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Some Conditions on Infinite Subsets of Infinite Groups

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Abstract. Let G be an infinite group. In this note we prove the following: For all $a, b \in G, (ab)^2 = (ba)^2$ if and only if every two infinite subsets X and Y of G contain elements x and y, respectively such that $(xy)^2 = (yx)^2$. Also if $n \in \{3, 6\} \cup \{2^k \mid k \in N\}$ then for all $a, b \in G, a^n b = ba^n$ if and only if every two infinite subsets X and Y of G contain elements x and y, respectively such that $x^n y = yx^n$.

1. Introduction

Let w be a word in the free group of rank $n > 0$. Let $V = V(w)$ be the variety of groups defined by the law $w(x_1, \cdots, x_n) = 1$. Define $V^* = V(w^*)$ to be the class of all groups G in which for any infinite subsets $X_1, \cdots, X_n$ there exist $x_i \in X_i$, $1 \le i \le n$, such that $w(x_1, \cdots, x_n) = 1$. In [11], P. Longobardi *et al.* posed the question of when the equality $F \cup V(w) = V(w^*)$ holds, where F is the class of finite groups.

There is no example, so far, of an infinite group in $V(w^*) \setminus V(w)$. In considering this question, many authors have obtained the equality for certain words (see [2], [4], [8], [11], [17], [18]) and for certain classes of groups (see [4]). The origin of these results is a question of P. Erdös, which was answered by B.H. Neumann [12]. Since this first paper, problems of similar nature have been the object of several articles, (for example [1], [3], [6], [7], [8], [10], [14], [15]). Let n be a positive integer and A_n and B_n be the varieties of groups generated by the laws $(xy)^n (yx)^{-n} = 1$ and $x^n y (yx^n)^{-1} = 1$, respectively. It is easy to see that $\mathsf{A}_n = \mathsf{B}_n$ for all $n \in N$. Our main results are

Theorem 1. *Every infinite* A_2^*- *group* is an A_2- *group*.

Theorem 2. *Let n be an integer in the set* $\{3, 6\} \cup \{2^k \mid k \in N\}$. *Then every infinite* B_n^*- *group* is a B_n- *group*.

2. Proofs

Let G be a group. We denote by $Z(G)$ the centre of G and by G^n the subgroup of G generated by n-th power elements of G. If X is a non-empty set, we denote by $X^{(m)}$ the set of all m element subsets of X.

However we do not need the following lemma as it states, but we have found it to be useful for other investigations on the problems of the similar nature.

We note that, by Lemma 3 in [4], every infinite A_n^*-group or B_n^*-group with infinite centre is an A_n-group.

Lemma 1. *Let G be an infinite $V(w^*)$-group, where w is a word in the free group of rank 2. Let A be an infinite abelian subgroup of G and $y_1, \cdots, y_n \in G$. Then there exists an infinite subset T of the set $B = \{a \in A \mid w(a, y_i) = w(y_i, a) = 1,\ \forall i = 1, \cdots, n\}$ such that $t_1 t_2^{-1} \in B$ for all distinct elements t_1, t_2 in T. Also, $A \setminus B$ is finite.*

