A Framed $f(3, -1)$ Structure on the Cotangent Bundle of a Hamilton Space

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Abstract. For the cotangent bundle $\left( T^*M, \tau^*, M \right)$ of a smooth manifold $M$, the kernel of a differential $\tau^*_u$ of the projection $\tau^*$ defines the vertical subbundle $VT^*M$ of the bundle $(TT^*M, \tau_{T^*M}, T^*M)$. A supplement $HT^*M$ of it is called a horizontal subbundle or a nonlinear connection on $M$, [6,7]. The direct decomposition $TT^*M = HT^*M \oplus VT^*M$ gives rise to a natural almost product structure $P$ on the manifold $T^*M$. A general method to associate to $P$ a framed $f(3, -1)$-structure of any corank is pointed out. This is similar to that given by us in [2] for the tangent bundle of a Lagrange space. When we endow $M$ with a regular Hamiltonian $H$ and use as the nonlinear connection that canonically induced by $H$, a framed $f(3, -1)$-structure $P_2$ of corank 2 naturally appears on $T^*M$. This reduces to that found by us in [3] when $H = K^2$, for $K$ the fundamental function of a Cartan space $K^* = (M, K)$. Then we show that on some conditions for $H$ the restriction of $P_2$ to the submanifold $H = 1$ of $T^*_0M$ provides an almost paracontact structure on this submanifold. The conditions taken on $H$ hold for the $\varphi$-Hamiltonians introduced by us in [4] as well as for $H = K^2$. The idea of this study has the origin in the paper [1] of M. Anastasiei.

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1. A framed $f(3, -1)$ structure on $T^*M$

Let $M$ be a smooth i.e. $C^\infty$ manifold of dimension $n$ with local coordinates $(x^i)$, $i, j, k, \cdots = 1, \cdots, n$. And let $(T^*M, \tau^*, M)$ be its cotangent bundle. On $T^*M$ we shall take as local coordinates $(x^i = x^i \circ \tau^*_u, p_i)$, where $(p_i)$ are the coordinates of a covector from $T^*_uM$, $x(x')$, in the natural cobasis $(dx^i)$.

The set $VT^*M = \cup_{u \in T^*M} V_uT^*M$ for $V_uT^*M = \ker \tau^*_u$, projected over $T^*M$ gives the vertical bundle over $T^*M$. A supplement $HT^*M$ of it is called horizontal bundle or a nonlinear connection on $M$. We have the decomposition
The distribution \( u \rightarrow V_u T^* M \) is locally spanned by \( \hat{\partial}^i = \frac{\partial}{\partial p_i} \) and one takes \( \delta_i = \partial_i + N_a(x, p) \hat{\partial}^k \) as a local basis for the horizontal distribution \( u \rightarrow H_u T^* M \). Thus the basis \( (\delta_i, \hat{\partial}^i) \) is adapted to the decomposition (1.1). The Einstein convention on summation over the indices \( i, j, k, \cdots \) is implied.

The linear operator \( P \) on \( T_u T^* M \) defined by

\[
P(\delta_i) = \delta_i, \quad P(\hat{\partial}^i) = -\hat{\partial}^i,
\]

(1.2)
gives an almost product on \( T^* M \) that is, \( P^2 = I \), where \( I \) is the identity operator.

The dual basis of \( (\delta_i, \hat{\partial}^i) \) is \((dx^i, \delta p_j = dp_j - N_{ij}(x, p) dx^j)\).

Let \( \xi_1, \xi_2, \cdots, \xi_r \) be \( r \) linearly independent horizontal vector fields and \( \zeta_1, \zeta_2, \cdots, \zeta_s \) be \( s \) linearly independent vertical vector fields on \( T^* M \), such that \( m = r + s < 2n \). We consider also the \( r \) horizontal 1-forms \( \omega_1, \omega_2, \cdots, \omega_r \) \((\omega_a = \omega_a dx^i, \alpha, \beta, \cdots = 1, \cdots, r)\)
and \( s \) vertical 1-forms \( \eta_1, \eta_2, \cdots, \eta_s \) \((\eta_a = \eta_a \delta p_j, a, b, \cdots = 1, \cdots, s)\) such that

\[
\omega_a(\xi_\beta) = \delta_{a\beta}, \quad \eta_a(\zeta_b) = \delta_{ab}.
\]

Notice that we have also

\[
\omega_a(\xi_\alpha) = 0, \quad \eta_a(\zeta_a) = 0.
\]

(1.3)'

