A Note on $\psi$-Operator

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Abstract. Studying $\psi$-operator closely, we introduce a new type of sets and consider the interrelation of such sets with some generalized open sets already known in literature.

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1. Introduction

An ideal $I$ as we know is a nonempty collection of subsets of $X$ closed with respect to finite union and hereditary. For a subset $A$ of $X$, $A^* = \{ x \in X: U \cap A \notin I, \text{ for every } U \in \tau(x) \}$ where $\tau(x)$ is the collection of all nonempty open sets containing $x$. $A^*$ is a closed subset for any $A \subseteq X$ [5]. Now theory of ideals gets a new dimension in case it satisfies $I \cap \tau = \{ \emptyset \}$ [2]. Such ideals have been termed as ‘codense ideal’ by Dontchev, Ganster and Rose in 1999 who have also defined a set $D \subseteq X$ as $I$-dense if $D^* = X$ [2]. Eventually $I$ is codense if and only if $X = X^*$. With the help of $()^*$-operator, another operator called $\Psi$-operator is defined as $\Psi(A) = X - (X - A)^*$ [3]. In this paper we have used the $\Psi$-operator to define an interesting generalized open sets and study its properties. A topological space with an ideal $I$ is denoted by $(X, \tau, I)$.

2. Set operator $\Psi$

In this section we discuss a few properties of the set operator $\Psi$. We first prove:

**Theorem 2.1.** Let $(X, \tau, I)$ be a topological space, then $U \subseteq \Psi(U)$ for every open set $U$ of $(X, \tau)$.

**Proof.** We know that $\Psi(U) = X - (X - U)^*$. Now $(X - U)^* \subseteq \text{cl}(X - U) = X - U$, since $X - U$ is closed. Therefore $X - (X - U)^* \supseteq X - (X - U) = U$ implying $U \subseteq \Psi(U)$.

Now we give an example of a set $A$ which is not open but satisfies $A \subseteq \Psi(A)$.  

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Example 2.1. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\}, I = \{\emptyset, \{c\}\}$. Now $\Psi (\{a\}) = X - \{X - \{a\}\} = X - \{b, c\} = \{a, c\}$. Therefore $\{a\} \subset \Psi (\{a\})$, but $\{a\}$ is not open.

Corollary 2.1. Let $(X, \tau, I)$ be a space, then int $A \subset \Psi (A)$ for any subset $A$ of $X$.

Proof. We know that int $A$ is open, then, by Theorem 2.1,

\[(2.1) \quad \text{int} A \subset \Psi (\text{int} A)\]

Again int $A \subset A$, therefore (see [3])

\[(2.2) \quad \Psi (\text{int} A) \subset \Psi (A)\]

From (2.1) and (2.2), int $A \subset \Psi (A)$.

Our next result on $\Psi$-operator seems to be interesting.

Theorem 2.2. Let $(X, \tau, I)$ be a space, where $I$ is codense. Then for $A \subset X$, $\Psi (A) \subset A^*$.

Proof. Suppose $\alpha \in \Psi (A)$ but $\alpha \notin A^*$. Then there exists a nonempty neighborhood $U_\alpha$ of $\alpha$ such that $U_\alpha \cap A \in I$. Since $\alpha \in \Psi (A)$, therefore $\alpha \in \cup \{M \in \tau : M - A \in I\}$ [3], which implies that there exists $V \in \tau$ such that $\alpha \in V$ and $V - A \in I$. Now $U_\alpha \cap V$ is a neighborhood of $\alpha$. Now $U_\alpha \cap V \cap A \in I$, by heredity. Again $U_\alpha \cap V - A \in I$, by heredity. Write $U_\alpha \cap V = (U_\alpha \cap V \cap A) \cup (U_\alpha \cap V - A) \in I$, by finite additivity. Since $U_\alpha \cap V$ is nonempty open, a contradiction to $I$ being codense. Therefore $\alpha \in A^*$. This implies that $\Psi (A) \subset A^*$.

Corollary 2.2. Let $(X, \tau, I)$ be a topological space, where $I$ is codense. Then for $A \subset X$, $\Psi (A) \subset \text{cl} A$.

Proof. This follows from Theorem 2.2 and the fact that $A^* \subset \text{cl} A$ for any $A \subset X$.

We shall now prove Theorem 2.3. Some of the results in the theorem have been proved by Hamlett and Jankovic [3]. However using Theorem 2.2 and Corollary 2.2, the proofs have become much simpler.

