Almost Contra-Precontinuous Functions

ERDAL EKICI
Department of Mathematics, Canakkale Onsekiz Mart University, Terzioglu Campus, 17100 Canakkale, Turkey
e-mail: eekici@comu.edu.tr

Abstract. In this paper, we present and study almost contra-precontinuity as a new generalization of regular set-connectedness, contra-precontinuity, contra-continuity, almost $s$-continuity and perfectly continuity. Furthermore, we obtain basic properties and preservation theorems of almost contra-precontinuity and relationships between almost contra-precontinuity and $P$-regular graphs.

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1. Introduction

In 1996, Dontchev introduced contra-continuous functions. Recently, Dontchev, Ganster and Reilly introduced a new class of functions called regular set-connected functions (in 1999) and Jafari and Noiri introduced and studied a new form of functions called contra-precontinuous functions (in 2002). We introduced and studied a new class of functions called almost contra-precontinuous functions which generalize classes of regular set-connected [5], contra-precontinuous [9], contra-continuous [4], almost $s$-continuous [19] and perfectly continuous [17] functions. Moreover, we obtain basic properties and preservation theorems of almost contra-precontinuous functions and relationships between almost contra-precontinuity and $P$-regular graphs.

2. Preliminaries

Throughout this paper, all spaces $X$ and $Y$ (or $(X, \tau)$ and $(Y, \upsilon)$) are always topological spaces.

A subset $A$ of a space $X$ is said to be regular open (respectively regular closed) if $A = \text{int}(\text{cl}(A))$ (respectively $A = \text{cl}(\text{int}(A))$) where $\text{cl}(A)$ and $\text{int}(A)$ denote the closure and interior of $A$ [26]. A subset $A$ of a space is called preopen if $A \subset \text{int}(\text{cl}(A))$ [14]. The complement of a preopen set is said to be preclosed.

The family of all regular open (respectively regular closed, preopen, preclosed) sets of $X$ is denoted by $\text{RO}(X)$ (respectively $\text{RC}(X)$, $\text{PO}(X)$, $\text{PC}(X)$).
A subset $A$ of a space $X$ is said to be semi open if $A \subset cl(\text{int}(A))$. The complement of a semi open set is called semi closed [2]. The intersection of all semi closed sets containing $A$ is called the semi closure [2] of $A$ and is denoted by $scl(A)$. The semi interior of $A$ is defined by the union of all semi open sets contained in $A$ and is denoted by $s\text{-int}(A)$.

**Definition 1.** A function $f : X \to Y$ is called contra-precontinuous if $f^{-1}(V)$ is preclosed in $X$ for each open set $V$ of $Y$ [9].

**Definition 2.** A function $f : X \to Y$ is called contra-continuous if $f^{-1}(V)$ is closed in $X$ for each open set $V$ of $Y$ [4].

**Definition 3.** A function $f : X \to Y$ is said to be regular set-connected if $f^{-1}(V)$ is clopen for every $V \in RO(Y)$ [5].

**Definition 4.** A function $f : X \to Y$ is said to be perfectly continuous if $f^{-1}(V)$ is clopen in $X$ for every open set $V$ of $Y$ [17].

**Definition 5.** A function $f : X \to Y$ is called almost s-continuous if for each $x \in X$ and each $V \in SO(Y)$ with $f(x) \in V$, there exists an open set $U$ in $X$ containing $x$ such that $f(U) \subset scl(V)$ [19].

**Definition 6.** A function $f : X \to Y$ is said to be almost precontinuous if $f^{-1}(V)$ is preopen in $X$ for every regular open set $V$ of $Y$ [16].

**Definition 7.** A function $f : X \to Y$ is said to be precontinuous if $f^{-1}(V)$ is preopen in $X$ for every open set $V$ of $Y$ [13].

**Definition 8.** A function $f : X \to Y$ is called M-preopen (M-preclosed) if image of each preopen (resp. preclosed) set is preopen (resp. preclosed) [14].

**3. Almost contra-precontinuous functions**

**Definition 9.** A function $f : X \to Y$ is said to be almost contra-precontinuous if $f^{-1}(V) \in PC(X)$ for each $V \in RO(Y)$.

