The Neutrix Product of the Distributions $x^\lambda \ln x$ and $x^{-\lambda - r}$

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Abstract. The neutrix product of the distributions $x^\lambda \ln x$ and $x^{-\lambda - r}$ is evaluated for $\lambda = 0, \pm 1, \pm 2, \cdots$ and $r = 1, 2, \cdots$.

In the following, we let $N$ be the neutrix, see van der Corput [1], having domain $N = \{1, 2, \cdots, n, \cdots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ell n^{-1} n, \ell n^r n : \lambda > 0, r = 1, 2, \cdots$$

and all functions which converge to zero in the normal sense as $n$ tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) \, dx = 1$.

Putting $\delta_n(x) = n \rho(nx)$ for $n = 1, 2, \cdots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. 
Now let $D$ be the space of infinitely differentiable functions with compact support and let $D'$ be the space of distributions defined on $D$. Then if $f$ is an arbitrary distribution in $D'$, we define

$$f_n(x) = \left( f^* \delta_n \right)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

**Definition 1.** Let $f$ and $g$ be distributions in $D'$ for which on the interval $(a, b)$, $f$ is the $k$-th derivative of a locally summable function $F$ in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of $f$ and $g$ is defined on the interval $(a, b)$ by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

The following definition for the non-commutative neutrix product of two distributions was given in [3] and generalizes Definition 1.

**Definition 2.** Let $f$ and $g$ be distributions in $D'$ and let $g_n = g^* \delta_n$. We say that the neutrix product $f \circ g$ of $f$ and $g$ exists and is equal to the distribution $h$ on the interval $(a, b)$ if

$$N - \lim_{n \to \infty} \left\{ fg_n, \phi \right\} = \left\langle h, \phi \right\rangle,$$

for all functions $\phi$ in $D$ with support contained in the interval $(a, b)$. Note that if

$$\lim_{n \to \infty} \left\{ fg_n, \phi \right\} = \left\langle h, \phi \right\rangle,$$

we simply say that the product $f \cdot g$ exists and equals $h$.

This definition of the neutrix product is in general non-commutative. It is obvious that if the product $f \cdot g$ exists then the neutrix product $f \circ g$ exists and $f \cdot g = f \circ g$.

Further, it was proved in [3] that if the product $fg$ exists by Definition 1 then the product $f \circ g$ exists by Definition 2 and $fg = f \circ g$.

The next two theorems were proved in [3].
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**Theorem 1.** Let $f$ and $g$ be distributions and suppose that the neutrix products $f \circ g$ and $f \circ g'$ exist on the interval $(a, b)$. Then the neutrix product $f' \circ g$ exists and

$$ (f \circ g)' = f' \circ g + f \circ g', $$
on the interval $(a, b)$.

**Theorem 2.** The neutrix product $x^l_+ \circ x_-^r$ exists and

$$ x^l_+ \circ x_-^r = -\frac{\pi \csc (\pi \lambda)}{2 (r-1)!} \delta^{(r-1)}(x) \quad (1) $$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

We now prove the following theorem:

**Theorem 3.** The neutrix product $x^l_+ \ln x_+ \circ x_-^r$ exists and

$$ x^l_+ \ln x_+ \circ x_-^r = -\frac{\pi \csc (\pi \lambda)}{2 (r-1)!} \left[ 2 \psi(\lambda + r) - \Gamma'(1) \right] \delta^{(r-1)}(x) \quad (2) $$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$ where $\Gamma$ denotes the Gamma function and

$$ \psi(\lambda + r) = \frac{\Gamma'(\lambda + r)}{\Gamma(\lambda + r)} $$

**Proof.** We will first of all suppose that $-1 < \lambda < 0$. Then $x^l_+ \ln x_+$ and $x_-^{\lambda-1}$ are locally stumble functions and

$$ x_-^{\lambda-1} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)} (x_-^{\lambda-1})^{(r-1)} $$

