On Certain Representation of Topological Groups

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In this note classes of groups representations of which have either invariant vectors or invariant functionals are introduced. Connection between these classes of groups is established.

Let $E$ be a separable topological vector space over the field of complex numbers $\mathbb{C}$ and $GL(E)$ be the group of all linear automorphisms of $E$ and $G$ be a separable topological group and $\rho : G \to GL(E)$ be a linear representation of the group $G$ in $E$. We denote by $E^G$ the subspace of $\rho(G)$-invariant elements in $E$, that is $E^G = \{x \in E : \rho(g)x = x \text{ for all } g \in G\}$, and denote by $E'$ the space of continuous linear functionals on the space $E$.

A functional $f \in E'$ is called $\rho(G)$-invariant if $f(gx) = f(g^{-1}x)$ for all $g \in G$.

A linear representation $\rho$ of topological group $G$ in $E$ is called continuous if the mapping $G \times E \to E$ defined by the formula $g(x) \to \rho(g)x$ is continuous.

Everywhere we consider continuous representations.

**Definition 1.** A linear representation $\rho$ is called linear reductive (or $\alpha$-representation) if for each $x \in E^G$, $x \neq 0$ there exists continuous $\rho(G)$-invariant linear functional $f \in E'$ such that $f(x) \neq 0$. In the case of $E^G = \{0\}$ the representation $\rho$ is also called $\alpha$-representation.

**Definition 2.** A linear representation $\rho$ of the group $G$ in $E$ is said to be $\beta$-representation if for arbitrary nontrivial continuous $\rho(G)$-invariant functional $f \in E'$ there exists element $x \in E^G$ such that $f(x) \neq 0$.

We recall that a closed subspace $E_1$ in $E$ is said to be $t$-complementable if there exists a closed subspace $E_2$ in $E$ such that $E_1 \cap E_2 = \{0\}$ and $E$ is the topological direct sum of subspaces $E_1$ and $E_2$. Here one assumes that there exist continuous projections acting from $E$ onto $E_1$ and $E_2$ [2]. In this case subspace $E_2$ is called $t$-complementable to $E_1$ and the notation $E = E_1 \oplus E_2$ is used.
Definition 3. A linear representation $\rho$ of the group $G$ in $E$ is said to be $t_\gamma$-representation (or semi-simple), if for every $\rho(G)$-invariant subspace $E_1$ in $E$ there exists on $\rho(G)$-invariant complementary subspace $E_2$.

We refer to topological group $G$ as the group of class $t\alpha$ (respectively $t\beta$ and $t\gamma$), if its every continuous linear representation is a $t\alpha$-(resp. $t\beta$- and $t\gamma$-) representation.

We refer to topological group $G$ as the $ft\alpha$ (respectively $ft\beta$ and $ft\gamma$) group, if its every finite dimensional continuous representation is a $t\alpha$-(resp. $t\beta$- and $t\gamma$-) representation.

Proposition. Every $t_\gamma$-group is also a $t\alpha$-group.

Proof. Let $G$ be a $t_\gamma$-group and $0 \neq x \in E^G$. Since $p$ is a $t_\gamma$-representation, then there exists a $G$-invariant closed subspace $E_x$ complementing the one dimensional invariant subspace $Cx$, where $C$ denotes the field of complex number. Let us consider the linear functional $f$ in $E$ defined in the following way: $f(y) = 0$, at $y \in E_x$ and $f(x) = 1$. Then $f$ is invariant with respect to $G$, $f \in E'$ and $f(x) \neq 0$. Consequently, $G$ is a $t\alpha$-group.

Theorem 1. A locally compact group $G$ is a $t\beta$-group if and only if $G$ is compact.

Proof. Since $G$ is a locally compact group, there exists a right-invariant nontrivial Haar measure $dg$ on $G$. We consider the Banach space $V = L_1(G, dg)$ with the norm $\|\varphi\| = \left[ \int_G |\varphi| dg \right]$, $\varphi \in V$. We give a representation $G$ in $V$ by setting

$$(T_{g\varphi})(t) = \varphi(tg), \quad \varphi \in V, \quad g, \ t \in G.$$

Since $dg$ is right-invariant measure, then nontrivial linear functional $f(\varphi) = \int_G \varphi(t)dg$ on $V$ is $\rho(G)$-invariant. Since the group $G$ is a $t\beta$-group, there exists nontrivial $\varphi_0 \in V^G$ such that $f(\varphi_0) \neq 0$. But if $G$ is not a compact group, then $V^G = \{0\}$. Consequently, $G$ is compact.

