Entire Functions and their Derivatives Share Two Finite Sets

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Abstract. In this paper, we study the uniqueness of entire functions. We mainly obtain the following result: Let \( f(z) \) and \( g(z) \) be two non-constant entire functions, \( n \geq 5 \), \( k \) two positive integers, and let \( \{z: z^n = 1\}, \{a, b, c\} \) where \( a, b, c \) are nonzero finite distinct constants satisfying \( a^2 \neq bc, b^2 \neq ac, c^2 \neq ab \). If \( E(S_1, f) = E(S_2, g) \), \( E(S_2, f^{(k)}) = E(S_2, g^{(k)}) \), then \( f(z) = g(z) \).

1. Introduction and main results

Let \( f(z) \) be a non-constant meromorphic function in the whole complex plane. In this paper we use the following standard notations of value distribution theory,

\[ T(r, f), m(r, f), N(r, f), N\left(r, \frac{1}{f}\right), \ldots \]

(see Hayman [7], Yang [9]). We denote by \( S(r, f) \) any function satisfying

\[ S(r, f) = o(T(r, f)) \]

as \( r \to +\infty \), possibly outside of a set with finite measure.

Let \( S \) be a set of complex numbers. Set

\[ E(S, f) = \bigcup_{a \in S} \{z : f(z) - a = 0\} \]

where a zero point with multiplicity \( m \) is counted \( m \) times in the set.

In 1977, Gross [5] posed the following question.
Question 1. Can one find two finite sets \( S_j (j = 1, 2) \) such that any two non-constant entire functions \( f \) and \( g \) satisfying \( E(S_j, f) = E(S_j, g) \) for \( j = 1, 2 \) must be identical?

Yi [11] gave a positive answer to the question. He proved

**Theorem A.** Let \( f(z) \) and \( g(z) \) be two non-constant entire functions, \( n \geq 5 \) a positive integer, and let \( S_1 = \{ z : z^n = 1 \}, S_2 = \{ a \} \), where \( a \neq 0 \) is a constant satisfying \( a^{2n} \neq 1 \). If \( E(S_j, f) = E(S_j, g) \) for \( j = 1, 2 \), then \( f(z) = g(z) \).

In this paper, we have proved

**Theorem 1.** Let \( f(z) \) and \( g(z) \) be two non-constant entire functions, \( n \geq 5 \), \( k \) two positive integers, and let \( S_1 = \{ z : z^n = 1 \}, S_2 = \{ a, b, c \} \), where \( a, b, c \) are nonzero finite distinct constants satisfying \( a^2 \neq bc, b^2 \neq ac, c^2 \neq ab \). If \( E(S_1, f) = E(S_1, g) \), \( E(S_2, f^{(k)}) = E(S_2, g^{(k)}) \), then \( f(z) = g(z) \).

**Remark 1.** The following example shows that the condition that \( a, b, c \) are nonzero finite distinct constants satisfying \( a^2 \neq bc, b^2 \neq ac, c^2 \neq ab \) in Theorem 1 is necessary.

**Example 1.** Let \( S_1 = \{ z : z^n = 1 \}, S_2 = \{ a, b, \sqrt{ab} \} \), where \( a, b \) are two distinct nonzero constants. Taking \( f(z) = e^{\sqrt{ab}z}, g(z) = e^{-\sqrt{ab}z} \). Obviously, \( E(S_1, f) = E(S_1, g) = \{ z : z^n = 1 \} \). If \( E(S_2, f^*) = E(S_2, g^*) = \{ z : e^{\sqrt{ab}z} = 1 \} \), but \( f(z) \neq g(z) \).

When \( S_2 \) has two elements or one element, we have the following results.

**Theorem 2.** Let \( f(z) \) and \( g(z) \) be two non-constant entire functions, \( n \geq 5 \), \( k \) two positive integers, and let \( S_1 = \{ z : z^n = 1 \}, S_2 = \{ a, b \} \), where \( a, b \) are two nonzero finite distinct constants. If \( E(S_1, f) = E(S_1, g) \), \( E(S_2, f^{(k)}) = E(S_2, g^{(k)}) \), then one of the following cases must occur:

1. \( f(z) = g(z) \);
2. \( b = -a, f(z) = e^{cz+d}, g(z) = te^{-cz-d} \), where \( c, d, t \) are three constants satisfying \( t^n = 1 \) and \( (-1)^k te^{-2k} = a^2 \).
Proof of Theorem 1

First we consider the case when \( f(z) \) and \( g(z) \) are two transcendental entire functions. By Lemma 2 we know that either \( f(z)g(z) = t \) or \( f(z) = tg(z) \), where \( t \) is a constant satisfying \( t^n = 1 \). Next we divide two cases.

