Uniquely N-colorable and Chromatically Equivalent Graphs

CHONG-YUN CHAO
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260

Abstract. For each integer \( n \geq 3 \), we present a uniquely \( n \)-colorable graph with \( 2n + 1 \) vertices, and their generalizations. For each integer \( n \geq 3 \), we present two uniquely \( (n+1) \)-colorable graphs which are chromatically equivalent with \( 2n + 2 \) vertices, and with \( 2n + 3 \) vertices, and their generalizations.

1. Introduction

Let \( G \) be a simple graph, \( V(G) \), be its vertex-set and \( E(G) \) be its edge-set. An assignment of colors to the vertices of \( G \) in such a way that adjacent vertices are assigned with different colors is called a (proper) coloring of \( G \). The minimum number of colors needed to color \( G \), is called the chromatic number of \( G \), and is denoted by \( \chi(G) \). Let \( \lambda \) be a positive integer. Then a \( \lambda \)-coloring of \( G \) is a partition of \( V(G) \) into \( \lambda \) color classes such that the vertices in the same color class are not adjacent. If every \( \chi(G) \)-coloring of \( G \) gives the same partition of \( G \), then \( G \) is said to be a unique \( \chi(G) \)-colorable graph. A chromatic polynomial, \( P(G, \lambda) \), in \( \lambda \) is the number of ways of \( \lambda \)-coloring of \( G \). Two graphs \( G \) and \( H \) are said to be chromatically equivalent, if they are nonisomorphic and \( P(G, \lambda) = P(H, \lambda) \).

In [1], some families of uniquely 3-colorable graphs without triangles were presented. In [4], the author, Osterweil, presented some families of uniquely 3-colorable graphs with triangles. His method was to use the complements of certain graphs. He also stated that the techniques used here seem applicable to the more general study of unique \( n \)-colorability in graphs. Recently, Chia in [2], by using the same method as Osterweil's, extended Osterweil's result to the case of uniquely \( n \)-colorable graphs. Here, we shall also consider the complements of certain graphs to prove the following

**Theorem 1.**

(a) For each integer \( n \geq 3 \), there exists a uniquely \( n \)-colorable graph with \( 2n + 1 \) vertices.

(b) For each integer \( n \geq 3 \), there exists a uniquely \( n \)-colorable graph with \( 2n + 1 \) vertices.
Using our Theorem 1, we shall prove

**Theorem 2.**

(a) For each integer \( n \geq 3 \), there exist two uniquely \((n+1)\)-colorable graphs with \( 2n + 2 \) vertices which are chromatically equivalent.

(b) For each integer \( n \geq 3 \), there exist two uniquely \((n+1)\)-colorable graphs with \( 2n + 3 \) vertices which are chromatically equivalent.

We need the following well known Theorem (see p. 55 in [3]) for our proofs and examples: Let \( G \) be a graph. Then

\[
P(G, \lambda) = P(G - e, \lambda) - P(G / e, \lambda)
\]

(1)

where \( G - e \) is the graph obtained from \( G \) by deleting an edge \( e \) in \( G \), and \( G / e \) is the graph obtained from \( G \) by contracting the edge \( e \). Or

\[
P(G, \lambda) = P(G + e', \lambda) + P((G + e') / e', \lambda)
\]

(2)

where \( e' \notin E(G) \) and \( G + e' \) is the graph obtained from \( G \) by adding the edge \( e' \) into \( G \).

2. Examples, proofs and generalizations

The following examples lead to the proof for the general case, i.e., they lead to the proofs for Theorem 1(a) and (b).