Let $G$ be a compact group and $dg$ be a measure of Haar on $G$, normed by the condition $\int_G dg = 1$, and $\rho: \to GL(E)$ be a linear continuous representation in complete locally convex space $E$ and $E'$ be the conjugated space to $E$.

Let $f \in (E')^G$ be an arbitrary nonzero element. There is $x \in E$ such that $f(x) \neq 0$. We consider operator

$$p = \int_G \rho(g)dg$$

(1)
It is known (see, for example, [1], p.150) that this operator is a projection operator on $E^G$ (expression (1) is considered as an integral of function on $G$ with values in $GL(E)$ (see [1], p.150)). Therefore 

$$f(\nu) = \int_G (g(x))dg = \int_G f(g(x))dg = f(x)\int_G dg = f(x) \cdot 1 = f(x) \neq 0,$$

therefore $\rho$ is a $t\beta$-representation. This means that $G$ is a $\beta$-group.

**Theorem 2.** For the group $G$ the following conditions are equivalent:

(a) $G$ is a group of class $t\beta$

(b) $G$ is a group of class $t\gamma$.

**Proof.** $(a) \Rightarrow (b)$. Let $\rho: G \rightarrow GL(E)$ be an arbitrary continuous linear representation of the topological group $G$ of class $t\beta$ in a space $E$, and $E_0$ be a nontrivial $t$-complementable $\rho(G)$-invariant subspace in $E$.

We prove that there exists $\rho(G)$-invariant $t$-complement of $E_0$. Let $L(E) = \text{Hom}_c(E, E) = \text{End} E$ be the space of all continuous linear mappings of $E$ into $E$. $G$ operates in $L(E)$ as follows: 

$$g f = gf^{-1}, \text{ where } g \in G, f \in L(E).$$

In $L(E)$ we shall consider strong operator topology $st$: sequence $f_\alpha$ converges to $f$ strongly if for all $x \in E$ the sequence $f_\alpha(x)$ converges to $f(x)$ (and it is denoted by $f_\alpha \xrightarrow{st} f$).

Since $E_0$ is $t$-complementable vector space in $E$, there exists a closed vector subspace $E_1$ of $E$ such that $E$ is topological direct sum of $E_0$ and $E_1$.

Let $p_0(p_1)$ is projective operator on $E_0(E_1)$ parallel to the space $E_1$ (respectively $E_0$). Then $1 = p_0 + p_1$ and these operators are continuous [2].

The equality $g p_0 = p_0 g p_0$ is clear. Hence $\tilde{g} p_0 = p_0 \tilde{g} p_0$; indeed $p_0 \tilde{g} p_0 = p_0 g p_0 g^{-1} = g p_0 g^{-1} = \tilde{g} p_0$.

We shall consider the following linear subspaces

$$W_0 = l.s.\{\tilde{g} p_1 : g \in G\}, V_0 = l.s.\{(\tilde{g} - 1)p_1 : g \in G\},$$

where $l.s.$ means linear span.

We have

$$W_0 = V_0 + Cp_1.$$
We show that $V_0 \cap C_{P_1} = \{0\}$. Indeed, $p_1E = E_1$ and at the same time

\[(\tilde{g} - 1)p_1 = \tilde{g}p_1 - p_1 = \tilde{g}(1 - p_0) - p_1 = 1 - \tilde{g}p_0 - p_1 \]

\[= p_0 - \tilde{g}p_0 = p_0^2 - \tilde{g}p_0 = p_0^2 - p_0\tilde{g}p_0 = p_0(1 - \tilde{g})p_0.\]