Proof. Let us firstly prove, by induction on n, that $A \setminus B$ is finite. Let $n = 1$ and set $y_1 = y$. Consider the set $Y = \{y^a \mid a \in A\}$. If Y is finite, then the index $|A : C_A(y)|$ is finite too, hence $C_A(y)$ is infinite and contained in the centre of $H = \langle A, y \rangle$. This means that $Z(H)$ is infinite and so by Lemma 3 of [4], H is a $V(w)$-group, thus $A = B$. So we may assume, without loss of generality, that Y is infinite. Suppose now that the set $A \setminus B_1$ is infinite where $B_1 = \{a \in A \mid w(a, y) = 1\}$. Consider the two infinite sets Y and $A \setminus B_1$. By the hypothesis there are elements $a \in A \setminus B_1$ and $b \in A$ such that $w(a, y^b) = 1$ and so $w(a, y) = 1$, a contradiction. Thus $A \setminus B_1$ is finite. Similarly $A \setminus B_2$ is finite where $B_2 = \{a \in A \mid w(y, a) = 1\}$. Therefore $A \setminus B$ is finite since $B_1 \cap B_2 = B$. Now suppose, inductively, that $n > 1$ and $A \setminus C$ is finite where $C = \{a \in A \mid w(a, y_i) = w(y_i, a) = 1,\ i = 1, \cdots, n-1\}$ is finite. As in the case, $n = 1$, we have $A \setminus D$ is finite where $D = \{a \in A \mid w(a, y_n) = w(y_n, a) = 1\}$. Thus $A \setminus D$ is finite since $D \cap C = B$. Thus the induction is complete. Now we prove the first part of the Lemma. Suppose that A has a torsion-free element a and $S = \langle a \rangle \cap (A \setminus B)$, then S is finite. Since $\langle a \rangle$ is residually finite, there is a subgroup T of $\langle a \rangle$ of finite index such that $T \setminus \{1\} \cap S = \phi$. Thus $T \setminus \{1\} \subseteq B$ and the proof is complete in this case. Thus, we assume that A is a torsion group. In this case, the subgroup H generated by $A \setminus B$ is finite, and $A \setminus H$ is infinite. Choose a transversal T for H in A. This is an infinite subset of A contained in B, and for each pair t_1, t_2 of distinct elements of T, we have $t_1 t_2^{-1} \notin H$ and so $t_1 t_2^{-1} \in B$, since $A \setminus B \subseteq H$, and the proof is complete.

Lemma 2. *Let G be an infinite* A_n^**-group. If A is an infinite abelian subgroup of G then* $A^n \leq Z(G)$.

Proof. Let $y \in A$ and $x \in G$. We prove that $xy^n = y^n x$. Define the sets

$$\mathsf{X}_1 = \left\{ \{a, b\} \subset A \,\middle|\, (ab)^n \in C_G(x) \right\}, \ \mathsf{X}_2 = A^{(2)} \setminus \mathsf{X}_1.$$

By Ramsey's Theorem [13], there exists an infinite subset X of A such that $X^{(2)} \subseteq \mathsf{X}_1$ or $X^{(2)} \subseteq \mathsf{X}_2$. If $X^{(2)} \subseteq \mathsf{X}_2$, partition the set X into two disjoint infinite subsets X_1 and X_2. Considering infinite subsets $x^{-1}X_1$ and $X_2 x$, the property A_n^*, yields $a \in X_1$ and $b \in X_2$ such that $(x^{-1}abx)^n = (bxx^{-1}a)^n$ and so $(ab)^n \in C_G(x)$, thus $\{a, b\} \in \mathsf{X}_2$, a contradiction. Therefore $X^{(2)} \subseteq \mathsf{X}_1$. Now, fix an element a_0 of X, then $a_0^n b^n \in C_G(x)$ for all $b \in X \setminus \{a_0\}$. Let $M = \left\{ a_0^n b^n \,\middle|\, b \in X \setminus \{a_0\} \right\}$. If M is infinite, then the centre of $\langle A, x \rangle$ is infinite and so $\langle A, x \rangle$ is an A_n-group, thus $xy^n = y^n x$. Now assume that M is finite. Therefore there exist infinite subset T of $X \setminus \{a_0\}$ and $b_0 \in T$ such that $a_0^n b^n = a_0^n b_0^n$ for all $b \in T$. Thus $(bb_0^{-1})^n = 1$ for all $b \in T$. Hence $D = \left\{ x \in A \,\middle|\, x^n = 1 \right\}$ is an infinite abelian subgroup of A. Now let $B = \left\{ a \in D \,\middle|\, (ax)^n = (xa)^n \right\}$, then by Lemma 1, B is a cofinite set in A. Consider infinite subsets $x^{-1}yD$ and Dx. By the property A_n^*, there exist $d_1, d_2 \in D$ such that $(x^{-1}yd_1d_2x)^n = (d_2xx^{-1}yd_1)^n$, therefore $xy^n = y^n x$, since $y, d_1, d_2 \in A$ and $d_1^n = d_2^n = 1$.