Thus

$$ (x_-^{\lambda-1})_n = x_-^{\lambda-1} * \delta^n_n(x) = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)} \int_x^{1/n} (t-x)^{-\lambda-1} \delta^{(r-1)}(t) dt $$

for $r = 1, 2, \ldots$ and so
\[
\frac{\Gamma(\lambda + r)}{\Gamma(\lambda + 1)} \int_{-\infty}^{\infty} x^\lambda \ell_n x, \left(\frac{x^{r-1}}{r} \right) x^m \, dx = \int_{0}^{1/n} x^{\lambda + m} \ell_n x \int_{x}^{1/n} (t - x)^{-\lambda - 1} \delta_n(t) \, dt \, dx \\
= \int_{0}^{1/n} \delta_n^{(r-1)}(t) \int_{0}^{t} x^{\lambda + m} \ell_n x \, dx \, dt \\
= \int_{0}^{1/n} t^n \delta_n^{(r-1)}(t) \int_{0}^{1} \ell_n(tu)(1 - u)^{-\lambda - 1} \, du \, dt \\
= B(\lambda + m + 1, -\lambda) \int_{0}^{1/n} t^n \ell_n t \delta_n^{(r-1)}(t) \, dt \\
+ B_{1,0}(\lambda + m + 1, -\lambda) \int_{0}^{1/n} t^n \delta_n^{(r-1)}(t) \, dt,
\]

where the substitution \( x = tv \) has been made, \( B \) denotes the Beta function and in general

\[
B_{p,q}(\lambda, \mu) = \frac{\partial^{p+q}}{\partial \lambda^p \partial \mu^q} B(\lambda, \mu)
\]

Making the substitution \( nt = y \). We have

\[
\int_{0}^{1/n} t^n \delta_n^{(r-1)}(t) \, dt = n^{r-m-1} \int_{0}^{1} y^{m+n} \rho^{(r-1)}(y) \, dy,
\]  
(4)

\[
\int_{0}^{1/n} t^n \ell_n t \delta_n^{(r-1)}(t) \, dt = -n^{r-m-1} \ell_n \int_{0}^{1} y^{m+n} \rho^{(r-1)}(y) \, dy \\
+ n^{r-m-1} \int_{0}^{1} y^{m+n} \ell_n y \rho^{(r-1)}(y) \, dy
\]  
(5)

for \( m = 0, 1, 2, \ldots \).

In particular, when \( m = r - 1 \), it is easily proved by induction that

\[
\int_{0}^{1} y^{r-1} \rho^{(r-1)}(y) \, dy = \frac{1}{2} (-1)^{r-1} (r-1)!,
\]  
(6)

\[
\int_{0}^{1} y^{r-1} \ell_n y \rho^{(r-1)}(y) \, dy = (-1)^{r-1} (r-1)! \left[ \frac{1}{2} \phi(r-1) + c(\rho) \right],
\]  
(7)

for \( r = 1, 2, \ldots \), where

\[
\phi(r) = \begin{cases} 
\sum_{i=1}^{r-1} i, & r = 1, 2, \ldots \\
0, & r = 0
\end{cases}
\]
and
\[ c(\rho) = \int_0^1 \ell n t \rho(t) \, dt. \]

Further, putting
\[ K = - \frac{\Gamma(\lambda + 1)}{\lambda \Gamma(\lambda + r)} \sup_x \left\{ \left| \rho^{(r-1)}(x) \right| \right\} > 0, \]
we have
\[ \left| x^{-\lambda-r}_n \right| = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)} \left[ \int_{nx}^{\infty} (u - nx)^{-\lambda-1} u^{r-1} \rho^{(r-1)}(u) \, du \right] \]
\[ \leq -\lambda Kn_{\lambda+r}^{1+n} (u - nx)^{-\lambda-1} \, du \]
\[ = Kn_{\lambda+r}^{1+n} \]
and so when \( m = r, \) we have
\[ \left| \int_{-\infty}^{\infty} x^r \ell n x_n (x^{-\lambda-r}_n) x^r \, dx \right| \leq \int_0^{1/\lambda} \left| x^\lambda \ell n x (x^{-\lambda-r}_n) x^r \right| \, dx \leq -\lambda^{-1} Kn^{-1} \ell n n, \]
(8)

Now let \( \varphi \) be an arbitrary function in \( D. \) Then
\[ \varphi(x) = \sum_{m=0}^{r-1} \frac{x^m}{m!} \phi^{(m)}(0) + \frac{x^r}{r!} \phi^{(r)}(\xi x), \]
where \( 0 < \xi < 1 \) and so
\[ \left( x^r \ell n x_n (x^{-\lambda-r}_n) \varphi(x) \right) = \sum_{m=0}^{r-1} \frac{x^m}{m!} \phi^{(m)}(0) \left[ \int_{-\infty}^{\infty} x^r \ell n x_n (x^{-\lambda-r}_n) x^m \, dx \right] \]
\[ + \frac{1}{r!} \int_{-\infty}^{\infty} x^r \ell n x_n (x^{-\lambda-r}_n) x^r \phi^{(r)}(\xi x) \, dx. \]
(9)