**Theorem 1.** Let $(X, \tau)$ and $(Y, \upsilon)$ be topological spaces. The following statements are equivalent for a function $f : X \to Y$:
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\begin{enumerate}
\item $f$ is almost contra-precontinuous;
\item $f^{-1}(F) \in PO(X)$ for every $F \in RC(Y)$;
\item for each $x \in X$ and each regular closed set $F$ in $Y$ containing $f(x)$, there exists a preopen set $U$ in $X$ containing $x$ such that $f(U) \subset F$;
\item for each $x \in X$ and each regular open set $V$ in $Y$ non-containing $f(x)$, there exists a preclosed set $K$ in $X$ non-containing $x$ such that $f^{-1}(V) \subset K$;
\item $f^{-1}(\text{int}(\text{cl}(G))) \in PC(X)$ for every open subset $G$ of $Y$;
\item $f^{-1}(\text{cl}(\text{int}(F))) \in PO(X)$ for every closed subset $F$ of $Y$.
\end{enumerate}

\textbf{Proof.}

(1) $\iff$ (2): Let $F \in RC(Y)$. Then $Y \setminus F \in RO(Y)$. By (1), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \in PC(X)$. We have $f^{-1}(F) \in PO(X)$.

Reverse can be obtained similarly.

(2) $\Rightarrow$ (3): Let $F$ be any regular closed set in $Y$ containing $f(x)$. By (2), $f^{-1}(F) \in PO(X)$ and $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$. Then $f(U) \subset F$.

(3) $\Rightarrow$ (2): Let $F \in RC(Y)$ and $x \in f^{-1}(F)$. From (3), there exists a preopen set $U_x$ in $X$ containing $x$ such that $U \subset f^{-1}(F)$. We have $f^{-1}(F) = \bigcup_{x \in f^{-1}(F)} U_x$. Thus, $f^{-1}(F)$ is preopen.

(3) $\iff$ (4): Let $V$ be any regular open set in $Y$ non-containing $f(x)$. Then, $Y \setminus V$ is a regular closed set containing $f(x)$. By (3), there exists a preopen set $U$ in $X$ containing $x$ such that $f(U) \subset Y \setminus V$. Hence, $U \subset f^{-1}(Y \setminus V) \subset X \setminus f^{-1}(V)$ and then $f^{-1}(V) \subset X \setminus U$. Take $H = X \setminus U$. We obtain that $H$ is a preclosed set in $X$ non-containing $x$.

The converse can be shown easily.

(1) $\iff (5)$: Let $G$ be open subset of $Y$. Since $\text{int}(\text{cl}(G))$ is regular open, then by (1), it follows that $f^{-1}(\text{int}(\text{cl}(G))) \in PC(X)$.

The converse can be shown easily.

(2) $\iff (6)$: It can be obtained similar as (1) $\iff (5)$. 

Remark 1. The following diagram holds:

\[
\text{perfectly continuous} \Rightarrow \text{contra-continuous} \Rightarrow \text{contra-precontinuous} \\
\downarrow \hspace{2cm} \downarrow \\
\text{regular set-connected} \Rightarrow \text{almost contra-precontinuous} \\
\uparrow \\
\text{almost } s\text{-continuous}
\]

None of the implications is reversible for almost contra-precontinuity as shown by the following examples.

Example 1. Let \( X = \{a, b, c\} \), \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{c\}, \{a, b\}\} \) and \( \nu = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\} \). Then the identity function \( f : (X, \tau) \to (X, \nu) \) is almost contra-precontinuous. But it is not regular set-connected.

Example 2. Let \( X = \{a, b, c\} \), \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}\} \) and \( \nu = \{X, \emptyset, \{a\}, \{a, b\}\} \) and \( f : (X, \tau) \to (X, \nu) \) be the identity function. Then \( f \) is almost contra-precontinuous function which is not contra-precontinuous.

The other implications are not reversible as shown in [5, 6, 9].

Theorem 2. If \( f : X \to Y \) is almost contra-precontinuous function and \( A \) is a semi open subset of \( X \), then the restriction \( f \mid_A : A \to Y \) is almost contra-precontinuous.

Proof. Let \( F \in RC(Y) \). Since \( f \) is almost contra-precontinuous, then \( f^{-1}(F) \in PO(X) \). Since \( A \) is semi open in \( X \), it follows from ([13], Lemma 2.1) that \( (f \mid_A)^{-1}(F) = A \cap f^{-1}(F) \in PO(A) \). Therefore, \( f \mid_A \) is a almost contra-precontinuous function.

Remark 2. It should be noted that every restriction of an almost contra-precontinuous function is not necessarily almost contra-precontinuous.