**Example 1.** Let \( P_5 \) be the following path of length 5,

![Diagram of P5](image)

and \( G_6 \) be the complement of \( P_5 \) in the complete graph, \( K_6 \), with 6 vertices. Thus, \( G_6 \) is the following graph:

![Diagram of G6](image)
Since $G_6$ contains the triangles $K'_3 = <0, 2, 4>$ and $K''_3 = <1, 3, 5>$, $\chi(G_6) \geq 3$. We claim that $G$ can be colored by 3 colors, $\alpha, \beta$ and $\gamma$ with color indifference. We shall use the following notations: $i(\alpha)$ means the vertex $i$ is colored by the color $\alpha$, and $\rightarrow i(\alpha)$ means the vertex $i$ is forced to be colored with the color $\alpha$. We color $1(\alpha)$, $3(\beta)$ and $5(\gamma)$. Since the neighborhood of the vertex $0, N(0) \supset \{3(\beta), 5(\gamma)\}, \rightarrow 0(\alpha)$. Similarly, since $N(2) \supset \{0(\alpha), 5(\gamma)\}, \rightarrow 2(\beta)$, and since $N(4) = \{1(\alpha), 2(\beta), 0(\alpha)\}, \rightarrow 4(\gamma)$. Thus, $\chi(G_6) = 3$. Since there is no choice of colors for each vertex in $G_6$, $G_6$ is uniquely 3-colorable, and the 3 color classes are

\{0, 1\}, \{2, 3\}, \{4, 5\}.

We also can show that $G_6$ is a uniquely 3-colorable graph by using its chromatic polynomial. The chromatic polynomial of $G_6$ is, by adding $e_1 = (0, 1)$ to $G_6$ and using (2) and deleting $e_2 (1, 4)$ in $G_6 + e_1$ and using (1),

$$P(G_6, \lambda) = P(G_6 + e_1, \lambda) + P(G_6 / e_1, \lambda)$$

$$= P((G_6 + e_1) - e_2, \lambda) - P(G_6 + e_1 / e_2, \lambda) + P(G_6 / e_1, \lambda)$$

$$= \lambda(\lambda - 1)(\lambda - 2)^2(\lambda - 3) - \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^2 + \lambda(\lambda - 1)(\lambda - 2)^3.$$

The Lemma 1 in [1] states: Let $\chi(G) = k$. Then $P(G, k) = k! \cdot t$ for some positive integer $t$, and $t$ is the number of ways of coloring $G$ in exactly $k$ colors with color indifference. Furthermore, $t = 1$ if and only if $G$ is a uniquely $k$-colorable graph. Hence, with $\lambda = 3$, $P(G_6, 3) = 0 - 0 + 3 \cdot 2 \cdot 1^3 = 3!$, and $G_6$ is a uniquely 3-colorable graph.

**Example 2.** Let $H'_7$ be the graph with 7 vertices consisting of a 7-cycle and a triangle, i.e., $V(H'_7) = \{0, 1, \cdots, 6\}$ and $E(H'_7) = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 0), (6, 1)\}$, and $H_7$ be the complement of $H'_7$ in $K_7$. Thus, $H_7$ is the following graph:

![Graph](image-url)
Since $H$ contains triangles, $\chi(H) \geq 3$. We claim that $H$ can be colored by 3 colors $\alpha, \beta$ and $\gamma$ with color indifference. We color $l(\alpha), 3(\beta)$ and $5(\gamma)$. Then since $N(0) \supseteq \{3(\beta), 5(\gamma)\}, \to 0(\alpha)$.

Similarly, since $N(2) \supseteq \{0(\alpha), 5(\gamma)\} \to 2(\beta)$. Since $N(4) \supseteq \{0(\alpha), 2(\beta), l(\alpha)\}, \to 4(\gamma)$. Since $N(6) = \{2(\beta), 3(\beta), 4(\gamma)\}, \to 6(\alpha)$. Thus, $\chi(H) = 3$. Since there is no choice of colors for each vertex in $H$, $H$ is uniquely 3-colorable, and the 3 color classes are

$$\{0, 1, 6\}, \{2, 3\}, \{4, 5\}$$

We also show that $H$ is a uniquely 3-colorable graph by using its chromatic polynomial. Repeatedly using (1) and (2), we have

$$P(H, \lambda) = P(H-e_1, \lambda) - P(H/e_1, \lambda) \quad (\text{where } e_1 = (1, 4))$$
$$= (P(H-e_1 + e_2, \lambda) + P(H-e_1 / e_2, \lambda)) - (P(H/e_1 + e_3, \lambda) + P(H/e_1 / e_3, \lambda))$$
where $e_2 = (0, 1)$ and $e_3 = (2, 3)$

$$= (P((H-e_1 + e_2 - e_4, \lambda) - P((H-e_1 + e_2) / e_4, \lambda)) -$$

$$= (P((H / e_1 + e_3 - e_4, \lambda) - P((H / e_1 + e_3) / e_4, \lambda))$$
where $e_4 = (6, 3)$

$$= \lambda(\lambda - 1)(\lambda - 2)^4(\lambda - 3) - \lambda(\lambda - 1)(\lambda - 3)^3 + \lambda(\lambda - 1)(\lambda - 2)^4$$