Theorem 2.3. Let $(X, \tau, I)$ be a topological space and $I$ be codense. Then

(i) for any $A \subset X$, $\Psi (A) \subset \text{int} \text{cl} A$.
(ii) for any closed subset $A$, $\Psi (A) \subset A$.
(iii) for any $A \subset X$, $\text{int} \text{cl} A = \Psi (\text{int} \text{cl} A)$.
(iv) for any regular open subset $A$, $A = \Psi (A)$.
(v) for any $U \in \tau$, $\Psi (U) \subset \text{int} \text{cl} U \subset U^*$.
(vi) for $J \in I$, $\Psi (J) = \emptyset$.

Proof.

(i) From Corollary 2.2 $\Psi (A) \subset \text{cl} A$. Since $\Psi (A)$ is open, then $\Psi (A) \subset \text{int} \text{cl} A$.
(ii) Proof is obvious.
(iii) Now for any set $A$, $\Psi (\text{int} \text{cl} A) \subset \text{cl} \text{int} \text{cl} A$, by Corollary 2.2. Since $\Psi (\text{int} \text{cl} A)$ is open, $\Psi (\text{int} \text{cl} A) \subset \text{int} \text{cl} \text{int} \text{cl} A$ implying

\[(2.3) \quad \Psi (\text{int} \text{cl} A) \subset \text{int} \text{cl} A\]
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Since int cl $A$ is open, therefore by Theorem 2.1

\[(2.4)\] \quad \text{int cl } A \subset \Psi (\text{int cl } A).

From (2.3) and (2.4), int cl $A = \Psi (\text{int cl } A)$

(iv) If $A$ is regular open, therefore $A = \text{int cl } A$. Now from (iii), $A = \Psi(A)$.

(v) By Corollary 2.2, $\Psi (U) \subset \text{cl } U$. Since $\Psi(A)$ is open, therefore

\[(2.5)\] \quad \Psi(U) \subset \text{int cl } U

Here $I$ is codense and $U$ is open, therefore $U^* = \text{cl } U$ implies that

\[(2.6)\] \quad \text{int cl } U \subset U^*

From (2.5) and (2.6), $\Psi(U) \subset \text{int cl } U \subset U^*$.

(vi) Proof is follows from Theorem 2.2.

We now prove Theorem 2.4.

**Theorem 2.4.** Let $(X, \tau, I)$ be a topological space. Then for each $x \in X$, $X - \{x\}$ is $I$-dense if and only if $\Psi(\{x\}) = \emptyset$.

**Proof.** Proof follows from the definition of I-dense set, since $\Psi(\{x\}) = \emptyset$ if and only if $(X - \{x\})^* = X$.

3. $\Psi$ - C set

In this section, using $\Psi$-operator, we discuss a new class of sets which happens to contain the class of all open sets.

**Definition 3.1.** Let $(X, \tau, I)$ be a topological space and $A \subset X$, $A$ is said to be a $\Psi$-C set if $A \subset \text{cl } \Psi (A)$. The collection of all $\Psi$-C sets in $(X, \tau, I)$ is denoted by $\Psi (X, \tau)$.

**Theorem 3.1.** Let $(X, \tau, I)$ be a topological space. If $A \in \tau$ then $A \in \Psi (X, \tau)$.

**Proof.** The proof follows from Theorem 2.1. From Theorem 3.1 it follows that $\tau \subset \Psi (X, \tau)$ holds in a topological space $(X, \tau, I)$.

Now we give an example which shows that the reverse inclusion is not true.

**Example 3.1.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c, d\}\}$, $I = \{\emptyset, \{c\}\}$ denoting $C(\tau)$ the closed sets in $(X, \tau)$. Therefore $C(\tau) = \{\emptyset, X, \{a, b\}\}$. Now $\Psi(\{a, d\}) = X - \{b, c\}^* = X - \{a, b\} = \{c, d\}$. Thus $\text{cl } \Psi(\{a, d\}) = X$. Therefore $\{a, d\} \subset \text{cl } \Psi(\{a, d\})$, but $\{a, d\}$ is not open in $\tau$.

We give an example which shows that any closed set in $(X, \tau, I)$ may not be a $\Psi$-C set.

**Example 3.2.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$, $I = \{\emptyset, \{a\}\}$. $C(\tau) = \{\emptyset, X, \{a, c\}, \{c\}, \{a\}\}$. Now $\Psi(\{a\}) = X - \{b, c\}^* = X - \{a, b, c\} = \emptyset$. Therefore $\{a\}$ is closed in $(X, \tau)$ but $\{a\} \subset \text{cl } \Psi(\{a\})$.

Now we prove that the arbitrary union of $\Psi$-C sets is a $\Psi$-C.