Example 3. Let \( X = \{a, b, c, d\} \), \( \sigma = \{X, \emptyset, \{a, b\}\} \), and \( \tau = \{X, \emptyset, \{a\}, \{b, c, d\}\} \). The identity function \( f : (X, \sigma) \to (X, \tau) \) is almost contra-precontinuous, but, if \( A = \{a, c, d\} \) where \( A \) is not semi open in \( (X, \sigma) \) and \( \sigma_A \) is the relative topology on \( A \) induced by \( \sigma \), then \( f \mid_A : (A, \sigma_A) \to (X, \tau) \) is not almost contra-precontinuous.
Note that \{b, c, d\} is regular closed in \((X, \tau)\), but that \((f \mid_A)^{-1}(\{b, c, d\}) = \{c, d\}\) is not preopen in \((A, \sigma_A)\).

**Definition 10.** A cover \(\sum = \{U_\alpha : \alpha \in I\}\) of subsets of \(X\) is called a p-cover if \(U_\alpha\) is preopen for each \(\alpha \in I\).

**Lemma 1.** If \(U \in PO(X)\) and \(V \in PO(U)\), then \(V \in PO(X)\) [13].

**Theorem 3.** Let \(f : X \to Y\) be a function and \(\sum = \{U_\alpha : \alpha \in I\}\) be a p-cover of \(X\). If for each \(\alpha \in I, f \mid_{U_\alpha}\) is almost contra-precontinuous, then \(f : X \to Y\) is an almost contra-precontinuous function.

**Proof.** Let \(V \in RC(Y)\). Since \(f \mid_{U_\alpha}\) is almost contra-precontinuous for each \(\alpha \in I\), \((f \mid_{U_\alpha})^{-1}(V) \in PO(U_\alpha)\). Since \(U_\alpha \in PO(X)\), by the previous lemma, \((f \mid_{U_\alpha})^{-1}(V) \in PO(X)\) for each \(\alpha \in I\). Then \(f^{-1}(V) = \bigcup_{\alpha \in I} (f \mid_{U_\alpha})^{-1}(V) \in PO(X)\). This gives \(f\) is an almost contra-precontinuous.

**Theorem 4.** Let \(f : X \to Y\) be a function and let \(g : X \to X \times Y\) be the graph function of \(f\), defined by \(g(x) = (x, f(x))\) for every \(x \in X\). If \(g\) is almost contra-precontinuous, then \(f\) is almost contra-precontinuous.

**Proof.** Let \(V \in RC(Y)\), then \(X \times V = X \times cl(int(V)) = cl(int(X)) \times cl(int(V)) = cl(int(X \times V))\). Therefore, \(X \times V \in RC(X \times Y)\). Since \(g\) is almost contra-precontinuous, then \(f^{-1}(V) = g^{-1}(X \times V) \in PO(X)\). Thus, \(f\) is almost contra-precontinuous.

**Theorem 5.** Let \(f : X \to Y\) and \(g : Y \to Z\) be functions. Then, the following properties hold:

1. If \(f\) is almost contra-precontinuous and \(g\) is regular set-connected, then \(g \circ f : X \to Z\) is almost contra-precontinuous and almost precontinuous.

2. If \(f\) is almost contra-precontinuous and \(g\) is perfectly continuous, then \(g \circ f : X \to Z\) is precontinuous and contra-precontinuous.

3. If \(f\) is contra-precontinuous and \(g\) is regular set-connected, then \(g \circ f : X \to Z\) is almost contra-precontinuous and almost precontinuous.
Proof. (1) Let $V$ be any regular open set in $Z$. Since $g$ is regular set-connected, $g^{-1}(V)$ is clopen. Since $f$ is almost contra-precontinuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is preopen and preclosed. Therefore, $g \circ f$ is almost contra-precontinuous and almost precontinuous.

(2) and (3) can be obtained similarly.

**Theorem 6.** If $f : X \to Y$ is a surjective $M$-preopen and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is almost contra-precontinuous, then $g$ is almost contra-precontinuous.

**Proof.** Let $V$ be any regular closed set in $Z$. Since $g \circ f$ is almost contra-precontinuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is preopen. Since $f$ is surjective $M$-preopen, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is preopen. Therefore, $g$ is almost contra-precontinuous.

**Theorem 7.** If $f : X \to Y$ is a surjective $M$-preclosed and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is almost contra-precontinuous, then $g$ is almost contra-precontinuous.