Since
\[ \left| \int_{-\infty}^{\infty} x^r \ell n x_n (x^{-\lambda-r}_n) x^r \phi^{(r)}(\xi x) \, dx \right| \leq \sup_x \left\{ \phi^{(r)}(x) \right\} \left| (-\lambda^{-1}) Kn^{-1} \ell n n, \right. \]
it follows from equations (3) to (9) that
\[ N = \lim_{n \to \infty} \frac{\Gamma(\lambda + r)}{\Gamma(\lambda + 1)} \{ x_+^\lambda \ln x_+ + x_-^{-\lambda - r} \} \phi(x) \]
\[ = (-1)^{r-1} B(\lambda + r, -\lambda) \left[ \frac{1}{2} \phi(r-1) + c(\rho) \right] \phi^{(r-1)}(0) + \frac{1}{2} (-1)^{r-1} B_{1,0}(\lambda + r, -\lambda) \phi^{(r-1)}(0). \]  

(10)

Differentiating the identity
\[ B(\lambda, \mu) = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda + \mu)} \]

partially with respect to \( \lambda \), it follows that
\[ B_{1,0}(\lambda + r, -\lambda) = \frac{\Gamma'(\lambda + r) \Gamma(-\lambda)}{(r-1)!} - \frac{\Gamma(\lambda + r) \Gamma'(-\lambda) \Gamma'(r)}{[(r-1)!]^2} \]  

(11)

and taking logs and differentiating the identity
\[ \Gamma(\lambda + r) = (\lambda + r - 1) \cdots (\lambda + 1) \Gamma(\lambda + 1) \]

gives
\[ \psi(\lambda + r) = \frac{\Gamma'(\lambda + r)}{\Gamma(\lambda + r)} = \sum_{i=1}^{r-1} (\lambda + r - i)^{-1} + \psi(\lambda + 1). \]  

(12)

In particular, we have
\[ \frac{\Gamma'(r)}{(r-1)!} = \phi(r-1) + \Gamma'(1). \]  

(13)

It now follows from equations (11) and (13) that
\[ \frac{\Gamma'(\lambda + 1)}{\Gamma(\lambda + r)} B_{1,0}(\lambda + r, -\lambda) = \frac{\Gamma(\lambda + 1) \Gamma(-\lambda)}{(r-1)!} \left[ \frac{\Gamma'(\lambda + r)}{\Gamma(\lambda + r)} - \frac{\Gamma'(r)}{(r-1)!} \right] \]
\[ = -\frac{\pi \text{cosec}(\pi \lambda)}{(r-1)!} \left[ \psi(\lambda + r) - \phi(r-1) - \Gamma'(1) \right]. \]  

(14)
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Further,

$$\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)} B(\lambda + r, -\lambda) = -\frac{\pi \cosec (\pi \lambda)}{(r-1)!}$$  \hspace{1cm} (15)$$

and equation (2) now follows from equations (10), (14) and (15) for the case $-1 < \lambda < 0$.

Now let us suppose that equation (2) holds when $-k < \lambda < -k + 1$ and $r = 1, 2, \cdots$, where $k$ is a positive integer. This is true when $k = 1$. Thus if $-k - 1 < \lambda < -k$ it follows from our assumption that

$$x^\lambda_+ \ln x_+ o x^{-\lambda-1-\rho} = \frac{\pi \cosec (\pi \lambda)}{2(r-1)!} \left[ 2c(\rho) + \psi (\lambda + 1 + r) - \Gamma'(1) \right] \delta^{(r-1)}(x),$$

for $r = 1, 2, \cdots$. It follows from Theorem 1 that

$$\left[ (\lambda + 1) x^\lambda_+ \ln x_+ + x^\lambda_+ \right] o x^{-\lambda-r-1} + (\lambda + r + 1) x^{\lambda+1}_+ \ln x_+ o x^{-\lambda-r-2}$$

$$= \frac{\pi \cosec (\pi \lambda)}{2(r-1)!} \left[ 2c(\rho) + \psi (\lambda + r + 1) - \Gamma'(1) \right] \delta^{(r)}(x)$$

$$= \frac{(\lambda + 1) x^\lambda_+ \ln x_+ o x^{-\lambda-r-1} - \frac{\pi \cosec (\pi \lambda)}{2r!} \delta^{(r)}(x)}{\delta^{(r)}(x)}$$

$$+ \frac{(\lambda + r + 1) \pi \cosec (\pi \lambda)}{2r!} \left[ 2c(\rho) + \psi (\lambda + 2r + 1) - \Gamma'(1) \right] \delta^{(r)}(x).$$