Hence, $P(H, 3) = 0 - 0 + 3 \times 2 \times 1^4 - 0 - 0 = 3!$, and $H$ is uniquely 3-colorable.

The proof of Theorem 1(a) goes as follows. For any integer $n \geq 3$, let $P_{2n-1}$ be the following (simple) path of length $2n-1$:

$$P_{2n-1} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad 0 \quad 1 \quad 2 \quad \cdots \quad \cdots \quad 2n-2 \quad 2n-1$$

and $G_{2n}$ be the complement of $P_{2n-1}$ in the complete graph $K_{2n} = <0, 1, \cdots, (2n-1)>$. Thus, $G_{2n}$ contains two complete subgraphs with $n$ vertices, namely,

$$K'_n = <0, 2, \cdots, 2t, \cdots, (2n-2)> \quad \text{and}$$

$$K''_n = <1, 3, \cdots, (2t+1), \cdots, (2n-1)>.$$
Clearly, $\chi(G_{2n}) \geq n$. We claim that $G_{2n}$ can be colored by $n$ colors, $\alpha_1, \alpha_2, \ldots, \alpha_n$ with color indifference. We color $1(\alpha_1), 3(\alpha_2), \ldots, (2k-1)(\alpha_k), \ldots, (2n-1)(\alpha_n)$. Since $N(0) \supset \{(2k-1)(\alpha_k)\}$ for $k = 2, 3, \ldots, n$, $\rightarrow \{0(\alpha_i)\}$. Similarly, since $N(2t) \supset \{(2p-2)(\alpha_p)\}$ for $p = 1, 2, \ldots, t$, and $(2k-1)(\alpha_k)$ for $k = t+2, t+3, \ldots, n$, $\rightarrow 2t(\alpha_{t+1})$ for $t = 1, 2, \ldots, n-2$. Also, since $N(2n-2) = \{(2p-2)(\alpha_p)\}$ for $p = 1, 2, \ldots, n-1$ and $(2k-1)(\alpha_k)$ for $1, 2, \ldots, n-1$, $\rightarrow (2n-2)(\alpha_n)$. Thus, $\chi(G_{2n}) = n$. Since there is no choice of colors for each vertex in $G_{2n}$, $G_{2n}$ is uniquely $n$-colorable, and the $n$ color classes are

$$\{0, 1\}, \{2, 3\}, \ldots, \{2k, 2k+1\}, \ldots, \{2n-2, 2n-1\}.$$

The proof of Theorem 1(b) goes as follows. For any integer $n \geq 3$, let $H'_{2n+1}$ be the graph with $2n+1$ vertices consisting of a $(2n+1)$-cycle and a triangle, i.e., $V(H'_{2n+1}) = \{0, 1, \ldots, 2n\}$ and $E(H'_{2n+1}) = \{(0, 1), (1, 2), \ldots, (i, i+1), \ldots, (2n, 0), (2n, 1)\}$, and $H_{2n+1}$ be the complement of $H'_{2n+1}$ in the complete graph $K_{2n+1} = < 0, 1, \ldots, 2n >$. Thus, $H_{2n+1}$ contains two complete subgraphs with $n$ vertices, namely,

$$K'_n = < 0, 2, \ldots, 2t, \ldots, (2n-2) > \quad \text{and} \quad K''_n = < 1, 3, \ldots, (2t+1), \ldots, (2n-1) >.$$