Hence $T(E) \in E_0$ for arbitrary $T \in V_0$ and therefore $V_0 \cap C_{P_1} = \{0\}$. Thus $W_0 = V_0 + C_{P_1}$. Now we show that $W = V \oplus C_{P_1}$, where $W = W_0$, $V = V_0$ and the closures are taken on topology st. Assume that the sequence $x_\alpha \in W_0$ converges to $x \in W_0 = W$. Then

\[x_\alpha = y_\alpha + c_\alpha p_1 \xrightarrow{\text{st}} x, \quad y_\alpha \in V_0, \ c_\alpha \in C.\]

By applying $p_1$ to

\[y_\alpha + c_\alpha p_1 \xrightarrow{\text{st}} x\]

we get

\[p_1(y_\alpha + c_\alpha p_1) = p_1y_\alpha + c_\alpha p_1 = 0 + c_\alpha p_1 \xrightarrow{\text{st}} p_1x.\]

Since $c_\alpha p_1 \in C_{P_1}$, then $p_1x \in C_{P_1}$. On the other hand from the convergence $x_\alpha \xrightarrow{\text{st}} x$ we get that

\[p_0 x_\alpha = p_0(y_\alpha + c_\alpha p_1) = p_0y_\alpha + p_0c_\alpha p_1 = y_\alpha + 0 \xrightarrow{\text{st}} p_0x,\]

i.e., $p_0x \in V_0 = V$. Therefore $x = (p_0 + p_1)x = p_0x + p_1x \in V + C_{P_1}$.

Assume that $z \in V_0 = V$, then there exists $y_\alpha \in V_0$, such that $y_\alpha \xrightarrow{\text{st}} z$.

From here we have

\[p_0y_\alpha = y_\alpha \xrightarrow{\text{st}} p_0z \quad \text{and} \quad z = p_0z,\]

i.e., $z(E) \subset E_0$. But $C_{P_1}(E) \subset E_1$. Hence $V \cap C_{P_1} = \{0\}$. Thus $W = V \oplus C_{P_1}$.

Since $V$ is st-closed and dim $C_{P_1} = 1$, there exists $f \in W'$ such that $ker f = V$. Let $\omega$ be an arbitrary element of $W$, i.e., $\omega = \lambda p_1 + \omega_1$, $\lambda \in C$, $\omega_1 \in V$. Then

\[f(\omega) = \lambda f(p_1) + f(\omega_1) = \lambda f(p_1).\]

It can be assumed that $f(p_1) = 1$. Taking it into account, we get that

\[\lambda = f(\omega) \text{ and } \omega_1 = \omega - f(\omega)p_1.\]

Thus $\omega = f(\omega)p_1 + (\omega - f(\omega)p_1)$. From the construction of $W$ it follows that this space is $\rho(G)$-invariant. We consider restriction of the representation of the group $G$ to $W$ and show that $f$ is a $\rho(G)$-invariant functional. Indeed, $f(\lambda p_1 + \nu) = \lambda$, where $\nu \in V$. Therefore
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\[ f(\tilde{g}(\lambda p_1 + \nu)) = f(\lambda \tilde{g} p_1 + \tilde{g} \nu) = f(\lambda p_1 - \lambda(1 - \tilde{g}) p_1 + \tilde{g} \nu) = f(\lambda p_1 + \lambda(\tilde{g} - 1)p_1 + \tilde{g} \nu) = \lambda, \]

because \( \lambda(\tilde{g} - 1)p_1 + \tilde{g} \nu \in V \). This means that \( f \) is \( \rho(G) \)-invariant.

Since \( G \) is a \( TB \) group, for this functional there exists a \( \rho(G) \)-invariant element \( p_2 = p_1 + Q_0 \in W(Q_0 \in V) \) such that \( f(p_2) = 1 \neq 0 \). We define \( L = \overline{p_2(E_1)} \) (closure in topology \( st \)).

Since \( p_2 \) is \( \rho(G) \)-invariant, \( L \) is also \( \rho(G) \)-invariant: Indeed \( \tilde{g} p_2 = g p_2 g^{-1} = p_2 \), i.e., \( gp_2 = p_2 g \). Taking it into account we get that

\[ g(p_2(E)) = g p_2 g^{-1}(g(E)) = p_2(g(E)) \subset p_2 E. \]

By using the continuity of \( g \) and taking the limit we get that \( \rho(L) \subset L \), i.e., \( L \) is \( \rho(G) \)-invariant.