Case 1. \( f(z)g(z) = t \), where \( t \) is a constant satisfying \( t^n = 1 \). Obviously, \( f \neq 0 \). Hence we have

\[
f(z) = e^{b(z)}, \quad g(z) = te^{-b(z)}
\]
where \( h(z) \) is a non-constant entire function. Thus we have

\[
f^{(k)} = P(h',\cdots,h^{(k)})e^h, \quad g^{(k)} = tQ(h',\cdots,h^{(k)})e^{-h}
\]

(3.2)

where \( P, \ Q \) are polynomials of \( h', h'', \cdots, h^{(k)} \). Set

\[
P(h', h'', \cdots, h^{(k)}) (z) = P(h'(z), h''(z), \cdots, h^{(k)}(z)),
\]

\[
Q(h', h'', \cdots, h^{(k)}) (z) = Q(h'(z), \cdots, h^{(k)}(z)).
\]

Obviously there exists \( z_0 \) such that \( f^{(k)}(z_0) = a \). Then by \( E(S_2, f^{(k)}) = E(S_2, g^{(k)}) \) and (3.2) we deduce that one of the following cases must occur:

(i) \( g^{(k)}(z_0) = a, P(h', h'', \cdots, h^{(k)})(z_0) Q(h', h'', \cdots, h^{(k)})(z_0) - \frac{a^2}{t} = 0 \);

(ii) \( g^{(k)}(z_0) = b, P(h', h'', \cdots, h^{(k)})(z_0) Q(h', h'', \cdots, h^{(k)})(z_0) - \frac{ab}{t} = 0 \);

(iii) \( g^{(k)}(z_0) = c, P(h', h'', \cdots, h^{(k)})(z_0) Q(h', h'', \cdots, h^{(k)})(z_0) - \frac{ac}{t} = 0 \).

Next we consider four sub-cases.

**Case 1.1.** \( P(h'(z), h''(z), \cdots, h^{(k)}(z)) Q(h'(z), h''(z), \cdots, h^{(k)}(z)) - \frac{a^2}{t} \neq 0 \),

\[
P(h'(z), h''(z), \cdots, h^{(k)}(z)) Q(h'(z), h''(z), \cdots, h^{(k)}(z)) - \frac{ab}{t} \neq 0,
\]

\[
P(h'(z), h''(z), \cdots, h^{(k)}(z)) Q(h'(z), h''(z), \cdots, h^{(k)}(z)) - \frac{ac}{t} \neq 0.
\]

Then by \( E(S_2, f^{(k)}) = E(S_2, g^{(k)}) \) we obtain

\[
\bar{N}\left(r, \frac{1}{f^{(k)} - a}\right) \leq \bar{N}\left(r, \frac{1}{P(h', h'', \cdots, h^{(k)})(z) Q(h', h'', \cdots, h^{(k)})(z) - \frac{a^2}{t}}\right) + \bar{N}\left(r, \frac{1}{P(h', h'', \cdots, h^{(k)})(z) Q(h', h'', \cdots, h^{(k)})(z) - \frac{ab}{t}}\right) + \bar{N}\left(r, \frac{1}{P(h', h'', \cdots, h^{(k)})(z) Q(h', h'', \cdots, h^{(k)})(z) - \frac{ac}{t}}\right). \tag{3.3}
\]
By Logarithmic derivative lemma (see [7,9]), we have

$$T(r, h') = m(r, h') = m\left( r, \frac{f'}{f} \right) = S(r, f).$$

Obviously,

$$T(r, h^{(j)}) \leq T(r, h') + S(r, h') = S(r, f), \ (j = 2, \ldots, k).$$

Hence we get

$$T(r, P(h', h'', \ldots, h^{(k)})) = S(r, f), \ T(r, Q(h', h'', \ldots, h^{(k)})) = S(r, f). \quad (3.4)$$

Thus by (3.3), (3.4) and Nevanlinna first fundamental theorem we have

$$\frac{1}{f^{(k)}} - a \leq T\left( \frac{1}{f^{(k)}} - a \right) \leq T\left( \frac{1}{f^{(k)}} - a \right) + O(1) = S(r, f).$$

By Milloux’s inequality (see [7,9]) we obtain

$$T(r, f) \leq \overline{N}(r, f) + N\left( r, \frac{1}{f} \right) + \overline{N}\left( r, \frac{1}{f^{(k)} - a} \right) + S(r, f).$$

Hence by the above two formulas and (3.3)-(3.4) we deduce a contradiction: $T(r, f) = S(r, f)$.

