On Strongly 0-Prime Ideals in Near-Rings

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Abstract. In this paper we introduce the notion of strongly 0-prime ideals in near-rings similar to the notion introduced in rings. We give some characterizations of a near-ring $N$ whose unique maximal nil ideal $N_r(N)$ coincides with the set of all its nilpotent elements $N(N)$ by using its minimal strongly 0-prime ideals.

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1. Introduction

Throughout this paper $N$ stands for a near-ring with identity. We use $N_r(N)$ and $N(N)$ to represent the unique maximal nil ideal and the set of all nilpotent elements of $N$ respectively. Observe that $N_r(N) = N(N)$ if and only if $N_r(N)$ is a completely semiprime ideal of $N$ (i.e., $a^2 \in N_r(N)$ implies $a \in N_r(N)$ for $a \in N$).

An ideal $P$ of $N$ is 0-prime if for any two ideals $A$ and $B$ of $N$, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal $P$ of $N$ is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for any $a, b \in N$ [1]. An ideal $P$ of $N$ is said to be strongly 0-prime if $P$ is 0-prime and $N/P$ has no non-zero nil ideals. A near-ring $N$ is said to be strongly 0-prime if the ideal $\{0\}$ is strongly 0-prime. We have shown that if $M = \{x, x^2, x^3, \cdots\}$ where $x$ is not a nilpotent element of $N$, then there exists a strongly 0-prime ideal $P$ of $N$ such that $P \cap M = \emptyset$. An ideal $P$ of a near-ring is minimal strongly 0-prime ideal if $P$ is minimal among strongly 0-prime ideals of $N$. Observe that every completely prime ideal of $N$ is strongly 0-prime and every strongly 0-prime ideal is 0-prime but the converses do not hold.

An ideal $I$ of $N$ is said to have the insertion of factors property (or) simply $IFP$ if $xy \in I$ implies $xNy \subseteq I$ for $x, y \in N$. An ideal $I$ of $N$ has the strict $IFP$ if $xy \in I$ implies $\langle x \rangle N \langle y \rangle \subseteq I$ for $x, y \in N$. Observe that every completely semiprime ideal of $N$ has the $IFP$. In a ring, $IFP$ implies strict $IFP$ but in a near-ring $IFP$ does not imply strict $IFP$.
Recently, Kim and Kwak [3] characterized 2-primal rings in terms of their minimal prime ideals. Hong and Kwak [2] characterized a ring satisfying \( N'_r(R) = N(R) \) in terms of its minimal strongly prime ideals. So, in this paper we give some characterizations of a near-ring \( N \) whose unique maximal nil ideal \( N'_r(N) \) coincides with the set of all its nilpotent elements \( N(N) \) by using its minimal strongly 0-prime ideals. For the basic definition and terminology we refer to [4].

**Example 1.1.** Consider the near-ring \((N, +, \cdot)\) defined on the Klein’s four group \((N, +)\) with \( N = \{0, a, b, c\} \) where \( \cdot \) is defined as follows (as per scheme 2, p.408 [4]).

\[
\begin{array}{cccc}
  . & 0 & a & b & c \\
  0 & 0 & 0 & 0 & 0 \\
  a & 0 & 0 & a & a \\
  b & 0 & a & b & b \\
  c & 0 & a & c & c \\
\end{array}
\]

Clearly \( \{0, a\} \) is a strongly 0-prime ideal, since the ideals are \( \{0\}, \{0, a\} \) and \( \{0, a, b, c\} \).

Let \( N \) be a near-ring and let \((m)\text{Spec}(N)\) be the set of all (minimal) strongly 0-prime ideals of \( N \). For \( P \in \text{Spec}(N) \), we put

\[
O(P) = \{a \in N \mid aN^b = 0 \text{ for some } b \in N \setminus P \}.
\]

\[
\overline{O}(P) = \{a \in N \mid a^m \in O(P) \text{ for some positive integer } m \}.
\]

\[
O_p = \{a \in N \mid ab = 0 \text{ for some } b \in N \setminus P \}.
\]

\[
\overline{O}_p = \{a \in N \mid a^m \in O_p \text{ for some positive integer } m \}.
\]

\[
N(P) = \{a \in N \mid aN^b \subseteq N'_r(N) \text{ for some } b \in N \setminus P \}.
\]

\[
\overline{N}(P) = \{a \in N \mid a^m \in N(P) \text{ for some positive integer } m \}.
\]