Clearly, $\chi(H_{2n+1}) \geq n$. We claim that $H_{2n+1}$ can be colored by $n$ colors $\alpha_1, \alpha_2, \ldots, \alpha_n$ with color indifference. We color $1(\alpha_1), 3(\alpha_2), \ldots, (2k-1)(\alpha_k), \ldots, (2n-1)(\alpha_n)$. Since $N(0) \supset \{(2k-1)(\alpha_k)\}$ for $k = 2, 3, \ldots, n$, $\rightarrow \{0(\alpha_i)\}$. Similarly, since $N(2t) \supset \{(2p-2)(\alpha_p)\}$ for $p = 1, 2, \ldots, t$, and $(2k-1)(\alpha_k)$ for $k = t+2, t+3, \ldots, n$, $\rightarrow 2t(\alpha_{t+1})$ for $t = 1, 2, \ldots, n-1$. Also, since $N(2n) = \{(2p-2)(\alpha_p)\}$ for $p = 2, 3, \ldots, n$, and $(2k-1)(\alpha_k)$ for $k = 2, 3, \ldots, n-1$, $\rightarrow 2n(\alpha_1)$. Thus, $\chi(H_{2n+1}) = n$. Since there is no choice of colors for each vertex in $H_{2n+1}$, $H_{2n+1}$ is uniquely $n$-colorable, and the $n$ color classes are

$$\{0, 1, 2n\}, \{2, 3\}, \ldots, \{2k, 2k+1\}, \ldots, \{2n-2, 2n-1\}.$$

Following Osterweil's idea about 6-clique rings in [3], we let $C_i$ be the complete graph with $m_i$ vertices where $m_i \geq 1$. For $i \neq j$, we say that $C_i$ and $C_j$ are adjacent, denoted by $C_i \leftrightarrow C_j$, if every vertex in $C_i$ is adjacent to all vertices in $C_j$. 


For each integer \( n \geq 3 \), let \( \overline{P}_{2n-1} \) be the graph with \( V(\overline{P}_{2n-1}) = \bigcup_{i=1}^{2n-1} V(C_i) \) and 
\[
E(\overline{P}_{2n-1}) = \bigcup_{i=0}^{2n-2} E(C_i \leftrightarrow C_{i+1}).
\]
Thus, \( \overline{P}_{2n-1} \) has \( m = \sum_{i=0}^{2n-1} m_i \) vertices and 
\[
E(\overline{P}_{2n-1}) \text{ has } \sum_{i=0}^{2n-1} \left( \frac{m_i}{2} \right) + \sum_{i=0}^{2n-2} (m_i)(m_{i+1}) \text{ edges.}
\]
Let \( \overline{G}_{2n} \) be the complement of \( \overline{P}_{2n-1} \) in the complete graph \( K_m \). Then we have:

**Corollary 1(a).** \( \overline{G}_{2n} \) is a uniquely \( n \)-colorable graph with \( m \) vertices.

**Proof.** Let \( N_i \) be the null graph with \( m_i \) vertices for \( i = 0, 1, \ldots, 2n-1 \). The proof is similar to the proof of Theorem 1(a) by replacing the vertices \( i \) in \( G_{2n} \) by \( N_i \) for \( i = 0, 1, \ldots, 2n-1 \).

For each integer \( n \geq 3 \), let \( \overline{H}_{2n+1} \) be the graph with \( V(\overline{H}_{2n+1}) = \bigcup_{i=0}^{2n} V(C_i) \) and 
\[
E(\overline{H}_{2n+1}) = \bigcup_{i=0}^{2n-1} E(C_i \leftrightarrow C_{i+1}) \cup (E(C_{2n} \leftrightarrow C_0)) \cup (E(C_{2n} \leftrightarrow C_1)), \quad \text{and} \quad \overline{H}_{2n+1}
\]
be the complement of \( \overline{H}_{2n+1} \) in the complete graph \( K_q \) where \( q = \sum_{i=0}^{2n} m_i \). Then we have:

**Corollary 1(b).** \( \overline{H}_{2n+1} \) is a uniquely \( n \)-colorable graph with \( q \) vertices.

**Proof.** Let \( N_i \) be the null graph with \( m_i \) vertices for \( i = 0, 1, \ldots, 2n \). The proof is similar to the proof of Theorem 1(b) by replacing the vertices \( i \) in \( H_{2n+1} \) by \( N_i \) for \( i = 0, 1, \ldots, 2n \).