Thus

$$(\lambda + 1) x^\lambda_+ \ln x_+ o x^{-\lambda-r-1} =$$

$$-\frac{(\lambda + 1) \pi \cosec (\pi \lambda)}{2r!} \left[ 2c(\rho) + \psi (\lambda + r + 1) - \psi (\lambda + r + 2) \right] \delta^{(r)}(x) +$$

$$+ \frac{\pi \cosec (\pi \lambda)}{2(r-1)!} \left[ r^{-1} + \psi (\lambda + r + 1) - \psi (\lambda + r + 2) \right] \delta^{(r)}(x),$$

$$-\frac{(\lambda + 1) \pi \cosec (\pi \lambda)}{2r!} \left[ 2c(\rho) + \psi (\lambda + r + 1) - \Gamma'(1) \right] \delta^{(r)}(x),$$

since, from equation (12), we have

$$\psi (\lambda + r + 2) - (\lambda + r + 1)^{-1} = \psi (\lambda + r + 1)$$
and so
\[ r^{-1} + \psi(\lambda + r + 1) - \psi(\lambda + r + 2) = \frac{\lambda + 1}{r(\lambda + r + 1)}. \]

Equation (2) now follows by induction for \( \lambda < 0, \lambda \neq -1, -2, \ldots \) and \( r = 2, 3, \ldots \).

To cover the case \( r = 1 \), we note the product \( x_{-}^{\lambda+1} \ell n x_{-} \cdot x_{-}^{\lambda-1} \) exists by Definition 1 and
\[ x_{-}^{\lambda+1} \ell n x_{-} \cdot x_{-}^{\lambda-1} = 0 \]
for all \( \lambda \).

Let us suppose that equation (2) holds when \( -k < \lambda < -k + 1 \) and \( r = 1 \), where \( k \) is a positive integer. This is true when \( k = 1 \). Thus if \( -k + 1 < \lambda < -k \), it follows from our assumption that
\[ x_{-}^{\lambda+1} \ell n x_{-} o x_{-}^{\lambda-2} = \frac{1}{2} \pi \cosec (\pi \lambda) \left[ 2c(\rho) + \psi(\lambda + 2) - \Gamma'(1) \right] \delta(x). \]

It follows from equation (16) and Theorem 1 that
\[
\left[ (\lambda + 1)x_{+}^{\lambda+1} \ell n x_{+} + x_{+}^{\lambda+1} \right] o x_{-}^{\lambda-1} + (\lambda + 1)x_{+}^{\lambda+1} \ell n x_{+} o x_{-}^{\lambda-2} = 0 \\
= (\lambda + 1)x_{+}^{\lambda+1} \ell n x_{+} o x_{-}^{\lambda-1} - \frac{1}{2} \pi \cosec (\pi \lambda) \delta(x) \\
+ \frac{1}{2} (\lambda + 1) \pi \cosec (\pi \lambda) \left[ 2c(\rho) + \psi(\lambda + 2) - \Gamma'(1) \right] \delta(x) \\
= (\lambda + 1)x_{+}^{\lambda+1} \ell n x_{+} o x_{-}^{\lambda-1} \\
+ \frac{1}{2} (\lambda + 1) \pi \cosec (\pi \lambda) \left[ 2c(\rho) + \psi(\lambda + 1) - \Gamma'(1) \right] \delta(x)
\]

Equation (2) now follows by induction for \( \lambda < 0, \lambda \neq -1, -2, \ldots \) and \( r = 1 \).

Now let us suppose that equation (2) holds when \( k - 1 < \lambda < k \) and \( r = 1, 2, \ldots \), where \( k \) is a positive integer. This true when \( k = 0 \). Then for an arbitrary function \( \phi \) in \( D \) we have
\[
\left\{ x_{+}^{\lambda+1} \ell n x_{+} (x_{-}^{\lambda-1} - r) \cdot \phi(x) \right\} = \left\{ x_{+}^{\lambda+1} \ell n x_{+} (x_{-}^{\lambda-1} - r) \cdot \psi(x) \right\},
\]
where \( \psi(x) = x \varphi(x) \) is also in \( D \). It follows from our assumption with \( k-1 < \lambda < k \) that