\[
N_p = \{a \in N \mid ab \in N'_r(N) \text{ for some } b \in N \setminus P \}.
\]

\[
\overline{N}_p = \{a \in N \mid a^m \in N_p \text{ for some positive integer } m \}.
\]
Hong and Kwak [2] have defined \( O(P) \) in a ring \( R \) as \( O(P) = \{ a \in R \mid aRb = 0 \} \) for some \( b \in R \setminus P \}. But we have defined \( O(P) = \{ a \in R \mid aR\{b\} = 0 \} \) for some \( b \in R \setminus P \}. These two definitions coincide in rings. Suppose \( a \in O(P) \). Then \( aRb = 0 \) for some \( b \in R \setminus P \} \implies b \in (0 : aR) = \{ x \in R \mid aRx = 0 \} \), which is an ideal if \( R \) is a ring. Thus \( aRb = 0 \) implies \( aR\{b\} = 0 \).

We have shown that \( O(P) \) and \( N(P) \) are ideals of \( N \) and they are subsets of \( P \). Clearly \( O(P) \subseteq O_P \subseteq \overline{O}_P \), \( N(P) \subseteq N_P \subseteq \overline{N}_P \), \( \overline{O}(P) \subseteq \overline{O}_P \) and \( \overline{N}(P) \subseteq \overline{N}_P \). If \( P \) is a completely prime ideal of \( N \), then \( \overline{N}_P \) is a subset of \( P \). For a reduced near-ring \( N \), \( O(P) = O_p \) and \( \overline{O}(P) = \overline{O}_p = N(P) = N_p = \overline{N}(P) = \overline{N}_p \).

**Lemma 1.2.** \( O(P) \) and \( N(P) \) are ideals of \( N \) for any strongly 0-prime ideal \( P \) of \( N \).

**Proof.** Let \( P \) be any strongly 0-prime ideal of \( N \) and let \( a_1, a_2 \in O(P) \). Then \( a_1 N\{b_1\} = 0 \) for some \( b_1 \in N \setminus P \) and \( a_2 N\{b_2\} = 0 \) for some \( b_2 \in N \setminus P \). Since \( b_1 \), \( b_2 \in N \setminus P \) and \( N \setminus P \) is an \( m \)-system there exists \( b_1' \in \{b_1\} \) and \( b_2' \in \{b_2\} \) such that \( b_1' b_2' \in N \setminus P \). Let \( b_2 = b_1' b_2' \). For any \( n \in N \) and \( x \in \{b_2\} \), \( (a_1 - a_2)nx = 0 \) implies \( a_1 - a_2 \in O(P) \). Let \( x \in O(P) \). Then \( xN\{b\} = 0 \) for some \( b \in N \setminus P \). Thus for \( n, n' \in N \) and \( b' \in \{b\} \), we have \( (n(n' + x) - nn')n'b' = 0 \) implies \( n(n' + x) - nn' \) \( \in O(P) \) and \( (xn)n'b' = 0 \) implies \( xn \in O(P) \). Therefore \( O(P) \) is an ideal of \( N \). Similarly one can show that \( N(P) \) is an ideal of \( N \).

**Lemma 1.3.** For a near-ring \( N \) and \( P \in \text{Spec} \, (N) \), we have the following:

(i) If \( O_p(N_p) \) is an ideal of \( N \) for any strongly 0-prime ideal of \( N \), then \( O_p(N_p) \) is a completely semiprime ideal of \( N \) if and only if \( O_p = \overline{O}_p (N_p = \overline{N}_p). \)

(ii) \( O(P)(N(P)) \) is a completely semiprime ideal of \( N \) if and only if \( O(P) = \overline{O}(P)(N(P) = \overline{N}(P)) \).

**Proof.**

(i) Let \( P \) be any strongly 0-prime ideal of \( N \). Suppose that \( O_p \) is a completely semiprime ideal of \( N \). Let \( a \in \overline{O}_p \). Then \( a^m b = 0 \) for some positive integer \( m \) and for some \( b \in N \setminus P \). Thus \( a^m \in O_p \) and this implies \( a \in O_p \) as \( O_p \) is completely semiprime. Therefore \( \overline{O}_p \subseteq O_p \) and hence \( \overline{O}_p = O_p \). The converse is obvious. Proof of part (ii) is similar to that of (i).
Theorem 1.4. If \( M = \{x, x^2, x^3, \cdots\} \) where \( x \) is not a nilpotent element of \( N \), then there exist a strongly 0-prime ideal \( P \) of \( N \) such that \( P \cap M = \emptyset \).