**Example 3.** Let \( G_6 \) be the graph in our Example 1, and \( Q_8 \) be the graph with 
\[
V(Q_8) = V(G_6) \cup \{6, 7\} \quad \text{and} \quad E(Q_8) = E(G_6) \cup \{(6, 1), (6, 3)\} \cup \{(7, i) \text{ for } i = 0, 1, 2, 3, 4, 5, 6\}.
\]
Thus, \( Q_8 \) is the following graph:

![Graph Q_8](image-url)
Since $Q_8$ contains a complete graph with 4 vertices, $\chi(Q_8) \geq 4$. We claim that $Q_8$ is uniquely 4-colorable. Since $G_6$ is a uniquely 3-colorable graph with colors $\alpha$, $\beta$ and $\gamma$, the vertex 7 in $Q_8$ has to be colored by a new color $\delta$. Since $N(6) = \{1(\alpha), 3(\beta), 7(\delta)\}, \rightarrow 6(\gamma)$. Since there is no choice of colors for each vertex in $Q_8$, $Q_8$ is uniquely 4-colorable, and the 4 color classes are:

$\{0, 1\}, \{2, 3\}, \{4, 5, 6\}, \{7\}$.

Let $R_8$ be the graph with $V(R_8) = V(G_6) \cup \{6, 7\}$ and $E(R_8) = E(G_6) \cup \{(6, 1), (6, 3), (6, 5)\} \cup \{(7, i) \text{ for } i = 0, 1, 2, 3, 4, 5\}$. Thus, $R_8$ is the following graph:

Since $R_8$ contains a complete graph with 4 vertices, $\chi(R_8) \geq 4$. We claim that $R_8$ is uniquely 4-colorable. Since $G_6$ is a uniquely 3-colorable graph with colors $\alpha$, $\beta$ and $\gamma$, $\rightarrow 7(\delta)$. Since there is no choice of colors for each vertex in $R_8$, $R_8$ is uniquely 4-colorable, and the 4 color classes are:

$\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}$.

We claim that $Q_8$ and $R_8$ are chromatically equivalent. By using (2) and (1), we have

$$P(Q_8, \lambda) = P(Q_8 \cup (5, 6), \lambda) + P(Q_8 \cap (5, 6), \lambda), \text{ i.e.,}$$

$$P(Q_8, \lambda) = P(G_6, \lambda) + P(G_6, \lambda)$$

for $\lambda = 1, 2, 3, 4, 5, 6, 7$. Thus, $Q_8$ and $R_8$ are chromatically equivalent.
Since the polynomials on the right sides of (3) and (4) are the same, \( P(Q_8, \lambda) = P(R_9, \lambda) \). Since the degree of vertex 7 in \( Q_8 \) is 7 and no vertex in \( R_9 \) is of degree 7, \( Q_8 \neq R_9 \) and \( Q_8 \) and \( R_9 \) are chromatically equivalent.

By using some of the properties of chromatic polynomials (in [3]), we have

\[
P(Q_8, \lambda) = \lambda P(G_6, \lambda - 1) \times ((\lambda - 1) - 2)
\]

\[
= \lambda (\lambda - 1) (\lambda - 2) (\lambda - 3)^4 (\lambda - 4) - \lambda (\lambda - 1) (\lambda - 2) (\lambda - 3)^2 (\lambda - 4)^2 + \lambda (\lambda - 1) (\lambda - 2) (\lambda - 3)^4
\]

and

\[
P(R_9, \lambda) = \lambda P(G_6, \lambda - 1) \times (\lambda - 3)
\]

\[
= [\lambda (\lambda - 1) (\lambda - 2) (\lambda - 3)^3 (\lambda - 4) - \lambda (\lambda - 1) (\lambda - 2) (\lambda - 3)^2 (\lambda - 4)^2 + \lambda (\lambda - 1) (\lambda - 2) (\lambda - 3)^4]
\]

\[
= (\lambda (\lambda - 1) (\lambda - 2) (\lambda - 3)^4 (\lambda - 4) - \lambda (\lambda - 1) (\lambda - 2) (\lambda - 3)^2 (\lambda - 4)^2 + \lambda (\lambda - 1) (\lambda - 2) (\lambda - 3))^4
\]

Thus, \( P(Q_8, \lambda) = P(R_9, \lambda) \). Also, \( P(Q_8, 4) = P(R_9, 4) = 4! \), i.e., \( Q_8 \) and \( R_9 \) are uniquely 4-colorable graphs, and they are chromatically equivalent.