Lemma 3. *Let G be an infinite* A_2^**-group. Then* $C_G(x^2)$ *is infinite for all* $x \in G$.

Proof. Let g be an element of G. If $C_G(g)$ is infinite then $C_G(g^2)$ is also infinite. Now we may assume that $C_G(g)$ is finite so that the set $T = \left\{ g^x \,\middle|\, x \in G \right\}$ is infinite. Put $U = \left\{ \{x, y\} \subset T \,\middle|\, (xy)^2 = (yx)^2 \right\}$ and $V = T^{(2)} \setminus U$. Then by Ramsey's Theorem there exists an infinite subset T_0 of T such that $T_0^{(2)} \subseteq U$ or $T_0^{(2)} \subseteq V$. By the property A_2^*, one can see that $T_0^{(2)} \subseteq U$ and $(xy)^2 = (yx)^2$ for all $x, y \in T_0$. Now, put $W_1 = \left\{ \{x, y\} \subset T_0 \,\middle|\, (xy^{-1})^2 = (x^{-1}y)^2 \right\}$ and $W_2 = T_0^{(2)} \setminus W_1$. Since $(xy^{-1})^2 = (y^{-1}x)^2 \Leftrightarrow (x^{-1}y)^2 = (yx^{-1})^2$, $T_0^{(2)}$ is partitioned into the sets W_1 and W_2. By Ramsey's Theorem, there exists an infinite subset T_1 of T_0 such that

$T_1^{(2)} \subseteq W_1$ or $T_1^{(2)} \subseteq W_2$. Suppose, if possible, that $T_1^{(2)} \subseteq W_2$. Partition T_1 into two infinite subsets X and Y. Consider infinite subsets X and $Z = \{y^{-1} \mid y \in Y\}$, by the property A_2^*, there exist $x \in X$ and $y \in Y$ such that $(xy^{-1})^2 = (y^{-1}x)^2$. Thus $\{x, y\}$ lies in W_1, a contradiction. Therefore $T_1^{(2)} \subseteq W_1$ and we have $(xy)^2 = (yx)^2$ and $(xy^{-1})^2 = (y^{-1}x)^2$ for all $x, y \in T_1$. Now fix $x_0 \in T_1$, then $yx_0y \in C_G(x_0)$ and $y^{-1}x_0y^{-1} \in C_G(x_0)$ for all $y \in T_1$. Therefore $y^{-1}x_0^2y \in C_G(x_0)$ for all $y \in T_1$. Since $C_G(x_0)$ is finite, there exist infinite subset W of T_1 and $y_0 \in W$ such that $(x_0^2)^{y_0} = (x_0^2)^y$ for all $y \in W$. Hence $yy_0^{-1} \in C_G(x_0^2)$ for all $y \in W$, so $C_G(x_0^2)$ is infinite, but x_0^2 and g^2 are conjugate and so $C_G(g^2)$ is also infinite.

Corollary 4. *Let G be an infinite A_2^*-group. Then G has an infinite abelian subgroup.*

Proof. We show that in any infinite group $G \in \mathsf{A}_2^*$ there exists an element x with $C_G(x)$ infinite. Then the result will follow, arguing as in Corollary 2.5 of [5]. If there exists an element $g \in G$ such that $g^2 \neq 1$, then by Lemma 3, g^2 has the required property. If $x^2 = 1$ for all $x \in G$, then G is abelian and any non-trivial element of G has the required property.

