On Full Hilbert C*-Modules

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Abstract. Let $M$ be both a full Hilbert $A$-module and a full Hilbert $B$-module. In this paper we prove that a map $\phi : A \to B$ is an isometrically *-isomorphism iff it satisfies $ax = \phi(a)x$ and $\phi(<x,y>_A) = <x,y>_B$ where $a \in A, x, y \in M$. We also show that the fullness condition can not be dropped.

1. Introduction

Hilbert C*-modules are used as a powerful tool in C*-algebraic quantum group theory, K- and KK-theory, induced representations of C*-algebras and Morita equivalence. Some sources of references to the subject are [1] and [2].

The goal of this paper is to show that if $M$ is full Hilbert C*-modules over C*-algebras $A$ and $B$ and $\phi : A \to B$ is a map, then $\phi$ is *-isomorphism iff $ax = \phi(a)x$ and $\phi(<x,y>_A) = <x,y>_B$ where $a \in A, x, y \in M$. We show that without any one of the assumptions of $M$ being full the result does not in general hold. Our result is interesting in its own.

Definition 1.1. Suppose $A$ is a C*-algebra. Let $M$ be a complex linear space which is a right $A$-module and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in C, a \in A$ and $x \in M$. $M$ is called a pre-Hilbert $A$-module if there exists an (A-valued) inner product $<\cdot,\cdot>: M \times M \to A$ satisfying:

1. $\langle x, x \rangle \geq 0$
2. $\langle x, x \rangle = 0$, iff $x = 0$
3. $\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle$
4. $\langle x, y \rangle = \langle y, x \rangle^*$
5. $\langle ax, y \rangle = a \langle x, y \rangle$.
A pre-Hilbert $A$-module is called a Hilbert $A$-module or Hilbert $C^*$-module over $A$, if it is complete with respect to the norm $\|x\| = \sqrt{\langle x, x \rangle}$. $M$ is said to be full if the linear span of $\{\langle x, y \rangle; x, y \in M\}$ is dense in $A$.

Example 1.2. Let $A$ be a $C^*$-algebra. Then $A$ together with its product as the usual action is a left $A$-module. In addition it equipped with inner product $\langle a, b \rangle = ab^*$ is a full Hilbert $A$-module.

2. Main theorem

Let $M$ be a (full) Hilbert $B$-module and $\phi : A \rightarrow B$ a *-isomorphism of $C^*$-algebras. Define the module action by $xaax \phi = a$ and $A$-valued inner product by $\langle \phi(x), \phi(y) \rangle_B = \langle x, y \rangle_A$. Then it is straightforward to show that $M$ is a (full) Hilbert $A$-module. We are going to establish a converse statement to the above.

Lemma 2.1. Let $N$ be a full Hilbert $C^*$-module over $C$ and $u \in C$. Then $ux = 0$ for all $x \in N$ iff $u = 0$.

Proof. Since $N$ is full, there exists $\{u_n\}$ in $\langle N, N \rangle$ such that $u = \lim_n u_n$.

Each $u_n$ is of the form $u_n = \sum_{i=1}^k <x_{i,n}, y_{i,n}>$ in which $x_{i,n}, y_{i,n} \in N$. Hence

$$uu^* = u \lim_n u_n^* = \lim_n uu_n^* = \lim_n \left( u \sum_{i=1}^k <y_{i,n}^*, x_{i,n}> \right) = \lim_n \sum_{i=1}^k <u_{i,n}^* y_{i,n}, x_{i,n}> = 0.$$ 

Hence $u = 0$.

Theorem 2.2. Let $M$ be both a full Hilbert $A$-module and a full Hilbert $B$-module and there exist a map $\phi : A \rightarrow B$ in such a way that $ax = \phi(a)x$ and $\phi(x, y)_A = <x, y>_B$. Then $\phi$ is an (isometrically) *-isomorphism.

Proof. If $a_n \rightarrow 0$ and $\phi(a_n) \rightarrow b$, then $a_nx \rightarrow 0$ and $\phi(a_n)x \rightarrow bx$. But $\phi(a_n)x \rightarrow 0$. Hence $bx = 0$. By Lemma 2.1 $b = 0$. Thus $\phi$ is continuous. $\phi(ab) = \phi(a)\phi(b)$.

Similarly one can show that $\phi$ is linear.
If \( a \in A \), then we may assume that \( a = \lim_{n} u_n, u_n = \sum_{i=1}^{k_n} <x_{i,n}, y_{i,n}>_A \) where \( x_{i,n}, y_{i,n} \in M \). Hence

\[
\phi(a^*) = \lim_{n} \phi(u_n^*) = \lim_{n} \sum_{i=1}^{k_n} \phi< y_{i,n}, x_{i,n}>_A = \lim_{n} \sum_{i=1}^{k_n} \{ y_{i,n}, x_{m} \}_B
\]

\[
= \left( \lim_{n} \sum_{i=1}^{k_n} \{ x_{i,n}, y_{i,n} \}_B \right)^* = (\phi(a))^*
\]

If \( \phi(a) = 0 \), then \( ax = \phi(a)x = 0 \) for all \( x \in M \). Hence \( a = 0 \). \( \phi \) is therefore one to one.

Given \( b \in B \) and \( \varepsilon > 0 \), there are \( \{x_i\}_{1 \leq i \leq n}, \{y_i\}_{1 \leq i \leq n} \) in \( M \) such that

\[
\left\| b - \sum_{i=1}^{n} <x_i, y_i>_B \right\| < \varepsilon, \text{ Hence } \left\| b - \phi \sum_{i=1}^{n} <x_i, y_i>_A \right\| < \varepsilon. \text{ Therefore } \phi \text{ has a dense range. But } \phi \text{ is a } ^*\text{-homomorphism from } A \text{ into } B, \text{ so that its range is closed. Thus } \phi \text{ is a } ^*\text{-isomorphism.}
\]

**Remark 2.3.** The result may fail, if any one of the conditions of \( M \) being full is dropped.

For example, first, take \( A \) to be a von Neumann algebra acting on a Hilbert space which has a central projection \( p \neq 0, I \). Put \( B = M = Ap \) and consider \( M \) as a Hilbert \( B \)-module and a Hilbert \( A \)-module with the usual actions and the inner products \( <x, y>_A = xy^* \). Clearly \( M \) is not full \( A \)-module. Then \( \phi: A \rightarrow B, \phi(a) = ap \) has evidently the properties \( ax = \phi(a)x \) and \( \phi(<x, y>_A) = <x, y>_B \), but is not one to one (and hence is not isometry).

Second, let \( A \) and \( B \) be arbitrary \( C^* \)-algebras and \( A \) be a proper subset of \( B \). Put \( M = A \) and consider it as a Hilbert \( A \)-module and a Hilbert \( B \)-module such above. Clearly \( M \) is not full \( B \)-module. Then the inclusion map \( \phi: A \rightarrow B \) satisfies obviously \( ax = \phi(a)x \) and \( \phi(<x, y>_A) = <x, y>_B \), but is not surjective.

**References**


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