Proof. Let \( M = \{x, x^2, x^3, \cdots\} \) and \( S = \{I \mid I \cap M = \emptyset\} \), where \( I \) is an ideal of \( N \). Then \( S \) is non-empty as \( \{0\} \in S \). By Zorn’s Lemma, \( S \) has a maximal element say \( P \).

We claim that \( P \) is strongly 0-prime. First we show that \( P \) is 0-prime. Suppose \( I_1 \) and \( I_2 \) are ideals of \( N \) such that \( I_1 \supset P \) and \( I_2 \supset P \). Let \( a \in I_1 \cap M \) and \( b \in I_2 \cap M \). Then we have \( a = x^n \) and \( b = x^m \) for some positive integers \( n, m \). Therefore \( ab = x^{n+m} \in I_1 I_2 \cap M \) implies \( I_1 I_2 \cap M \neq \emptyset \) and hence \( I_1 I_2 \subseteq P \). Therefore \( P \) is 0-prime. If \( I / P \) is a non-zero nil ideal of \( N / P \), then \( I \subseteq P \) and so \( I \cap M \neq \emptyset \).

Let \( y \in I \cap M \). Then \( y = x^k \) for some positive integer \( k \). Since \( I / P \) is a nil ideal, \( x^k P = P \) for some positive integer \( m \). Thus \( x^{km} \in P \) which is a contradiction. Therefore \( P \) is a strongly 0-prime ideal of \( N \) such that \( P \cap M = \emptyset \).

Lemma 1.5. For a near-ring \( N \), \( N_r(N) = \cap \{P \mid P \) is a strongly 0-prime ideal of \( N\} \) = \( \cap \{P \mid P \) is a minimal strongly 0-prime ideal of \( N\} \).

Proof. Suppose \( N_r(N) \subseteq P \) for some \( P \in \text{Spec}(N) \). Then \( N_r(N) / P \) is a non-zero nil ideal of \( N / P \) which is a contradiction that \( P \) is a strongly 0-prime ideal of \( N \). Thus \( N_r(N) \subseteq P \) for all strongly 0-prime ideals \( P \) of \( N \) and so \( N_r(N) \subseteq \cap \{P \mid P \) is a strongly 0-prime ideal of \( N\} \).

If \( x^k \neq 0 \) for any positive integer \( k \) and if \( M = \{x, x^2, x^3, \cdots\} \) then by Theorem 1.4, there exists a strongly 0-prime ideal \( P \) such that \( P \cap M = \emptyset \). Thus \( x \notin P \) which is a contradiction. Therefore \( x^k = 0 \) for some positive integer \( k \). So \( x \in N_r(N) \). The other equality is obvious.

Lemma 1.6. For a near-ring, we have the following:

(i) \( N(N) \subseteq \cap_{P \in \text{Spec}(N)} \overline{O}(P) \subseteq \cap_{Q \in \text{Spec}(N)} \overline{O}(Q) \).

(ii) \( N_r(N) \subseteq \cap_{P \in \text{Spec}(N)} N(P) = \cap_{Q \in \text{Spec}(N)} N(Q) \).

Proof. (i) Let \( a \in N(N) \). Then \( a^n = 0 \) for some positive integer \( n \). Let \( P \) be any strongly 0-prime ideal and let \( b \in N \setminus P \). Since \( a^n = 0 \), \( a^n N(b) = 0 \). Thus \( a^n \in O(P) \).
and hence $a \in \overline{O}(P)$. Therefore $a \in \bigcap_{P \in \text{Spec}(N)} \overline{O}(P)$. The other inclusion is obvious.

(ii) Let $a \in N_r(N)$. Let $P$ be any strongly 0-prime ideal of $N$. Then $aN \{b\} \subseteq N_r(N)$ for any $b \in N \setminus P$ which implies $a \in N(P)$. Thus $a \in P \bigcap_{P \in \text{Spec}(N)} N(P)$. But $P \bigcap_{P \in \text{Spec}(N)} N(P) \subseteq Q \bigcap_{Q \in \text{Spec}(N)} N(Q)$ always. Since $N(Q) \subseteq Q$ for each $Q \in m \text{ Spec}(N), \bigcap_{Q \in m \text{ Spec}(N)} N(Q) \subseteq N_r(N)$.

2. Strongly 0-prime ideals

Now we prove our main Theorem.

**Theorem 2.1.** For a near-ring $N$, the following statements are equivalent.

(i) $N_r(N) = N(N)$.