Lemma 5. *Let G be an infinite A_2^*-group. Then $C_G(x)$ is infinite for all $x \in G$.*

Proof. Let x be an arbitrary element in G. By Corollary 4, there exists an infinite abelian subgroup A, and by Lemma 2, $A^2 \leq Z(G)$. If A^2 is infinite, then $C_G(x)$ is also infinite. Now we may assume that A^2 is finite and so $D = \{a \in A \mid a^2 = 1\}$ is an infinite elementary abelian 2-group. Let $B = \{a \in D \mid (ax)^2 = (xa)^2\}$, then by Lemma 1, B is infinite. Therefore $x^a \in C_G(x)$ for all $a \in B$, since $a^2 = 1$. Now suppose, for a contradiction, that $C_G(x)$ is finite, then there exist infinite subset B_0 of B and element $a_0 \in B_0$ such that $x^a = x^{a_0}$ for all $a \in B_0$, therefore $aa_0^{-1} \in C_G(x)$ for all $a \in B_0$. This is a contradiction.

Since by Lemma 5, for any infinite subgroup H of an infinite A_2^*-group and any h in H, $C_H(h)$ is infinite, we have

Corollary 6. *Let G be an infinite A_2^*-group. Then for every element x of G there exists an infinite abelian subgroup containing x.*

Proof of Theorem 1. Let G be an infinite A_2^*-group and $x, y \in G$. It suffices to prove that $x^2 y = yx^2$. By Corollary 6, there exists an infinite abelian subgroup A containing x. By Lemma 2, $A^2 \leq Z(G)$ and so, $x^2 y = yx^2$, which completes the proof.

Lemma 7. *Let G be an infinite* B_n^**-group. Then* $C_G(a^n)$ *is infinite for all* $a \in G$.

Proof. Suppose, for a contradiction, that $C_G(a^n)$ is finite for some $a \in G$. Thus the set $X = \{a^g \mid g \in G\}$ is infinite. List the elements of X as $x_1, x_2, \cdots$ under some well order $\leq$ so that $x_i < x_j$ if $i < j$. For each $s \in X^{(2)}$ list the elements x_{i_1}, x_{i_2} of s in ascending order given by $\leq$ and write $\tilde{s} = (x_{i_1}, x_{i_2})$. Create three sets, one U_σ for each $\sigma \in S_2$ and V. For each $s \in X^{(2)}$, $\tilde{s} = (x_{i_1}, x_{i_2})$, put $s \in U_\sigma$ if $x^n_{i_{\sigma(1)}} x_{i_{\sigma(2)}} = x_{i_{\sigma(2)}} x^n_{i_{\sigma(1)}}$. Put $s \in V$ if $s \notin U_\sigma$ for any σ. By Ramsey's Theorem, there exists an infinite subset X_0 of X such that $X_0^{(2)} \subseteq U_\sigma$ for some σ or $X_0^{(2)} \subseteq V$. Suppose, if possible, that $X_0^{(2)} \subseteq V$. Partition X_0 into the infinite subsets Y and Z. Thus by the property B_n^*, there exist $y \in Y$ and $z \in Z$ such that $y^n z = zy^n$ and so $\{y, z\} \in U_\sigma$ for some σ, a contradiction. Therefore, $X_0^{(2)} \subseteq U_\sigma$ for some σ. By restricting the order $\leq$ to X_0, we may assume that $X_0 = \{x_1, x_2, \cdots\}$ and $x_i < x_j$ if $i < j$. Therefore for any $i_1 < i_2, x^n_{i_{\sigma(1)}} x_{i_{\sigma(2)}} = x_{i_{\sigma(2)}} x^n_{i_{\sigma(1)}}$. If $\sigma = 1$ then for any $i_1 < i_2, x^n_{i_1} x_{i_2} = x_{i_2} x^n_{i_1}$. Now fix i_1 and allow i_2 to vary over all indices i_2 greater than i_1. So $\{x_{i_2} \mid i_2 > i_1\} \subseteq C_G(x^n_{i_1})$, a contradiction. If $\sigma \neq 1$ then for any $i_1 < i_2, x^n_{i_2} x_{i_1} = x_{i_1} x^n_{i_2}$. Thus $x_{i_1} \in C_G(x^n_{i_2})$. Now since X_0 is infinite, there exists a sequence with t elements as $i_t > i_{t-1} > \cdots > i_1$ where t is an integer greater than $\left| C_G(a^n) \right|$, thus $\left| C_G(x^n_{i_t}) \right| < t$. Since a^n and $x^n_{i_t}$ are conjugate, $\left| C_G(x^n_{i_t}) \right| > t$, a contradiction.