Case 1.2. $P(h', h'', \ldots, h^{(k)}) = a^2$ and $f^{(k)}(z) = a$. Then by (3.2) we deduce that $f^{(k)}(z) = a^2$ and $f^{(k)}(z) = a$. Thus we obtain that if $f^{(k)}(z) = b$, then either $g^{(k)}(z) = b$ or $g^{(k)}(z) = c$. If $g^{(k)}(z) = b$ if and only if $f^{(k)}(z) = c$. Hence we deduce that $a^2 = b^2, a^2 = c^2$. Thus we get either $a = b$ or $a = c$ or $b = c$, which is a contradiction.

If there exists $z_1$ such that $f^{(k)}(z_1) = b, g^{(k)}(z_1) = c$, then we get $a^2 = bc$, a contradiction.
Case 1.3. \( P(h',h'',\ldots,h^{(k)}) = Q(h',h'',\ldots,h^{(k)}) = \frac{ab}{t} = 0 \). Then by (3.2) we deduce that \( f^{(k)}(z)g^{(k)}(z) = ab \) and \( f^{(k)}(z) = a \) if and only if \( g^{(k)}(z) = b \), \( f^{(k)}(z) = b \) if and only if \( g^{(k)}(z) = a \). Hence by \( E(S_2,f^{(k)}) = E(S_2,g^{(k)}) \) we deduce that \( f^{(k)}(z) = c \) if and only if \( g^{(k)}(z) = c \). Thus by (3.2) we get \( c^2 = ab \), a contradiction.

Case 1.4. \( P(h',h'',\ldots,h^{(k)}) = Q(h',h'',\ldots,h^{(k)}) = \frac{ab}{t} = 0 \). In this case, by using the same argument as do in Case 1.3 we get a contradiction. Hence we deduce that \( f(z)g(z) = t \) is impossible.

Case 2. \( f(z) = tg(z) \), where \( t \) is a constant satisfying \( t^n = 1 \). Hence we have \( f^{(k)} = tg^{(k)} \).

We claim that \( t = 1 \). Without loss of generality, we assume that there exist \( z_1 \) and \( z_2 \) such that \( f^{(k)}(z_1) = a \) and \( f^{(k)}(z_2) = b \). Suppose that \( t \neq 1 \), then by \( E(S_2,f^{(k)}) = E(S_2,g^{(k)}) \) and \( f^{(k)} = tg^{(k)} \) we deduce that either \( g^{(k)}(z_1) = b \) or \( g^{(k)}(z_1) = c \) and that either \( g^{(k)}(z_2) = a \) or \( g^{(k)}(z_2) = c \). Now we discuss the following four cases.

(2.1) \( g^{(k)}(z_1) = b \), \( g^{(k)}(z_2) = a \). Then by \( f^{(k)}(z) = tg^{(k)}(z) \) we get \( a = tb \) and \( b = ta \). Thus we get \( b = -a \), \( t = -1 \). If there exists \( z_3 \) such that \( f^{(k)}(z_3) = c \) then \( g^{(k)}(z_3) = -c \). Hence by \( E(S_2,f^{(k)}) = E(S_2,g^{(k)}) \) we deduce that \( -c = a \) or \( -c = b \) or \( -c = c \). Thus by \( b = -a \) we get \( c = b \) or \( c = a \) or \( c = 0 \), which is a contradiction. If there exists \( z_3 \) such that \( g^{(k)}(z_3) = c \), then we can similarly deduce a contradiction. If \( f^{(k)}(z) \neq c \) and \( g^{(k)}(z) \neq c \), then by \( f^{(k)}(z) = -g^{(k)}(z) \) we get \( f^{(k)}(z) \neq c, -c \), which contradicts Picard’s theorem.

(2.2) \( g^{(k)}(z_1) = b \), \( g^{(k)}(z_2) = c \). Then by \( f^{(k)}(z) = tg^{(k)}(z) \) we get \( a = tb \) and \( b = tc \). Thus we get \( b^2 = ac \), a contradiction.

(2.3) \( g^{(k)}(z_1) = c \), \( g^{(k)}(z_2) = a \). Then by \( f^{(k)}(z) = tg^{(k)}(z) \) we get \( a^2 = bc \), a contradiction.

(2.4) \( g^{(k)}(z_1) = c \), \( g^{(k)}(z_2) = c \). Then by \( f^{(k)}(z) = tg^{(k)}(z) \) we get \( a = b \), a contradiction.