(ii) $N_r(N)$ is a completely semiprime ideal of $N$.

(iii) $N(P)$ is a completely semiprime ideal of $N$ for each $P \in m \text{ Spec}(N)$.

(iv) $\overline{N}_p = \overline{N}(P) = N(P)$ for each $P \in m \text{ Spec}(N)$.

(v) $N(P) = N_p$ for each $P \in m \text{ Spec}(N)$.

(vi) $N_p \subseteq P$ for each $P \in m \text{ Spec}(N)$.

(vii) $N_{P \setminus N_r(N)} \subseteq P / N_r(N)$ for each $P \in m \text{ Spec}(N)$.

**Proof.**

(i)⇒(ii) Since $N_r(N) = N(N)$ for any $x$ in $N$, $x^2 \in N_r(N)$ implies $x^2$ is nilpotent and hence $x \in N(N) = N_r(N)$. Therefore $N_r(N)$ is a completely semiprime ideal of $N$.

(ii)⇒(iii) Let $P$ be a minimal strongly 0-prime ideal of $N$. Let $x \in N$ be such that $x^2 \in N(P)$. Then $x^2 \in N_r(N)$ for some $b \in N \setminus P$. Since $N_r(N)$ is a completely semiprime ideal, it has the IJP. So $xN \{b\} \subseteq N_r(N)$ which implies $xN \{b\} \subseteq N_r(N)$. Thus $x \in N(P)$ and hence $N(P)$ is completely semiprime.

(iii)⇒(i) Let $a \in N(N)$. Then $a^n = 0$ for some positive integer $n$. If $a \notin N_r(N)$, then there exists a minimal strongly 0-prime ideal $P$ of $N$ such that $a \notin P$. Since $N(P)$ is a completely semiprime ideal, $a^n = 0 \in N(P)$ implies $a \in N(P) \subseteq P$, a contradiction. So $a \in N_r(N)$.
(ii)⇒(iv) Let \( P \) be a minimal strongly 0-prime ideal of \( N \) and let \( a \in N_p \) for some \( a \in N \). Then \( a^n \in N_p \) for some positive integer \( n \). Thus \( a^n b \in N_r(N) \) for some \( b \in N \setminus P \). Since \( N_r(N) \) is a completely semiprime ideal of \( N \), it has the IFP. So we have \( (ab)^n \in N_r(N) \) implies \( ab \in N_r(N) \) and hence \( a \in N(P) \). Thus \( \overline{N}_p \subseteq N(P) \).

But \( N(P) \subseteq N_p \subseteq \overline{N}_p \) and \( \overline{N}(P) = \overline{N}_p \). Therefore \( \overline{N}_p = \overline{N}(P) = N(P) \) for each \( P \in m \text{Spec}(N) \).

(iv)⇒(v)⇒(vi) These are obvious.

(vi)⇒(vii) Suppose that \(\overline{N} = N \setminus N_r(N) \) is not reduced. Then there exists \( a \in N \) such that \( \overline{a} \neq \overline{0} \). Hence \( a \notin N_r(N) \). So there exists some strongly 0-prime ideal \( P \) of \( N \) such that \( a \notin \overline{P} \) and this implies \( a \in \overline{N} \setminus \overline{P} \). But \( \overline{a}^2 = \overline{0} \) implies \( a \in N_p \subseteq \overline{P} \), a contradiction. Therefore \( N_r(N) = N(N) \).

Note that if \( R \) is a ring, then \( N_r(R) \) has the IFP if and only if \( N_r(R) \) is completely semiprime. Let us assume that \( N_r(R) \) has IFP and let \( x^2 \in N_r(R) \). Then \( xRx \subseteq N_r(R) \) and hence \( xRx \subseteq P \) for every strongly 0-prime ideal \( P \) of \( N \). So \( x \in P \) for every strongly 0-prime ideal \( P \). Therefore \( x \in N_r(R) \). Thus we have the following Corollary.

**Corollary 2.2.** [2, Theorem 8] For a ring \( R \), the following statements are equivalent.

(i) \( N_r(R) = N(R) \).

(ii) \( N_r(R) \) has the IFP.

(iii) \( N(P) \) has the IFP for each \( P \in m \text{Spec}(R) \).

(iv) \( \overline{N}_p = \overline{N}(P) = N(P) \) for each \( P \in m \text{Spec}(R) \).

(v) \( N(P) = N_p \) for each \( P \in m \text{Spec}(R) \).