Similarly, we may use the graph \( H_7 \) in our Example 2 to construct graphs \( Q_9 \) and \( R_9 \) as follows. Let \( V(Q_9) = V(H_7) \cup \{7, 8\} \), and \( E(Q_9) = E(H_7) \cup \{(7, 1), (7, 3), (7, 5), i \} \) for \( i = 0, \cdots, 7 \). \( \cup \{(7, 1), (7, 3) \} \cup \{8, i\} \) Also, let \( V(R_9) = V(H_7) \cup \{7, 8\} \), and \( E(R_9) = E(H_7) \cup \{(7, 1), (7, 3), (7, 5) \} \cup \{8, i\} \) for \( i = 0, 1, \cdots, 6 \). Then \( P(Q_9, \lambda) = \lambda (P(H_7, \lambda - 1) \cdot ((\lambda - 1) - 2) \), and \( P(R_9, \lambda) = (\lambda P(H_7, \lambda - 1)) \cdot (\lambda - 3) \). Clearly, \( Q_9 \) and \( R_9 \) are not isomorphic. We can show that \( P(Q_9, \lambda) = P(R_9, \lambda) \) and \( P(Q_9, 4) = P(R_9, 4) = 4! \), i.e., \( Q_9 \) and \( R_9 \) are uniquely 4-colorable, and they are chromatically equivalent.

The proof of Theorem 2 (a) goes as follows. The graph \( G_{2n} \) in Theorem 1 (a) is a uniquely \( n \)-colorable graph with \( 2n \) vertices containing two complete subgraphs with \( n \) vertices, namely,

\[
K_n' = 0, 2, \cdots, 2r, \cdots, (2n - 2) \quad \text{and}
\]

\[
K_n'' = 1, 3, \cdots, (2t + 1), \cdots, (2n - 1) >.
\]
We construct a graph $Q_{2n+2}$ with $V(Q_{2n+2}) = V(G_{2n}) \cup \{2n, 2n+1\}$, and $E(Q_{2n+2}) = E(G_{2n}) \cup \{(2n,i)\text{ for } i = 1, 3, \cdots, (2t+1), \cdots, (2n-3)\} \cup \{(2n+1,j)\text{ for } j = 0, 1, \cdots, 2n\}$. Since $G_{2n}$ is uniquely $n$-colorable and the vertex $2n$ is incident with every vertex in $K''_{n}$ except the vertex $(2n-1)(\alpha_n)$, $\rightarrow (2n)(\alpha_n)$. Since the vertex $2n+1$ is incident with every vertex $i$ for $i = 0, 1, \cdots, 2n, \rightarrow (2n+1)(\alpha_{n+1})$. Thus, $Q_{2n+2}$ is uniquely $(n+1)$-colorable with the color classes.

\[\{0, 1\}, \{2, 3\}, \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1, 2n\}, \{2n+1\}\]

Let $R_{2n+2}$ be the graph with $V(R_{2n+2}) = V(G_{2n}) \cup \{2n, 2n+1\}$, and $E(R_{2n+2}) = E(G_{2n}) \cup \{(2n,i)\text{ for } i = 1, 3, \cdots, (2t+1), \cdots, (2n-1)\} \cup \{(2n+1,j)\text{ for } j = 0, 1, \cdots, (2n-1)\}$.

Since $G_{2n}$ is uniquely $n$-colorable and the vertex $2n$ is incident with every vertex in $K''_{n}, \rightarrow (2n)(\alpha_{n+1})$. Since the vertex $2n+1$ is incident with every vertex in $G_{2n}, \rightarrow (2n+1)(\alpha_{n+1})$. Thus, $R_{2n+2}$ is uniquely $(n+1)$-colorable with the color classes.

\[\{0, 1\}, \{2, 3\}, \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1, 2n\}, \{2n, 2n+1\}\]

We claim that $Q_{2n+2}$ and $R_{2n+2}$ are not isomorphic. In $Q_{2n+2}$, the degree of vertex $2n+1$ is $2n+1$, and in $R_{2n+2}$, none of the vertices is of degree $2n+1$. Hence, $Q_{2n+2}$ and $R_{2n+2}$ are not isomorphic.