**Theorem 3.2.** Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of nonempty $\Psi$-C sets in a topological space $(X, \tau, I)$ then $\bigcup \alpha A_\alpha \in \Psi (X, \tau)$.
Proof. For each \( \alpha \),
\[
A_\alpha \subset \text{cl } \Psi (A_\alpha) \subset \text{cl } \Psi \left( \bigcup_{\alpha \in \Delta} A_\alpha \right).
\]
This implies that
\[
\bigcup_{\alpha} A_\alpha \subset \text{cl } \Psi \left( \bigcup_{\alpha} A_\alpha \right).
\]
Thus \( \bigcup_{\alpha \in \Delta} A_\alpha \in \Psi (X, \tau) \).

Following example shows that intersection of two \( \Psi \)-C sets in \((X, \tau, I)\) may not be a \( \Psi \)-C set.

**Example 3.3.** Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\} \), \( I = \{\emptyset, \{c\}\} \), \( C(\tau) = \{\emptyset, X, \{b, c, d\}, \{a, d\}, \{d\}\} \).

Now \( \Psi (\{a, d\}) = X - \{b, c\}^* = X - \{b, c, d\} = \{a\} \).

Therefore \( \text{cl } \Psi (\{a, d\}) = \{a\} \), implies that \( \{a, d\} \subset \text{cl } \Psi (\{a, d\}) \). Again \( \Psi (\{b, c, d\}) = X - \{a\}^* = X - \{a, d\} = \{b, c\} \), implies that \( \text{cl } \Psi (\{b, c, d\}) = \{b, c\} \).

Therefore \( \{b, c\} \subset \text{cl } \Psi (\{b, c, d\}) \). Now \( \{b, c\} \cap \{a, d\} = \{d\} \) and \( \Psi (\{d\}) = X - \{a, b, c\}^* = X - \{a, b, c, d\} = \emptyset \).

Therefore \( \{d\} \notin \text{cl } \Psi (\{d\}) \).

Recall that a subset \( A \subset X \) is semi-open if \( A \subset \text{cl } \text{int } A \).

The collection of all semi-open sets in a topological space \((X, \tau)\) is denoted by \( SO (X, \tau) \).

Now we give the relation between \( SO (X, \tau) \) and \( \Psi (X, \tau) \) in \((X, \tau)\).

**Theorem 3.3.** Let \((X, \tau, I)\) be a topological space, then \( SO (X, \tau) \subset \Psi (X, \tau) \).

Proof. Let \( A \in SO (X, \tau) \), therefore \( A \subset \text{cl } \text{int } A \). We know that \( \text{int } A \subset \Psi (A) \) by Corallary 2.1. Therefore \( \text{cl } \text{int } A \subset \text{cl } \Psi (A) \). Thus \( A \subset \text{cl } \text{int } A \subset \text{cl } \Psi (A) \). Hence the theorem.

That the reverse inclusion of the above theorem fails to hold follows from Example 3.1 where \( \{a, d\} \in \Psi (X, \tau) \) where \( \{a, d\} \) is not a semi-open set.

Now we recall the definition of a semi-preopen set.

**Definition 3.2.** [1] A subset \( A \) of \( X \) is said to be a semi-preopen set if \( A \subset \text{cl } \text{int } A \). The collection of all semi-preopen sets in \((X, \tau)\) is denoted by \( SPO (X, \tau) \).

Theorem 3.3 and Example 3.4 show that if \( I \) is codense \( \Psi (X, \tau) \) in general is a larger class than the class of semi-open sets in \((X, \tau)\). However we shall show that the class of semi-preopen sets forms even a larger class than the class of \( \Psi \)-C sets.

**Theorem 3.4.** Let \( A \) be a \( \Psi \)-C set in a topological space \((X, \tau, I)\), where \( I \) is codense. Then \( A \in SPO (X, \tau) \).

Proof. Proof follows directly from Theorem 2.3(i), since \( \Psi (A) \subset \text{cl } A \) implies \( A \subset \text{cl } \text{int } cl A \).

By the above theorem we get \( \Psi (X, \tau) \subset SPO (X, \tau) \). However the inequality in the other direction fails to hold.

**Example 3.4.** Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a, b\}\} \), \( I = \{\emptyset, \{a\}\} \) and \( C(\tau) = \{\emptyset, X, \{c\}\} \).

Now \( \Psi (\{a\}) = X - \{b, c\}^* = X - \{a, b, c\} = \emptyset \).

Therefore \( \{a\} \notin \text{cl } \Psi (\{a\}) \), i.e., \( \{a\} \) is not a \( \Psi \)-C set. But \( \{a\} \subset \text{cl } \text{int } \{a\} \), therefore \( \{a\} \) is a semi-preopen set.
Corollary 3.1. \( SO (X, \tau) \subset \Psi (X, \tau) \subset SPO (X, \tau) \), when \( I \) is a codense ideal.