We clearly have \( P(\xi_a) = \xi_a, \quad P(\zeta_a) = -\zeta_a, \forall \alpha, a \) and

**Lemma 1.1.** \( \omega_a \circ P = \omega_a, \quad \eta_a \circ P = -\eta_a, \forall \alpha, a \).

Now we put

\[
P_m = P - \sum_a \omega_a \otimes \xi_a + \sum_a \eta_a \otimes \zeta_a
\]

(1.4)
and we have

**Theorem 1.1.** The triple \( F_m = (P_m, (\xi_a, \eta_a), (\omega_a, \eta_a)) \) defines a framed \( f(3, -1) \)-structure on \( T^*M \), that is, we have
Proof. One uses (1.3), (1.3)' and the Lemma 1.1.

This result is completed by

**Theorem 1.2.** The operator $P_m$ is of rank $2n - m$ and it satisfies

$$P_m^3 - P_m = 0.$$  \hspace{1cm} (1.6)

Proof. The equality (1.6) follows from (1.5). In order to prove that rank $P_m = 2n - m$, we show that ker $P_m$ is spanned by the vector fields $\left(\xi_a, \zeta_a\right)$, $a = 1, \ldots, r$, $a = 1, \ldots, s$, $r + s = m$. By (1.5), $\text{Span} \left(\xi_a, \zeta_a\right)$ is contained in ker $P_m$. For proving the converse inclusion, let be $Z = X^i \delta_i + Y^j \delta^j \in \ker P_m$. Then by (1.4),

$$P_m(Z) = X^i \delta_i - Y^j \delta^j - \sum_a \left(\omega_{ak} X^k\right) \xi_a \delta_i + \sum_a \left(\eta_a^k Y^k\right) \zeta_a \delta^j$$

and $P_m(Z) = 0$ gives

$$X^i = \sum_a \left(\omega_{ak} X^k\right) \xi_a, \quad Y^j = \sum_a \left(\eta_a^k Y^k\right) \zeta_a.$$  

It follows

$$Z = \sum_a \left(\omega_{ak} X^k\right) \xi_a + \sum_a \left(\eta_a^k Y^k\right) \zeta_a,$$

hence $Z \in \text{Span} \left(\xi_a, \zeta_a\right)$.

Theorem 1.2 says that the framed $f(3, -1)$-structure $F_m$ is of corank $m$. The term $f(3, -1)$-structure is suggested by (1.6). We refer to the book [5] for an account of framed $f(3, -1)$-structures and the other related structures.

The existence of $F_m$ is heavily based on the existence of linearly independent vector fields $\xi_a, \zeta_a$.

In the next section we shall exhibit a natural framed $f(3, -1)$-structure on $T^* M$ when $M$ is a Hamilton space.

2. A framed $f(3, -1)$-structure on $T^* M$, when $M$ is a Hamilton space

A Hamilton space is a pair $(M, H)$, where $H : T^* M \to \mathbb{R}$ is a smooth regular Hamiltonian. This means that the matrix with the entries
is of rank \( n \).

The regular Hamiltonian \( H \) induces (see Ch. 4 in [7]) a nonlinear connection of local coefficients

\[
N_j(x, p) = \frac{1}{4} \left\{ \{g_{ij}, H\} - \frac{1}{4} \left( g_{ik} \partial^k \partial_j H + g_{jk} \partial^k \partial_i H \right) \right\},
\]

where \( \{,\} \) denotes the usual Poisson brackets and \( g_{ij} \) denotes the inverse of the matrix \( (g_{jk}) \). Thus we may consider the almost product structure \( P \) completely determined by \( H \).

Assume that \( g^{ij}(x, p) p_i p_j > 0 \) on the slit cotangent bundle \( T_0^*M = T^*M \setminus 0 \) and set \( \varepsilon^2 = g^{ij}(x, p) p_i p_j \). From now on we restrict our considerations to \( T_0^*M \).

We consider the vector fields

\[
\xi = \frac{1}{\varepsilon} p^i \partial_i, \quad \zeta = \frac{1}{\varepsilon} p_i \partial^i
\]

and the 1-forms

\[
\omega = \frac{1}{\varepsilon} \left( g_{ij} p^i \right) dx^j, \quad \eta = \frac{1}{\varepsilon} \left( g^{ij} p_j \right) dp_i.
\]

It follows that

\[
\omega(\xi) = 1, \quad \eta(\zeta) = 1
\]

and the Lemma 1 holds for \( \alpha = a = 1, \xi_1 = \xi, \zeta_1 = \zeta, \omega_1 = \omega, \eta_1 = \eta \).