We claim that $Q_{2n+2}$ and $R_{2n+2}$ are chromatically equivalent. By using (2), we have

\[P(Q_{2n+2}, \lambda) = P(Q_{2n+2} + (2n, 2n-1), \lambda) + P(Q_{2n+2} / (2n, 2n-1), \lambda), \text{ and} \]

\[P(R_{2n+2}, \lambda) = P(R_{2n+2} + (2n, 2n+1), \lambda) + P(R_{2n+2} / (2n, 2n+1), \lambda). \text{ Since} \]

\[P(Q_{2n+2} + (2n, 2n-1), \lambda) = P(R_{2n+2} + (2n, 2n+1), \lambda) \text{ and} \]

\[P(Q_{2n+2} / (2n, 2n-1), \lambda) = P(R_{2n+2} / (2n, 2n+1), \lambda), P(Q_{2n+2}, \lambda) = P(R_{2n+2}, \lambda). \]

Hence, $Q_{2n+2}$ and $R_{2n+2}$ are chromatically equivalent, and are uniquely $(n+1)$-colorable.

The proof of Theorem 2(b) goes as follows. The graph $H_{2n+1}$ in Theorem 1(b) is a uniquely $n$-colorable graph with $2n+1$ vertices containing two complete subgraphs with $n$ vertices, namely,

\[K'_{n} = < 0, 2, \cdots, 2t, \cdots, (2n-2) > \text{ and} \]

\[K''_{n} = < 1, 3, \cdots, (2t+1), \cdots, (2n-1) >. \]
Let $Q_{2n+3}$ be the graph with $V(Q_{2n+3}) = V(H_{2n+1}) \cup \{2n+1, 2n+2\}$ and $E(Q_{2n+3}) = E(H_{2n+1}) \cup \{(2n+1, i) \mid i = 1, 3, \cdots, (2t+1), \cdots, (2n-3)\} \cup \{(2n+2, j) \mid j = 0, 1, \cdots, 2n+1\}$, and $R_{2n+3}$ be the graph with $V(R_{2n+3}) = V(H_{2n+1}) \cup \{2n+1, 2n+2\}$ and $E(R_{2n+3}) = E(H_{2n+1}) \cup \{(2n+1, i) \mid i = 1, 3, \cdots, (2t+1), \cdots, (2n-1)\} \cup \{(2n+2, j) \mid j = 0, 1, \cdots, 2n\}$. Similar to the proof of Theorem 2(a), $Q_{2n+3}$ and $R_{2n+3}$ are uniquely $(n+1)$-colorable with the color classes.

Then $0, 1, 2n, \{2, 3\}, \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1, 2n+1\}, \{2n+2\}$ in $Q_{2n+3}$, and

$$
\{0, 1, 2n\}, \{2, 3\}, \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1, 2n+1\}, \{2n+2\} \text{ in } R_{2n+3}.
$$

Also, similar to the proof in Theorem 2(a), $Q_{2n+3}$ and $R_{2n+3}$ are not isomorphic, and $P(Q_{2n+3}, \lambda) = P(R_{2n+3}, \lambda)$. Hence, $Q_{2n+3}$ are $R_{2n+3}$ chromatically equivalent, and are uniquely $(n+1)$-colorable.