We note that \( T(E_0) = 0 \) for any \( T \in V_0 \) and therefore \( T(E_0) = 0 \) for all \( T \in V \).

From here \( Q_0(E_0) = 0 \) and \( p_1(E_0) = 0 \). Then

\[ p_2(E_0) = (p_1 + Q_0) E_0 = p_1 E_0 + Q_0 E_0 = 0 \]

Thus

\[ p_2 E = p_2(E_0 + E_1) = p_2 E_0 + p_2 E_1 = p_2 E_1. \]

From here we get that \( L = \overline{p_2(E_1)} \). For every \( x \in E \) we have

\[ x = p_0 x + p_1 x = p_0 x + (p_2 - Q_0) x = (p_0 - Q_0) x + p_2 x. \]

Taking into account \( (p_0 - Q_0) x \in E_0 \) and \( p_2 x \in L \), we get that \( E = E_0 + L \). If \( z \in E_0 \cap p_2 E_1 \), then \( z = p_2 y (y \in E_1) \) and \( z \in E_0 \), in particular \( p_1 z = 0 \). Then

\[ 0 = p_1 (p_2 y) = p_1(p_1 + Q_0) y = (p_1^2 + p_1 Q_0) y = (p_1 + p_1 Q_0) y = p_1 y + p_1 Q_0 y = p_1 y = y. \]

From here we get \( y = 0 \) and \( z = 0 \). Hence \( E_0 \cap p_2 E_1 = \{0\} \).

Now we assume that \( z \in E_0 \cap L \), then there exists \( z_n \in p_2 E = p_2 E_1 \) such that \( z_n \overset{\text{st}}{\to} z \). Since \( z \in E_0 \), then \( p_0 z = z \), \( p_1 z = 0 \). Hence \( p_1 z_n \overset{\text{st}}{\to} p_1 z = 0 \). Next \( z_n = p_2 y_n \), where \( y_n \in E_1 \). Hence \( 0 \overset{\text{st}}{\to} p_1 z_n = p_1(p_2 y_n) = p_1 y_n = y_n \), i.e., \( y_n \overset{\text{st}}{\to} 0 \).
From here we get \( z \mapsto z_n = p_2 y_n = p_1 y_n \mapsto 0 \), and therefore \( z = 0 \).

It means that \( E_0 \cap L = \{0\} \).

Thus \( E = E_0 \oplus L \) i.e., it is topological direct sum of \( \rho(G) \)-invariant subspaces \( E_0 \) and \( L \).

(b) \( \Rightarrow \) (a). We assume that \( 0 \neq f \in E' \), \( f \) is \( \rho(G) \)-invariant continuous linear functional, \( V = \ker f \).

Subspace \( V = \ker f \) is obviously \( \rho(G) \)-invariant. Then from the condition (t7) it follows that there exists closed \( \rho(G) \) invariant subspace \( H \subset E \) such that \( E = V \oplus H \).

Since \( \text{codim } V = 1 \), then \( \dim H = 1 \), i.e., \( H = \{ \lambda v \}_{1 \in C}, v \neq 0 \) and \( f(v) \neq 0 \).

Let \( g v = \lambda_1 v \). We have that \( 0 \neq f(v) = f(g v) = f(\lambda_1 v) = \lambda_1 f(v) \). Hence \( \lambda_1 = 1 \), and therefore \( g v = v \) for every \( g \in G \). Thus \( G \) is \( t\beta \)-group. Theorem 2 is proved.

Corollary. Every \( t\beta \)-group is a \( t\alpha \)-group.

The proof follows from the Proposition and Theorem 2.

Theorem 3. For the group \( G \) the following conditions are equivalent:

(a) \( G \) is a \( ft\alpha \)-group,
(b) \( G \) is a \( ft\beta \)-group,
(c) \( G \) is a \( ft\gamma \)-group.

Proof. (a) \( \Leftrightarrow \) (c) is analogous to the proof of the proposal 2.2.4 (see [3] p.27). That (b) \( \Leftrightarrow \) (c) can be proved in the same way as in Theorem 2.

References