We set

\[
P_2 = P - \omega \otimes \xi + \eta \otimes \zeta.
\]

Using (2.3), (2.3)’ and Lemma 1 for the present case, one gets

**Theorem 2.2.** The triple \( F_2 = (P_2, (\xi, \zeta), (\omega, \eta)) \) is a framed \( f(3; -1) \)-structure on \( T_0^*M \), that is,
The framed \( f(3, -1) \)-structure \( F_2 \) is of corank 2 and depends only on the Hamiltonian \( H \) on \( T^*_0 M \).

We consider \( T^*_0 M \) as a Riemannian manifold with the Sasaki type metric

\[
G(x, p) = g_{ij} dx^i \otimes dx^j + g^{ij} \delta p_i \otimes \delta p_j.
\]

One easily checks that

\[
\omega(X) = G(X, \xi), \quad \eta(X) = G(X, \zeta), \quad \forall \, X \in \chi(T^*_0 M).
\]

We have

**Theorem 2.3.**  The Riemannian metric \( G \) satisfies

\[
G(P_X, P_Y) = G(X, Y) - \omega(X)\omega(Y) - \eta(X)\eta(Y), \quad \forall \, X, Y \in \chi(T^*_0 M).
\]

**Proof.**  First, we notice that \( G(\xi, \xi) = G(\zeta, \zeta) = 1 \) and \( G(\xi, \zeta) = 0 \) and we have that \( G(PX, \xi) = \omega(PX) = \omega(X), \, G(PX, \zeta) = -\eta(X) \) by (2.8) and Lemma 1.1. Then we have

\[
G(PX - \omega(X)\xi + \eta(X)\zeta, PY - \omega(Y)\xi + \eta(Y)\zeta) = G(PX, PY) - \omega(Y)G(PX, \xi) + \eta(Y)G(PY, \xi).
\]

because of \( G(PX, PY) = G(X, Y) \).

Theorem 2.3 says us that \( (F_2, G) \) is a Riemannian framed \( f(3, -1) \)-structure on \( T^*_0 M \).
3. On structure induced by $F_2$ on the indicatrix bundle over $T^n M$

The set $I_H = \{(x, p) \in T^n M \mid H(x, p) = 1\}$ is a $(2n - 1)$-dimensional submanifold on $T^n M$. We call it the indicatrix bundle of the Hamilton space $H^n = (M, H)$, extending a term used in Finsler geometry.

We consider again $T^n M$ as a Riemannian manifold with the Sasaki type metric $G$. We are interested to find the unit normal vector field to $I_H$. We recall that $G(\xi, \xi) = 1$ and $G(\xi, \zeta) = 1$. As for $H = K^2$, where $K$ is the fundamental function of a Cartan space it is known that $\zeta$ is the unit normal vector field to $I_K$, we look for conditions on $H$ such that $\zeta$ to be the unit normal vector field for the indicatrix bundle of the Hamilton space $H^n = (M, H)$. For the geometry of the Cartan spaces we refer to the Ch. 6 in [7].

Let be
\[ x^i = x^i(u^a), \]
\[ p_i = p_i(u^a), \quad \alpha = 1, 2, \ldots, 2n - 1 \] (3.1)
a parametrization of the submanifold $I_H$. The local vector fields $\frac{\partial}{\partial u^\alpha}$ that form a basis of the tangent space to $I_H$ can be put in the form
\[ \frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha} \delta_i + \left( \frac{\partial p_i}{\partial u^\alpha} - N_{ij}(x, p) \frac{\partial x^j}{\partial u^\alpha} \right) \delta^j. \] (3.2)

If one derives the identity $H(\ x(u^a), p(u^a) ) = 1$ with respect to $u^\alpha$, one obtains
\[ (\delta_i H) \frac{\partial x^i}{\partial u^\alpha} + \left( \frac{\partial}{\partial u^\alpha} \right) \left( \frac{\partial p_i}{\partial u^\alpha} - N_{ij}(x, p) \frac{\partial x^j}{\partial u^\alpha} \right) = 0. \] (3.3)

On using (3.2) we see that $\zeta$ is normal to $I_H$ if and only if
\[ G \left( \frac{\partial}{\partial u^\alpha}, \zeta \right) = \frac{1}{e} \left( g^{ij} p_j \right) \left( \frac{\partial p_i}{\partial u^\alpha} - N_{ij}(x, p) \frac{\partial x^j}{\partial u^\alpha} \right) = 0 \] (3.4)
for every $\alpha = 1, 2, \cdots, 2n - 1$.