**Proof.** Similarly as the previous theorem.

**Definition 11.** A function $f : X \to Y$ is called almost continuous if $f^{-1}(V)$ is open in $X$ for each regular open set $V$ of $Y$ [20].

**Theorem 8.** If a function $f : X \to Y$ is almost contra-precontinuous and almost continuous, then $f$ is regular set-connected.

**Proof.** Let $V \in \text{RO}(Y)$. Since $f$ is almost contra-precontinuous and almost continuous, $f^{-1}(V)$ is preclosed and open. Hence, $f^{-1}(V)$ is clopen. We obtain that $f$ is regular set-connected.

**Definition 12.** A filter base $\Lambda$ is said to be $p$-convergent (resp. rc-convergent) to a point $x$ in $X$ if for any $U \in \text{PO}(X)$ containing $x$ (resp. $U \in \text{RC}(X)$ containing $x$), there exists a $B \in \Lambda$ such that $B \subset U$.

**Theorem 9.** If a function $f : X \to Y$ is almost contra-precontinuous, then for each point $x \in X$ and each filter base $\Lambda$ in $X$ $p$-converging to $x$, the filter base $f(\Lambda)$ is rc-convergent to $f(x)$.
Proof. Let \( x \in X \) and \( \Lambda \) be any filter base in \( X \) \( p \)-converging to \( x \). Since \( f \) is almost contra-precontinuous, then for any \( V \in RC(Y) \) containing \( f(x) \), there exists \( U \in PO(X) \) containing \( x \) such that \( f(U) \subset V \). Since \( \Lambda \) is \( p \)-converging to \( x \), there exists a \( B \in \Lambda \) such that \( B \subset U \). This means that \( f(B) \subset V \) and therefore the filter base \( f(\Lambda) \) is rc-convergent to \( f(x) \).

Note that a function \( f : X \rightarrow Y \) is said to be almost contra-precontinuous at \( x \) if each regular closed set \( F \) in \( Y \) containing \( f(x) \), there exists a preopen set \( U \) in \( X \) containing \( x \) such that \( f(U) \subset F \).

**Theorem 10.** Let \( f : X \rightarrow Y \) be a function and \( x \in X \). If there exists \( U \in PO(X) \) such that \( x \in U \) and the restriction of \( f \) to \( U \) is almost contra-precontinuous at \( x \), then \( f \) is almost contra-precontinuous at \( x \).

Proof. Suppose that \( F \in RC(Y) \) containing \( f(x) \). Since \( f \big|_U \) is almost contra-precontinuous at \( x \), there exists \( V \in PO(U) \) containing \( x \) such that \( f(V) = (f \big|_U)(V) \subset F \). Since \( U \in PO(X) \) containing \( x \), it follows from ([13] 1982, Lemma 2.2) that \( V \in PO(X) \) containing \( x \). This shows clearly that \( f \) is almost contra-precontinuous at \( x \).

4. The preservation theorems

In this section, we investigate the relationships among almost contra-precontinuous functions, separation axioms, connectedness and compactness.

**Definition 13.** A space \( X \) is said to be weakly Hausdorff if each element of \( X \) is an intersection of regular closed sets [23].

**Definition 14.** A space \( X \) is said to be pre-\( T_0 \) if for each pair of distinct points in \( X \), there exists a preopen set of \( X \) containing one point but not the other [1,11].

**Definition 15.** A space \( X \) is said to be pre-\( T_1 \) if for each pair of distinct points \( x \) and \( y \) of \( X \), there exist preopen sets \( U \) and \( V \) containing \( x \) and \( y \) respectively such that \( y \notin U \) and \( x \notin V \) [1,11].

**Theorem 11.** If \( f : X \rightarrow Y \) is an almost contra-precontinuous injection and \( Y \) is weakly Hausdorff, then \( X \) is pre-\( T_1 \).
Proof. Suppose that $Y$ is weakly Hausdorff. For any distinct points $x$ and $y$ in $X$, there exist $V, W \in RC(Y)$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since $f$ is almost contra-precontinuous, $f^{-1}(V)$ and $f^{-1}(W)$ are preopen subsets of $X$ such that $x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that $X$ is pre-$T_1$.

**Definition 16.** A topological space $X$ is called $p$-ultra-connected if every two non-void preclosed subsets of $X$ intersect.