Hence we deduce that \( t = 1 \), that is \( f(z) = g(z) \).
Now we consider the case when \( f(z) \) and \( g(z) \) are two polynomials. Thus by
\[
E(S_1, f) = E(S_1, g) \text{ we have} \quad f^n(z) - 1 = k[g^n(z) - 1],
\]
(3.5)
where \( k \) is a constant. Hence we have

\[
f^{n-1}(z)f'(z) = kg^{n-1}(z)g'(z).
\]
(3.6)
Thus by (3.6) and \( n \geq 5 \) we deduce that there exists \( z_0 \) such that \( f(z_0) = g(z_0) = 0 \). Substituting this into (3.5) we get \( k = 1 \), that is \( f^n(z) = g^n(z) \). Hence we get

\[
f(z) = tg(z),
\]
(3.7)
where \( t \) is a constant satisfying \( t^n = 1 \). Thus we have

\[
f^{(k)}(z) = t g^{(k)}(z).
\]
(3.8)
Next by using the similar argument to Case 2 we get \( f(z) = g(z) \). The proof of the theorem is complete.

4. \textit{Proofs of theorems 2-3}

As the proof of Theorem 2 and Theorem 3 is similar, we only give the

\textit{Proof of Theorem 2}. First we consider the case when \( f(z) \) and \( g(z) \) are two transcendental entire functions.

By Lemma 2 we know that either \( f(z)g(z) = t \), or \( f(z) = tg(z) \), where \( t \) is a constant satisfying \( t^n = 1 \). Next we divide two cases.

\textbf{Case 1.} \( fg = t \). Obviously, \( f \neq 0 \). Hence we have

\[
f(z) = e^{h(z)}, \quad g(z) = te^{-h(z)},
\]
(4.1)
where \( h(z) \) is a non-constant entire function. In the following we consider two sub-cases.

\textbf{Case 1.1.} \( k = 1 \). Thus we by (4.1) have

\[
f'(z) = h'(z)e^{h(z)}, \quad g(z) = -th'(z)e^{-h(z)}.
\]
(4.2)
Obviously there exists $z_0$ such that $f'(z_0) = a$. Then by $E(S_2, f') = E(S_2, g')$ and (4.2) we deduce that one of the following cases must occur:

(i) $g'(z_0) = a$, $[h'(z_0)]^2 + \frac{x^2}{t} = 0$;
(ii) $g'(z_0) = b$, $[h'(z_0)]^2 + \frac{ab}{t} = 0$.

Next we consider three sub-cases.

**Case 1.1.** $(h'(z))^2 + \frac{a^2}{t} \neq 0$, $(h'(z))^2 + \frac{ab}{t} \neq 0$. Then by using the same argument as do in Case 1.1 of the proof of Theorem 1 we deduce a contradiction.

**Case 1.2.** $(h'(z))^2 + \frac{a^2}{t} = 0$. Then we have $h(z) = cz + d$, where $c, d$ are two constants satisfying $-tcz^2 = a^2$. Thus we get

$$f(z) = e^{cz + d}, g(z) = te^{-cz - d}.$$ 

Hence we have

$$f'(z) = ce^{cz+d}, g'(z) = -tce^{-cz-d}.$$ 

Obviously, $f'(z) = a$ if and only if $g'(z) = a$. Thus by $E(S_2, f') = E(S_2, g')$ we deduce that $f'(z) = b$ if and only if $g'(z) = b$. Hence we get $a^2 = b^2$, that is $b = -a$. Thus the conclusion (2) occurs.

**Case 1.3.** $(h'(z))^2 + \frac{ab}{t} = 0$. Then we have $h(z) = cz + d$, where $c, d$ are two constants satisfying $-tcz^2 = ab$. Thus we get

$$f(z) = e^{cz + d}, g(z) = te^{-cz - d}.$$ 

Hence we have

$$f'(z) = ce^{cz+d}, g'(z) = -tce^{-cz-d}.$$ 

Obviously, $f'(z) = a$ if and only if $g'(z) = b$, $f'(z) = b$ if and only if $g'(z) = a$. Thus the conclusion (3) occurs.

**Case 2.** $k \geq 2$. Then by (4.1) we have

$$f^{(k)} = P(h', \cdots, h^{(k)})e^h, g^{(k)} = tQ(h', \cdots, h^{(k)})e^{-h}$$ (4.3)

where $P, Q$ are polynomials of $h', h'', \cdots, h^{(k)}$. 

Obviously there exists \( z_0 \) such that \( f^{(k)}(z_0) = a \). Then by \( E(S_2, f^{(k)}) = E(S_2, g^{(k)}) \) and (4.3) we deduce that one of the following cases must occur:

(i) \( g^{(k)}(z_0) = a, P(h', h'', \ldots, h^{(k)})Q(h', h'', \ldots, h^{(k)})(z_0) - \frac{a^2}{r} = 0 \)

(ii) \( g^{(k)}(z_0) = b, P(h', h'', \ldots, h^{(k)})Q(h', h'', \ldots, h^{(k)})(z_0) - \frac{ab}{r} = 0 \).