Proof. The proof follows from Theorem 3.3 and Theorem 3.4.

Recall that Njastad in 1965 defined a set \( A \subset X \) to be an \( \alpha \)-set if \( A \subset \text{int cl int} \ A \) \[6\]. Denote the collection of all \( \alpha \)-sets as \( \tau^\alpha \).

In Example 3.3 it has been shown that intersection of two \( \Psi \)-C sets may not be a \( \Psi \)-C set. However we show that the intersection of a \( \Psi \)-C set and an \( \alpha \)-set is also a \( \Psi \)-C set.

Theorem 3.5. Let \( (X, \tau, I) \) be a topological space and \( A \in \Psi (X, \tau) \). If \( U \in \tau^\alpha \), then \( U \cap A \in \Psi (X, \tau) \).

Proof. First we note that if \( G \) is open, for any \( A \subset X, G \cap \text{cl} \ A \subset \text{cl}(G \cap A) \), as well as that \( \Psi(A \cap B) = \Psi(A) \cap \Psi(B) \). Hence if \( U \in \tau^\alpha \) and \( A \in \Psi(X, \tau) \) we have therefore \( U \cap A \subset \text{int}(\text{cl}(U)) \cap \text{cl} \Psi(A) \subset \text{int}(\text{cl}(\Psi(U)) \cap \text{cl}(\Psi(A))) = \text{cl}((\Psi(U) \cap \Psi(A))) = \text{cl}(\Psi(U \cap A)) \) and hence \( U \cap A \in \Psi(X, \tau) \).

From Theorem 3.5 we get the following corollary.

Corollary 3.2. Let \( (X, \tau, I) \) be a topological space and \( A \in \Psi (X, \tau) \). If \( U \in \tau \), then \( U \cap A \in \Psi (X, \tau) \).

Proof. It follows from the fact that \( \tau \subset \tau^\alpha \).

It is obvious that if \( A \in I \) is nonempty, where \( I \) is codense, then \( A \notin \Psi(X, \tau) \). [It follows from \( (vi) \) of Theorem 2.3].

However the following example shows that the converse need not hold.

Example 3.5. Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\} \), \( I = \{\emptyset, \{c\}\} \) and \( C(\tau) = \{\emptyset, X, \{b, c, d\}, \{a, d\}, \{d\}\} \). Now \( \Psi(\{a, c\}) = X - \{b, d\}^* = X - \{a, c\} = \{a\} \). Therefore \( \text{cl} \Psi(\{a, c\}) = \{a, d\} \). Thus \( \{a, c\} \notin \Psi(X, \tau) \), where as \( \{a, c\} \notin I \). Also recalling that \( \Psi(A) = X - (X - A)^* \), from the definition of \( I \)-dense set it follows that \( \Psi(A) = \emptyset \) if and only if \( (X - A) \) is \( I \)-dense. Therefore for a topological space \( (X, \tau, I) \) if \( I \) is codense \( A \neq \emptyset, A \notin \Psi(X, \tau) \) if \( A \in I \) or \( (X - A) \) is \( I \)-dense.

Calling a set \( D \) to be relatively \( I \)-dense in a set \( A \) if for every relatively nonempty open set \( U \cap A, U \in \tau \), it is true that \( (U \cap A) \cap D \notin I \). We now prove Theorem 3.6 giving a necessary and sufficient condition for \( A \notin \Psi(X, \tau) \).

Theorem 3.6. A set \( A \) does not belong to \( \Psi(X, \tau) \) if and only if there exists \( x \in A \) such that there is a neighborhood \( V_x \) of \( x \) for which \( X - A \) is relatively \( I \)-dense in \( V_x \).

Proof. Let \( A \notin \Psi(X, \tau) \). We are to show that there exists an element \( x \in A \) and a neighborhood \( V_x \) of \( x \) satisfying that \( X - A \) is relatively \( I \)-dense in \( V_x \). Since \( A \notin \text{cl} \Psi(A) \), there exists \( x \in X \) such that \( x \in A \) but \( x \notin \text{cl} \Psi(A) \). Hence there exists a neighborhood \( V_x \) of \( x \) such that \( V_x \cap \Psi(A) = \emptyset \). This implies that \( V_x \cap (X - (X - A)^*) = \emptyset \), therefore \( V_x \subset (X - A)^* \). Let \( U \) be any nonempty open set in \( V_x \). Since \( V_x \subset (X - A)^* \), therefore \( U \cap (X - A) \notin I \). This implies that \( (X - A) \) is relatively \( I \)-dense in \( V_x \).

Converse part follows by reversing the argument.
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