**Definition 17.** A topological space $X$ is called hyperconnected if every open set is dense [25].

**Theorem 12.** If $X$ is $p$-ultra-connected and $f : X \to Y$ is almost contra-precontinuous and surjective, then $Y$ is hyperconnected.

**Proof.** Assume that $Y$ is not hyperconnected. Then there exists an open set $V$ such that $V$ is not dense in $Y$. Then there exist disjoint non-empty regular open subsets $B_1$ and $B_2$ in $Y$, namely $\text{int}(cl(V))$ and $Y \setminus cl(V)$. Since $f$ is almost contra-precontinuous and onto, $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint non-empty preclosed subsets of $X$. By assumption, the $p$-ultra-connectedness of $X$ implies that $A_1$ and $A_2$ must intersect. By contradiction, $Y$ is hyperconnected.

**Definition 18.** A space $X$ is called preconnected provided that $X$ is not the union of two disjoint nonempty preopen sets [18].

**Theorem 13.** If $f : X \to Y$ is almost contra-precontinuous surjection and $X$ is preconnected, then $Y$ is connected.

**Proof.** Suppose that $Y$ is not connected space. There exist nonempty disjoint open sets $V_1$ and $V_2$ such that $Y = V_1 \cup V_2$. Therefore, $V_1$ and $V_2$ are clopen in $Y$. Since $f$ is almost contra-precontinuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are preopen in $X$. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that $X$ is not preconnected. This contradicts that $Y$ is not connected assumed. Hence, $Y$ is connected.

**Definition 19.** A space $X$ is said to be

(1) strongly compact if every preopen cover of $X$ has a finite subcover [8, 15].

(2) strongly countably compact if every countable cover of $X$ by preopen sets has a finite subcover.

(3) strongly Lindelöf if every preopen cover of $X$ has a countable subcover [15].
(4) $S$-Lindelof if every cover of $X$ by regular closed sets has a countable subcover [12].

(5) countably $S$-closed if every countable cover of $X$ by regular closed sets has a finite subcover [3].

(6) $S$-closed if every regular closed cover of $X$ has a finite subcover [27].

**Theorem 14.** Let $f : X \to Y$ be an almost contra-precontinuous surjection. Then the following statements hold:

(1) if $X$ is strongly compact, then $Y$ is $S$-closed.

(2) if $X$ is strongly Lindelof, then $Y$ is $S$-Lindelof.

(3) if $X$ is strongly countably compact, then $Y$ is countably $S$-closed.

**Proof.** We prove only (1), the proofs of (2) and (3) being entirely analogous.

Let \( \{V_\alpha : \alpha \in I\} \) be any regular closed cover of $Y$. Since $f$ is almost contra-precontinuous, then \( \{f^{-1}(V_\alpha) : \alpha \in I\} \) is a preopen cover of $X$ and hence there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Therefore, we have $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ and $Y$ is $S$-closed.

**Definition 20.** A space $X$ is said to be

(1) $P$-closed if every preclosed cover of $X$ has a finite subcover.

(2) countably $P$-closed if every countable cover of $X$ by preclosed sets has a finite subcover.

(3) $P$-Lindelof if every cover of $X$ by preclosed sets has a countable subcover.

(4) nearly compact if every regular open cover of $X$ has a finite subcover [21].

(5) nearly countably compact if every countable cover of $X$ by regular open sets has a finite subcover [7, 22].

(6) nearly Lindelof if every cover of $X$ by regular open sets has a countable subcover.

**Theorem 15.** Let $f : X \to Y$ be an almost contra-precontinuous surjection. Then the following statements hold:

(1) if $X$ is $P$-closed, then $Y$ is nearly compact.

(2) if $X$ is $P$-Lindelof, then $Y$ is nearly Lindelof.

(3) if $X$ is countably $P$-closed, then $Y$ is nearly countably compact.

**Proof.** We prove only (1), the proofs of (2) and (3) being entirely analogous.

Let \( \{V_\alpha : \alpha \in I\} \) be any regular open cover of $Y$. Since $f$ is almost contra-precontinuous, then \( \{f^{-1}(V_\alpha) : \alpha \in I\} \) is a preclosed cover of $X$. Since $X$ is $P$-closed,
there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, we have $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ and $Y$ is nearly compact.

**Definition 21.** A space $X$ is said to be mildly compact (mildly countably compact, mildly Lindelof) if every clopen cover (respectively, clopen countable cover, clopen cover) of $X$ has a finite (respectively, a finite, a countable) subcover [24].