(vi) \( N_p \subseteq P \) for each \( P \in m \text{Spec}(R) \).

(vii) \( N_{P/\overline{N}_p} \subseteq P / N_r(R) \) for each \( P \in m \text{Spec}(R) \).

**Corollary 2.3.** For a near-ring \( N \), assume that \( N_r(N) = N(N) \). If \( P = N(P) \) for each \( P \in \text{Spec}(N) \), then \( P \) is completely prime ideal of \( N \).
Proof. Let \( N_r(N) = N(N) \) and \( P = N(P) \) for each \( P \in \text{Spec}(N) \). Let \( ab \in P \) for \( a, b \in N \). Since \( N(P) \) is a completely semiprime ideal of \( N \), we have \( \langle a \rangle \langle b \rangle \subseteq P \) and hence \( a \in P \) or \( b \in P \). Therefore \( P \) is completely prime.

**Theorem 2.4.** For a near-ring \( N \), assume that \( N_r(N) = N(N) \). Then for each \( P \in \text{Spec}(N) \), the following statements are equivalent.

(i) \( P \in m \text{ Spec}(N) \).

(ii) \( (N) = P \).

Proof. (i)\( \Rightarrow \) (ii) Let \( \in m \text{ Spec}(N) \) and \( a \in P \). Suppose \( a \notin N(P) \). Let \( S = \{a, a^2, a^3, \ldots\} \). If \( 0 \in S \), then \( a^k = 0 \) for some positive integer \( k \) and hence \( a \in N(P) \), a contradiction. So \( 0 \notin S \). Let \( L = N \setminus P \) and let \( T = \{a^b b_1 a^b b_2 \cdots b_n a^b \neq 0 \mid b_i \in L, t_i \in [0] \cup Z^+ \} \), where \( Z^+ \) is the set of all positive integers. Then \( L \subseteq T \). Let \( M = S \cup T \). Let us show that \( M \) is an \( m \)-system. If \( x, y \in S \), then \( xay \in S \). Let \( x \in S \), \( y \in T \) with \( x = a^s, y = a^t b_1 a^t b_2 \cdots b_n a^t \). If \( xay \neq 0 \), then \( xay \in T \). Suppose \( xay = 0 \). Since \( b_1, b_2 \in L \), there exists \( b'_1 \in \{b_1\} \) and \( b'_2 \in \{b_2\} \) such that \( b'_1 b'_2 \in L \). Since \( b'_1 b'_2, b_3 \in L \), there exists \( b'_3 \in \langle b'_1 b'_2 \rangle \subseteq \langle b_1 \rangle \langle b_2 \rangle \) and \( b'_3 \in \{b_3\} \) such that \( b'_1 b'_2 b'_3 \in L \). Continuing this process we get \( b'_{13 \ldots n-1} b'_{n-1,1} b_0 \in L \). Then there exists \( b'_{13 \ldots n-1} \in \langle b'_{13 \ldots n-2} b'_{n-1,1} \rangle \subseteq \langle \langle b_1 \rangle \langle b_2 \rangle \rangle \cdots \langle b_{n-1} \rangle \rangle \rangle \rangle \rangle \) and \( b'_n \in \{b_n\} \) such that \( w = b'_{13 \ldots n-1} b'_n \in L \). Since \( xay = 0 \), \( xay \in N_r(N) \). Thus \( a^s a^t b_1 a^t b_2 \cdots b_n a^t \in N_r(N) \). Since \( N_r(N) = N(N) \), \( N_r(N) \) is a completely semiprime ideal of \( N \) and hence \( b_1 b_2 \cdots b_n a^{1+s+t_0+\cdots+t_n} \in N_r(N) \). Choose \( m = 1 + s + t_0 + \cdots + t_n \). Then \( b_1 b_2 \cdots b_n a^m \in N_r(N) \). Since \( N_r(N) \) has the IFP, \( \langle b_1 \rangle \langle b_2 \rangle \cdots \langle b_n \rangle \langle a^m \rangle \subseteq N_r(N) \). This implies \( \langle \langle b_1 \rangle \langle b_2 \rangle \rangle \cdots \langle b_n \rangle \langle a^m \rangle \rangle \subseteq N_r(N) \). Continuing this process, we get \( \cdots \\langle \langle b_1 \rangle \langle b_2 \rangle \rangle \cdots \langle b_{n-1} \rangle \rangle \langle b_n \rangle \langle a^m \rangle \rangle \subseteq N_r(N) \) and so \( b'_{13 \ldots n-1} b'_n a^m \in N_r(N) \). Hence \( wa^m \in N_r(N) \) where \( w = b'_{13 \ldots n-1} b'_n \). Since \( N_r(N) \) is a completely semiprime ideal, \( (aw)^m \in N_r(N) \) and hence \( aw \in N_r(N) \). Thus \( a \in N_r = N(P) \), which is a contradiction.