In the corollary 1(a) and (b), both of $\overline{G}_{2n}$ and $\overline{R}_{2n}$ contain two "generalized complete" subgraphs $\overline{K}_n$ and $\overline{K}_n$ with

$$V(\overline{K}_n) = \{N_i \mid N_i \text{ is a null graph with } m_i \text{ vertices for } i = 0, 2, \cdots, 2k, \cdots, (2n-2)\},$$

$$E(\overline{K}_n) = \{N_i \leftrightarrow N_j \mid i \neq j, i, j = 0, 2, \cdots, 2k, \cdots, (2n-2)\},$$

$$V(\overline{K}_n') = \{N_i \mid N_i \text{ is a null graph with } m_i \text{ vertices for } i = 0, 2, \cdots, 2k, \cdots, (2n-2)\},$$

and

$$E(\overline{K}_n') = \{N_i \leftrightarrow N_j \mid i \neq j \text{ and } i, j = 1, 3, \cdots, (2k-1), \cdots, (2n-1)\}.$$

**Corollary 2(a).** Let $\overline{G}_{2n} = \langle N_0, N_1, \cdots, N_{2n-1} \rangle$ be the graph in corollary 1(a), $Q_{2n+2}$ be the graph with $V(\overline{Q}_{2n+2}) = V(\overline{G}_{2n}) \cup \{N_{2n}, N_{2n+1}\}$ where $N_{2n}$ and $N_{2n+1}$ are null graphs with $m_{2n}$ and $m_{2n+1}$ vertices respectively, and $E(\overline{Q}_{2n+2}) = E(\overline{G}_{2n}) \cup \{N_{2n} \leftrightarrow N_i \mid i = 1, 3, \cdots, (2t+1), \cdots, (2n-3)\} \cup \{N_{2n+1} \leftrightarrow N_j \mid j = 0, 1, \cdots, 2n\}$, and $\overline{R}_{2n+2}$ be the graph with $V(\overline{R}_{2n+2}) = V(\overline{G}_{2n}) \cup \{N_{2n}, N_{2n+1}\}$ and $E(\overline{R}_{2n+2}) = E(\overline{G}_{2n}) \cup \{N_{2n} \leftrightarrow N_i \mid i = 1, 3, \cdots, (2t+1), \cdots, (2n-1)\} \cup \{N_{2n+1} \leftrightarrow N_j \mid j = 0, 1, \cdots, 2n-1\}$. Then $\overline{Q}_{2n+2}$ and $\overline{R}_{2n+2}$ are uniquely $(n+1)$-colorable, and are chromatically equivalent.
Corollary 2(b). Let $\overline{H}_{2n+1} = \langle N_0, N_1, \ldots, N_{2n} \rangle$ be the graph in corollary 1(b), $Q_{2n+3}$ be the graph with $V(Q_{2n+3}) = V(\overline{H}_{2n+1}) \cup \{N_{2n+1}, N_{2n+2}\}$ where $N_{2n+1}$ and $N_{2n+2}$ are null graphs with $m_{2n+1}$ and $m_{2n+2}$ vertices respectively, and $E(Q_{2n+3}) = E(\overline{H}_{2n+1}) \cup \{N_{2n+1} \leftrightarrow N_j \text{ for } i = 1, 3, \ldots, (2t+1), \ldots, (2n-3)\} \cup \{N_{2n+2} \leftrightarrow N_j \text{ for } j = 0, 1, \ldots, 2n+1\}$, and $\overline{R}_{2n+3}$ be the graph with $V(\overline{R}_{2n+3}) = V(\overline{H}_{2n+1}) \cup \{N_{2n+1}, N_{2n+2}\}$ and $E(\overline{R}_{2n+3}) = E(\overline{H}_{2n+1}) \cup \{N_{2n+1} \leftrightarrow N_j \text{ for } i = 1, 3, \ldots, (2t+1), \ldots, (2n-1)\} \cup \{N_{2n+2} \leftrightarrow N_j \text{ for } j = 0, 1, \ldots, 2n\}$. Then $Q_{2n+3}$ and $R_{2n+3}$ are uniquely $(n+1)$-colorable and are chromatically equivalent.

Proof.
(a) It is similar to the proof of Theorem 2(a) by replacing the vertices $i$ in $Q_{2n+2}$ and $R_{2n+2}$ in the Theorem 2(a) by $N_i$ for $i = 0, 1, \ldots, 2n+1$.
(b) It is similar to the proof of Theorem 2(b) by replacing the vertices $i$ in $Q_{2n+3}$ and $R_{2n+3}$ by $N_i$ for $i = 0, 1, \ldots, 2n+3$.

References