**Theorem 16.** If $f : X \to Y$ is an almost contra-precontinuous and almost continuous surjection and $X$ is mildly compact (resp. mildly countably compact, mildly Lindelof), then $Y$ is nearly compact (resp. nearly countably compact, nearly Lindelof) and $S$-closed (resp. countably $S$-closed, $S$-Lindelof).

**Proof.** Let $V \in RC(Y)$. Then since $f$ is almost contra-precontinuous and almost continuous, $f^{-1}(V)$ is preopen and closed in $X$ and hence $f^{-1}(V)$ is clopen. Let $\{V_\alpha : \alpha \in I\}$ be any regular closed (respectively regular open) cover of $Y$. Then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a clopen cover of $X$ and since $X$ is mildly compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, we obtain $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$. This shows that $Y$ is $S$-closed (respectively nearly compact).

The other proofs can be obtained similarly.

5. $P$-regular graphs

In this section, we define $P$-regular graphs and investigate the relationships between $P$-regular graphs and almost contra-precontinuous functions.

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of $f$ and is denoted by $G(f)$.

**Definition 22.** A graph $G(f)$ of a function $f : X \to Y$ is said to be $P$-regular if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a preclosed set $U$ in $X$ containing $x$ and $V \in RO(Y)$ containing $y$ such that $(U \times V) \cap G(f) = \emptyset$.

**Lemma 2.** The following properties are equivalent for a graph $G(f)$ of a function:

1. $G(f)$ is $P$-regular;
2. for each point $(x, y) \in (X \times Y) \setminus G(f)$, there exist a preclosed set $U$ in $X$ containing $x$ and $V \in RO(Y)$ containing $y$ such that $f(U) \cap V = \emptyset$. 
Proof. It is an immediate consequence of definition of $P$-regular graph and the fact that for any subsets $A \subseteq X$ and $B \subseteq Y$, $(A \times B) \cap G(f) = \emptyset$ if and only if $f(A) \cap B = \emptyset$.

Theorem 17. If $f : X \to Y$ is almost contra-precontinuous and $Y$ is $T_2$, then $G(f)$ is $P$-regular graph in $X \times Y$.

Proof. First, suppose that $Y$ is $T_2$. Let $(x, y) \in (X \times Y) \setminus G(f)$. It follows that $f(x) \neq y$. Since $Y$ is $T_2$, there exist open sets $V$ and $W$ containing $f(x)$ and $y$, respectively, such that $V \cap W = \emptyset$. We have $\text{int}(cl(V)) \cap \text{int}(cl(W)) = \emptyset$. Since $f$ is almost contra-precontinuous, $f^{-1}(\text{int}(cl(V)))$ is preclosed in $X$ containing $x$. Take $U = f^{-1}(\text{int}(cl(V)))$. Then $f(U) \subseteq \text{int}(cl(V))$. Therefore, $f(U) \cap \text{int}(cl(W)) = \emptyset$ and $G(f)$ is $P$-regular in $X \times Y$.

Theorem 18. Let $f : X \to Y$ have a $P$-regular graph $G(f)$. If $f$ is injective, then $X$ is pre-$T_1$.

Proof. Let $x$ and $y$ be any two distinct points of $X$. Then, we have $(x, f(y)) \in (X \times Y) \setminus G(f)$. By definition of $P$-regular graph, there exist a preclosed set $U$ of $X$ and $V \in RO(Y)$ such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \emptyset$; hence $U \cap f^{-1}(V) = \emptyset$. Therefore, we have $y \notin U$. Thus, $y \in X \setminus U$ and $x \notin X \setminus U$. We obtain that $X \setminus U \in PO(X)$. This implies that $X$ is pre-$T_1$.

Theorem 19. Let $f : X \to Y$ have a $P$-regular graph $G(f)$. If $f$ is surjective, then $Y$ is weakly $T_2$.

Proof. Let $y_1$ and $y_2$ be any distinct points of $Y$. Since $f$ is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) \setminus G(f)$. By definition of $P$-regular graph, there exist a preclosed set $U$ of $X$ and $F \in RO(Y)$ such that $(x, y_2) \in U \times F$ and $f(U) \cap F = \emptyset$; hence $y_1 \notin F$. Then $y_2 \notin Y \setminus F \in RC(Y)$ and $y_1 \in Y \setminus F$. This implies that $Y$ is weakly $T_2$.

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