$ be an open cover of $Y$. Then, there exists a compact subset $F_1$ of $\varphi(X)$ such that

$$\varphi(X) \subseteq \text{St}(F_1, U),$$

since $\varphi(X)$ is homomorphic to $X$. Since every infinite subset of $\omega$ has a limit point in $\Psi$, the set $F_2 = Y \setminus \text{St}(F_1, U)$ is finite. Consequently, if we put $K = F_1 \cup F_2$, then $K$ is a compact subset of $Y$ and $Y = \text{St}(K, U)$. Therefore, $Y$ is $\mathcal{K}$-starcompact completing the proof.

Concerning the image and preimage of a $\mathcal{K}$-starcompact space under a continuous map, Ikenaga and Tani [4] have proved the following two theorems.

Theorem 3.1. [4] Let $f : X \to Y$ be a continuous map from a $\mathcal{K}$-starcompact space $X$ onto a space $Y$. Then, $Y$ is a $\mathcal{K}$-starcompact space.

Theorem 3.2. [4] Let $f : X \to Y$ be an open perfect continuous map from a space $X$ onto a $\mathcal{K}$-starcompact space. Then, $X$ is a $\mathcal{K}$-starcompact space.

The following example shows that the condition of open perfect can not be replaced by perfect in the Theorem 3.2. We use the Alexandorff duplicate $A(X)$ of a space $X$. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0 \rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1 \rangle\})$, where $U$ is a neighborhood of $x$ in $X$.

Example 3.2. There exists a perfect onto map $f : X \to Y$ such that $Y$ is a $\mathcal{K}$-starcompact space, but $X$ is not $\mathcal{K}$-starcompact.
Proof. Let $Y$ be the space $X$ in the proof of Example 3.1. Then $Y$ is $\mathcal{K}$-starcompact and has the infinite discrete closed subset $F = D \times \{c^+\}$. Let $X$ be the Alexandroff duplicate $A(Y)$ of $Y$. Then, $X$ is not $\mathcal{K}$-starcompact, since $F \times \{1\}$ is an infinite discrete, open and closed set in $X$. Let $f : X \to Y$ be the natural map. Then, $f$ is a perfect map. This completes the proof.

By Corollary 3.2, we have the following corollary.

**Corollary 3.1.** The product of a $\mathcal{K}$-starcompact space and a compact space is $\mathcal{K}$-starcompact.

However, the product of two $\mathcal{K}$-starcompact spaces need not be $\mathcal{K}$-starcompact. In fact, the product of two countably compact spaces is not necessarily $\mathcal{K}$-star-ompact.

**Example 3.3.** There exist two countably compact spaces $X$ and $Y$ such that $X \times Y$ is not $\mathcal{K}$-starcompact.

**Proof.** We define $X = \bigcup_{\alpha < \omega_1} E_\alpha$, $Y = \bigcup_{\alpha < \omega_1} F_\alpha$, where $E_\alpha$ and $F_\alpha$ are the subsets of $\beta(\omega)$ which are defined inductively by the following conditions:

1. $E_\alpha \cap F_\beta = \omega$ if $\alpha \neq \beta$;
2. $|E_\alpha| \leq c$ and $|F_\alpha| \leq c$;
3. every infinite subset of $E_\alpha$ (resp. $F_\alpha$) has an accumulation point in $E_{\alpha+1}$ (resp. $F_{\alpha+1}$).

Those sets $E_\alpha$ and $F_\alpha$ are well-defined since every infinite closed set in $\beta(\omega)$ has the cardinality $2^c$. Then, $X \times Y$ is not $\mathcal{K}$-starcompact, because the diagonal $\{\langle n, n \rangle : n \in \omega\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $\omega$ and $\mathcal{K}$-starcompactness is preserved by open and closed subset.

We end this section by the following theorem. For a space $X$ let $l(X)$ denote the *Lindelöf number* of the space $X$; that is, the smallest cardinal number $\kappa$ such that every open cover of $X$ has an open refinement $\mathcal{V}$ with $|\mathcal{V}| \leq \kappa$.

**Theorem 3.3.** Every Tychonoff space can be embedded in a $\mathcal{K}$-starcompact Tychonoff space as a closed subspace.

**Proof.** Let $X$ be a Tychonoff space. If we put

$$Z = (\beta(X) \times (\tau^* + 1)) \setminus (\beta(X) \times \{\tau^*\}),$$

where $\tau$ is a regular cardinal and $\tau > l(X)$, then $X = X \times \{\tau^*\}$ is a closed subset of $Z$, which is homeomorphic to $X$. To show that $Z$ is $\mathcal{K}$-starcompact. Let $\mathcal{U}$ be an open cover of $Z$. Since $\beta(X) \times \tau^*$ is countably compact, there is a finite subset $F_1 \subseteq \beta(X) \times \tau^*$ such that

$$\beta(X) \times \tau^* \subseteq St(F_1, \mathcal{U}).$$

It remains to find a compact subset $F_2 \subseteq Z$ such that $\overline{X} \subseteq St(F_2, \mathcal{U})$. Denote by $\mathcal{V}$ the family of all sets of the from $V = (W(V) \times (\alpha(V), \tau^*)) \cap Z$ (where $\alpha(V) < \tau^*$ and $W(V)$ is an open set in $\beta(X)$) such that $V \subseteq U(V)$ for some $U(V) \in \mathcal{U}$. Then $\mathcal{V}$ is an open cover of $\overline{X}$. There exists a subcover $\mathcal{V}_0 \subseteq \mathcal{V}$ of cardinality $\leq l(X)$. If we put

$$\alpha^* = \sup\{\alpha(V) : V \in \mathcal{V}_0\} + 1,$$
then $F_2 = \beta(X) \times \{\alpha^*\}$ is a compact subspace of $Z$. Since every element of $\mathcal{V}_0$ intersects $F_2$, we have

$$X \subseteq \text{St}(F_2, \mathcal{V}_0) \subseteq \text{St}(F_2, \mathcal{U}).$$

Set $K = F_1 \cup F_2$; then $K$ is a compact subset of $Z$ and $Z \subseteq \text{St}(K, \mathcal{U})$, which completes the proof.

**Question 2.** Can a Tychonoff space be embedded in a $K$-starcompact Tychonoff space as a $G_\delta$-closed subspace?

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