Next we consider three sub-cases.

**Case 1.2.1.** \( P(h'(z), h''(z), \ldots, h^{(k)}(z))Q(h'(z), h''(z), \ldots, h^{(k)}(z)) - \frac{a^2}{r} \not= 0 \), and \( P(h'(z), h''(z), \ldots, h^{(k)}(z))Q(h'(z), h''(z), \ldots, h^{(k)}(z)) - \frac{ab}{r} \not= 0 \). Then by using the same argument as do in Case 1.1 of the proof of Theorem 2 we deduce a contradiction.

**Case 1.2.2.** \( P(h', h'', \ldots, h^{(k)})(Q(h', h'', \ldots, h^{(k)}) - \frac{a^2}{r} = 0 \). Thus by (4.3) we deduce that \( f^{(k)}(z)g^{(k)}(z) = a^2 \) and \( f^{(k)}(z) = a \) if and only if \( g^{(k)}(z) = a \). Hence we obtain \( f^{(k)}(z) \not= 0 \), thus by Lemma 1 we deduce that \( f(z) = e^{cz+d} \). Considering \( f(z)g(z) = t \), we get \( g(z) = te^{-cz-d} \). Thus we have

\[
 f^{(k)}(z) = c^k e^{cz+d}, \quad g^{(k)}(z) = (-1)^k tc^k e^{-cz-d},
\]

Obviously, \( c, d \) satisfies \( (-1)^k tc^{2k} = a^2 \). Thus the conclusion (2) occurs.

**Case 1.2.3.** \( P(h', h'', \ldots, h^{(k)})(Q(h', h'', \ldots, h^{(k)}) - \frac{ab}{r} = 0 \). Thus by (4.3) we deduce that \( f^{(k)}(z)g^{(k)}(z) = ab \) and \( f^{(k)}(z) = a \) if and only if \( g^{(k)}(z) = b \). Hence we have \( f^{(k)}(z) \not= 0 \), thus by Lemma 1 we deduce that \( f(z) = e^{cz+d} \). Considering \( f(z)g(z) = t \), we get \( g(z) = te^{-cz-d} \). Hence we have

\[
 f^{(k)}(z) = c^k e^{cz+d}, \quad g^{(k)}(z) = (-1)^k tc^k e^{-cz-d},
\]

Obviously, \( c, d \) satisfies \( (-1)^k tc^{2k} = ab \). Thus the conclusion (3) occurs.

**Case 2.** \( f = tg \). Then \( f^{(k)} = tg^{(k)} \). Without loss of generality, we assume that there exists \( z_1 \) such that \( f^{(k)}(z_1) = a \). Suppose that \( t \not= 1 \), then by \( E(S_2, f^{(k)}) = E(S_2, g^{(k)}) \) and \( f^{(k)} = tg^{(k)} \) we deduce that \( g^{(k)}(z_1) = b \). Hence we deduce that \( f^{(k)}(z) = a \) if and only if \( g^{(k)}(z) = b \) and that \( f^{(k)}(z) = b \) if and only if \( g^{(k)}(z) = a \).
If \( f^{(k)}(z) = b \) has solution, then by \( f^{(k)}(z) = tg^{(k)}(z) \) we get \( a = tb \) and \( b = ta \). Hence we get \( b = -a \) and \( t = -1 \). That is \( f(z) = -g(z) \), the conclusion (4) occurs.

If \( f^{(k)}(z) \neq b \), then \( g^{(k)}(z) \neq a \). Hence by \( f^{(k)}(z) = tg^{(k)}(z) \) we get \( f^{(k)}(z) \neq b, ta \). If \( b \neq ta \), then by Picard’s theorem we get a contradiction. If \( b = ta \), then by \( f^{(k)}(z_1) = a \) and \( g^{(k)}(z_1) = b \) we get \( a = tb \). Hence we get \( b = -a \) and \( t = -1 \). That is \( f(z) = -g(z) \), the conclusion (4) occurs.

Now we consider the case when \( f(z) \) and \( g(z) \) are two polynomials. Then by using same argument as do in Theorem 1 we get \( f(z) = tg(z) \). Thus we obtain \( f^{(k)}(z) = tg^{(k)}(z) \). Next by using the similar argument to Case 2 we obtain \( f(z) = g(z) \).

The proof of the theorem is complete.

References


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