Lemma 8. *Let G be an infinite* B_n^**-group. If* A *is an infinite abelian subgroup of G, then* $A^n \leq Z(G)$ *and* $G^n \leq C_G(A)$.

Proof. Let x be any element of G. We must prove that $xy^n = y^n x$ and $x^n y = yx^n$ for all $y \in A$. By Lemma 1, $B = \{a \in A \mid a^n x = xa^n, x^n a = ax^n\}$ is an infinite subset of A. Set $M = \{a^n \mid a \in B\}$ and $F = \{x^a \mid a \in A\}$. If M is infinite or F is finite then the centre of $H := \langle A, x \rangle$ is infinite and so H is a B_n-group, thus $xy^n = y^n x$ and $x^n y = yx^n$ for all $y \in A$. Now, we may assume that F is infinite and M is finite, therefore $D = \{a \in A \mid a^n = 1\}$ is an infinite group. Let y be an arbitrary element of A. Consider infinite subsets, yD and F, by the property $\mathcal{B}_n^*$, there exist $d \in D$ and $a \in A$ such that $(yd)^n x^a = x^a (yd)^n$ then $y^n x = xy^n$ since $a, d, y \in A$ and $d^n = 1$. Now consider infinite subsets F and yB then there exist $a \in A$ and $b \in B$ such that $(x^a)^n yb = yb(x^a)^n$ then $x^n yb = ybx^n$. Now, since $b \in B$, $x^n b = bx^n$ and so $x^n y = yx^n$.

Lemma 9. *Let n be a positive integer such that every infinite $\mathcal{B}_n^*$-group has an infinite abelian subgroup. Then every infinite $\mathcal{B}_n^*$-group is a $\mathcal{B}_n$-group.*

Proof. Let x and y be arbitrary elements of G, we prove that $x^n y = yx^n$. By Lemma 7, $C_G(x^n)$ is infinite and by the hypothesis, $C_G(x^n)$ has an infinite abelian subgroup, A say. If A^n is infinite then $Z(G)$ is infinite by Lemma 8. Now consider the infinite sets $xZ(G)$ and $yZ(G)$ then property $\mathcal{B}_n^*$, yields $x^n y = yx^n$. Thus we may assume that A^n is finite and so $D = \{a \in A \mid a^n = 1\}$ is an infinite subgroup of A. If $X = \{x^a \mid a \in D\}$ is finite, then $B = C_D(x)$ is infinite. Consider the infinite sets xB and yB, by the property $\mathcal{B}_n^*$, there exist $b_1, b_2 \in B$ such that $(xb_1)^n yb_1 = yb_1(xb_2)^n$, then $x^n y = yx^n$, since $b_1^n = b_2^n = 1$ and $xb_1 = b_1 x$. If X is infinite then consider Dy and X. By the property $\mathcal{B}_n^*$, there exist $d_1, d_2 \in D$ such that $d_1 y(x^{d_2})^n = (x^{d_2})^n d_1 y$ and so $x^n y = yx^n$. Therefore, in any case, we have $x^n y = yx^n$ and the proof is complete.

Proof of Theorem 2. Let $n \in \{3, 6\} \cup \{2^k \mid k \in \mathbb{N}\}$. We show that every infinite $\mathcal{B}_n^*$-group G, has an infinite abelian subgroup, then Theorem 2 follows from Lemma 9. It suffices to prove that G has a non-trivial element with infinite centralizer. If there is an element x of G such that $x^n \neq 1$, then x^n has the required property by Lemma 7. If G has exponent dividing $n \in \{3, 6\}$ then G is locally finite (see page 425 of [16]). Now if $n \in \{3, 6\} \cup \{2^k \mid k \in \mathbb{N}\}$ then G is an infinite locally finite or an infinite 2-group. Thus by Corollary 2.5 of [5], G has an infinite abelian subgroup A, therefore, in this case, every non-trivial element of A has the required property.

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