Comparing (3.3) with (3.4) it comes out that (3.4) holds if
\[ \delta_i H = 0, \quad \delta^i H = fg^{ij} p_j, \quad \text{for a smooth function on } T^n M. \] (3.5)
The conditions (3.5) are quite complicated. We noticed them having in mind the case $H = K^2$, for $K$ the fundamental function of a Cartan space. In such a case, it is well known that $\delta_i K^2 = 0$ and from the equality $K^2 = g^{ij}(x, p)p_i p_j$ it follows that $\dot{K}^2 = 2g^{ij}p_j$. The question is whether exist non-homogeneous Hamiltonians that satisfy (3.5).

We show now that the so-called $\varphi$-Hamiltonians introduced and studied by us in [4], fulfill the conditions (3.5).

Let $K^n = (M, K)$ be a Cartan space and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ a function of class $C^\infty$. Assume that $\varphi$ has the properties:

$$\varphi'(t) \neq 0, \varphi'(t) + 2\varphi''(t) \neq 0 \text{ for } t \in \text{Im}(K^2).$$

Then $H := \varphi(K^2)$ is a regular Hamiltonian on $T^*_q M$ called the $\varphi$-Hamiltonian associated to $K^n$.

As we have seen in [4] the Hamiltonians $H = \varphi(K^2)$ and $K^2$ define the same nonlinear connection. We have $\delta_i H = \varphi'(K^2) \delta_i K^2 = 0$. Hence the first condition (3.5) holds for any $\varphi$-Hamiltonian.

Let $g^{ij}(x, p) = \frac{1}{2} \frac{\partial K^2}{\partial p_i \partial p_j}$ be the metric tensor of $K^n$ and $g^{ij}(x, p)$ the metric tensor (2.1) of $H = \varphi(K^2)$. A direct calculation gives

$$g^{ij}(x, p) = \varphi' \left( g^{ij} + 2 \frac{\varphi''}{\varphi'} p^i p^j \right)$$

where

$$p^i = g^{ij}(x, p)p_j = \frac{1}{2} \frac{\partial K^2}{\partial p_i}.$$

We have

$$g^{ij} p_j = \varphi' \left( 1 + \frac{2\varphi''}{\varphi'} K^2 \right) p^i = \frac{\varphi' + 2\varphi'' K^2}{2\varphi'} \frac{\partial H}{\partial p_i}$$

because of

$$\frac{\partial H}{\partial p_i} = \varphi' \frac{\partial K^2}{\partial p_i}.$$

Thus the second condition (3.5) holds with $f = \frac{2\varphi'}{\varphi' + 2\varphi'' K^2} \neq 0$. 

Let us consider a Hamilton space $H^n = (M, H)$ such that $\zeta$ is the unit normal vector field of the indicatrix bundle $I_H$ defined by $H$.

We restrict to $I_H$ the elements of the triple $F_2$ and indicate this fact by a bar over those elements. We have

- $\bar{\xi} = \xi$ since $\xi$ is tangent to $I_H$,
- $\bar{\eta} = 0$ on $I_H$, since $\eta(X) = G(X, \zeta) = 0$ for any vector field $X$ tangent to $I_H$,
- $\bar{P}_2 = P - \omega \otimes \xi$ on $I_H$, because of $G(\bar{P}_2X, \zeta) = G(PX, \zeta) = \eta(PX) = -\eta(X) = 0$ for any vector field $X$ tangent to $I_H$.

We have

**Theorem 3.1.** The triple $(\bar{P}_2, \bar{\xi}, \bar{\omega})$ defines a Riemannian almost paracontract structure on $I_H$, that is,

1. $\bar{\omega}(\bar{\xi}) = 1$, $\bar{P}_2(\bar{\xi}) = 0$, $\bar{\omega} \circ \bar{P}_2 = 0$
2. $\bar{P}_2^2 = 1 - \bar{\omega} \otimes \bar{\xi}$ on $I_H$
3. $G(\bar{P}_2X, \bar{P}_2Y) = G(X, Y) - \bar{\omega}(X) \bar{\omega}(Y)$, for any vector fields $X, Y$ tangent to $I_H$.

**Proof.** All the assertions follow from Theorems 2.2 and 2.3.

For $L = K^2$ we regain our results from [3]. Concluding, we have enlarged the set of Hamiltonians for which Theorem 3.1 holds good.

**References**


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