\[
N = \lim_{n \to \infty} \left\{ x^\lambda_+ \ln x_+ (x^{\lambda-r-1}_-) \varphi(x) \right\} \\
= -\frac{(-1)^r \pi \csc (\pi \lambda)}{2r!} \left( 2c(\rho) + \psi(\lambda + r + 1) - \Gamma'(1) \right) \varphi^{(r)}(0)
\]

and so

\[
N = \lim_{n \to \infty} \left\{ x^{\lambda+1}_+ \ln x_+ (x^{\lambda-r-1}_-) \varphi(x) \right\} \\
= -\frac{(-1)^r \pi \csc (\pi \lambda)}{2(r-1)!} \left( 2c(\rho) + \psi(\lambda + r + 1) - \Gamma'(1) \right) \varphi^{(r-1)}(0)
\]

Equation (2) now follows by induction for \( \lambda > 0, \lambda \neq 1, 2, \cdots \) and \( r = 1, 2, \cdots \), completing the proof of the theorem.

**Corollary 3.1.** The neutrix product \( x^\lambda_+ \ln x_+ o x^{\lambda-r}_- \) exists and

\[
x^\lambda_+ \ln x_+ o x^{\lambda-r}_- = \frac{(-1)^r \pi \csc (\pi \lambda)}{2(r-1)!} \left( 2c(\rho) + \psi(\lambda + r) - \Gamma'(1) \right) \delta^{(r-1)}(x) \quad (17)
\]

for \( \lambda \neq 0, \pm 1, \pm 2, \) and \( r = 1, 2, \cdots \).

**Proof.** Equation (17) follows on replacing \( x \) by \(-x\) in equation (2).

**Theorem 4.** The neutrix product \( x^\lambda_+ o x^{\lambda-r}_- \ln x_+ \) exists and

\[
x^\lambda_+ \ln x_+ o x^{\lambda-r}_- = \frac{\pi \csc (\pi \lambda)}{2(r-1)!} \left( 2c(\rho) + \psi(-\lambda - r + 1) - \Gamma'(1) \right) \delta^{(r-1)}(x) \quad (18)
\]

for \( \lambda \neq 0, \pm 1, \pm 2, \cdots \) and \( r = 1, 2, \cdots \).

**Proof.** Differentiating equation (1) partially with respect to \( \lambda \) we get

\[
x^\lambda_+ \ln x_+ o x^{\lambda-r}_- - x^\lambda_+ o x^{\lambda-r}_- \ln x_- = \frac{\pi^2 \cot(\pi \lambda) \csc(\pi \lambda)}{2(r-1)!} \delta^{(r-1)}(x)
\]
and on using equation (2) it follows that

\[ x_+^\lambda \circ x_-^{\lambda-r} \ln x_- = -\frac{\pi \csc (\pi \lambda)}{2 (r-1)!} \left[ \pi \cot (\pi \lambda) + 2c (\rho) + \psi (\lambda + r) - \Gamma' (1) \right] \delta^{(r-1)} (x). \]  

(19)

Taking logs and differentiating the identity

\[ \Gamma (-\lambda) \Gamma (\lambda + 1) = (-1)^{r-1} \Gamma (-\lambda - r + 1) \Gamma (\lambda + r) = -\pi \csc (\pi \lambda) \]

gives

\[ -\psi (-\lambda - r + 1) + \psi (\lambda + r) = -\pi \cot (\pi \lambda) \]  

(20)

and equation (18) follows from equations (19) and (20).

**Corollary 4.1.** The neutrix product \( x_+^\lambda \circ x_-^{\lambda-r} \ln x_- \) exists and

\[ x_+^\lambda \circ x_-^{\lambda-r} \ln x_- = \frac{(-1)^r \pi \csc (\pi \lambda)}{2 (r-1)!} \left[ 2c (\rho) + \psi (-\lambda - r + 1) - \Gamma' (1) \right] \delta^{(r-1)} (x) \]  

(21)

for \( \lambda \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \).

**Proof.** Equation (21) follows on replacing \( x \) by \( -x \) in equation (18).

We finally note that if we replace \( \lambda \) by \( -\lambda - r \) in equation (21), we get

\[ x_-^{\lambda-r} \circ x_+^\lambda \ln x_+ = -\frac{\pi \csc (\pi \lambda)}{2 (r-1)!} \left[ 2c (\rho) + \psi (\lambda + 1) - \Gamma' (1) \right] \delta^{(r-1)} (x) \]

and we see that the product of the distributions \( x_+^\lambda \ln x_+ \) and \( x_-^{\lambda-r} \) is commutative only when \( r = 1 \).

**References**


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