Similarly, one can show that if \( x, y \in T \) then \( xay \neq 0 \) and \( xay \in T \). This shows that \( M \) is an \( m \)-system that is disjoint from \( (0) \). Hence there is a \( 0 \)-prime ideal \( Q \) that is disjoint from \( M \) such that \( a \notin Q \) and \( \subseteq P \). Now we claim that \( Q \) is strongly
0-prime. Suppose $I/Q$ is a non-zero nil ideal of $N/Q$. Since $Q \subset I$, $I \cap M \neq \emptyset$.

If $a^m \in I$ for some positive integer $m$, then $a^m + Q$ is a nilpotent element in $N/Q$. Thus $a^{mk} \in Q$ for some positive integer $k$, which is a contradiction. So we choose $x \in I \cap T$. Then $x \in T$ implies $0 \neq x' \in T$ for any positive integer $t$. Since $x + Q$ is nilpotent in $N/Q$, $x' \in Q$ for some positive integer $s$, which is again a contradiction.

Therefore $Q$ is a strongly 0-prime ideal of $N$ such that $a \notin Q$, a contradiction. Hence $N(P) = P$.

(ii)$\Rightarrow$(i) If $Q \subseteq P$ for $Q \in m \text{Spec}(N)$, then $N(P) \subseteq N(Q) \subseteq Q \subseteq P = N(P)$. Therefore $P \in m \text{Spec}(N)$.

**Corollary 2.5.** [2, Theorem 12] For a ring $R$, assume that $N_r(R) = N(R)$. Then for each $P \in \text{Spec}(R)$, the following statements are equivalent.

(i) $P \in m \text{Spec}(R)$.

(ii) $N(P) = P$.

A right ideal $I$ of a near-ring $N$ is called right (left) symmetric if $xyz \in I$ implies $xyz \in I(yxz \in I)$. An ideal $I$ of $N$ is symmetric if it is both right and left symmetric. An ideal $I$ of $N$ is called semi-symmetric if $x_1, x_2, \cdots x_n \in I$ implies $\langle x_1 \rangle \langle x_2 \rangle \cdots \langle x_n \rangle \subseteq I$.

**Theorem 2.6.** For a near-ring $N$, the following statements are equivalent.

(i) $N_r(N) = N(N)$.

(ii) $P$ is a completely prime ideal of $N$ for each $P \in m \text{Spec}(N)$.

(iii) $P$ is a completely semiprime ideal of $N$ for each $P \in m \text{Spec}(N)$.

(iv) $P$ has the strict IFP for each $P \in m \text{Spec}(N)$.

(v) $P$ is a symmetric ideal of $N$ for each $P \in m \text{Spec}(N)$.

(vi) $P$ is a semi-symmetric ideal of $N$ for each $P \in m \text{Spec}(N)$.

(vii) $ab \in P$ implies $bNa \subseteq P$ for $a, b \in N$ and $P \in m \text{Spec}(N)$.

**Proof.**

(i)$\Rightarrow$(ii). It follows from Theorem 2.4 and Corollary 2.3.

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv). These are obvious.

(iv)$\Rightarrow$(i). It follows from replacing $N(P)$ by $P$ in the proof of (c)$\Rightarrow$(a) of Theorem 2.1.

(ii)$\Rightarrow$(v)$\Rightarrow$(vi)$\Rightarrow$(vii) and (vii)$\Rightarrow$(ii) are trivial.
Corollary 2.7. [2, Corollary 13] For a ring $R$, the following statements are equivalent.

(i) $N_c(R) = N(R)$.
(ii) $P$ is a completely prime ideal of $R$ for each $P \in m\text{Spec}(R)$.
(iii) $P$ is a completely semiprime ideal of $R$ for each $P \in m\text{Spec}(R)$.
(iv) $P$ has the IFP for each $P \in m\text{Spec}(R)$.
(v) $P$ is a symmetric ideal of $R$ for each $P \in m\text{Spec}(R)$.
(vi) $xy \in P$ implies $yRx \subseteq P$ for $x, y \in R$ and $P \in m\text{Spec}(R)$.

References