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A New Approach to Solve a Nonlinear Wave Equation

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Abstract. We use the Adomian decomposition method to study a nonlinear wave equation. The decomposition method plays an important role in a wide class of differential and integral equations, stochastic and deterministic problems (linear or nonlinear) in Mathematics and Physics. The method provides solution without linearization, perturbation, or unjustified assumptions. An analytic solution of a nonlinear wave equation in the form of a series with easily computable components using the decomposition method will be determined. The nonhomogeneous equation is effectively solved by employing the phenomena of the self-canceling "noise" terms. The phenomena is useful in demonstrating a fast convergence of the exact solution.

1. Introduction

In this article we will concentrate on the goal of obtaining an analytic solution of a nonlinear wave equation in the form of a series by utilizing the Adomian decomposition method [1-4]. The method is a series solution technique that tackles any mathematical and physical problems directly and is relatively easy to obtain an accurate and rapidly convergent series solution. It is based on the Taylor Series, except that Adomian decomposition method expands the solution about a function, instead of a point. The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques and does not require unjustified assumptions, linearization, or perturbation.

The paper is organized as follows. In the remainder of this Section we briefly discuss the nonlinear wave equation (1.1). In Section 2 we give outline of the proposed method. In the final Section 3 we utilize two homogeneous and one nonhomogeneous nonlinear examples to illustrate the characteristic properties of the decomposition method.

We consider the nonlinear wave equation of the form

$$F(x, t, u, u_x, u_t) = u_t + u_x^2 = \phi(x, t), \tag{1.1}$$

with the initial condition

$$u(x, 0) = f(x).$$

The equation (1.1) is of the general form $u_t + c(u, u_x)u_x = 0$, of which two special cases $c = constant$ and $c = u$ were considered in [6-9,11]. The nonlinearity of the x-derivative term in (1.1) implies, as we demonstrate, that the solution $u = u(x,t)$ is not constant along the characteristic curves. However, the velocity of specific points x on the wave is related to the nature of the wave form at those points as was the case for the quasilinear equation which is considered in [7,9].

2. Outline of the method

To apply the decomposition method, we write equation (1.1) in an operator form

$$L_t(u(x,t)) = \phi(x,t) - Nu \tag{2.1}$$

with $Nu = (u_x)^2$ and where the differential operator L_t is given by

$$L_t = \frac{\partial}{\partial t}.$$

It is clear that L_t is invertible and L_t^{-1} is the one-fold integration defined by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot)dt.$$

Applying the inverse operator L_t^{-1} to both sides of (2.1) yields

$$L_t^{-1}L_t u(x,t) = L_t^{-1}(\phi(x,t)) - L_t^{-1}(Nu)$$

from which it follows that

$$u(x,t) = f(x) + L_t^{-1}(\phi(x,t)) - L_t^{-1}(Nu)\,. \tag{2.2}$$

The decomposition method [4] consists of decomposing the unknown function $u(x,t)$ into a sum of components defined by the decomposition series

$$u(x,t) = u_0(x,t) + u_1(x,t) + \cdots + u_n(x,t) + \cdots = \sum_{n=0}^{\infty} u_n(x,t)\,, \tag{2.3}$$

and the nonlinear term $Nu(u_x)^2$ be expressed in the A_n Adomian's polynomials; thus $Nu = \sum_{n=0}^{\infty} A_n$ where the A_n polynomials are formally generated by specific formulas

[1-3]. In the following, we outline the framework to generate these polynomials, where it was defined that

$$\begin{aligned} A_0 &= (u_{0_x})^2 \\ A_1 &= 2u_{0_x} u_{1_x} \\ A_2 &= u_{1_x}^2 + 2u_{0_x} u_{2_x} \\ A_3 &= 2u_{1_x} u_{2_x} + 2u_{0_x} u_{3_x} \\ &\vdots \end{aligned} \tag{2.4}$$

It is important to note that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2 etc.

To solve (2.2), we substitute (2.3) and (2.4) into (2.2) to obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L_t^{-1}(\phi(x,t)) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right). \tag{2.5}$$

The components $u_0, u_1, u_2, \cdots,$ of $u(x,t)$ in (2.3) are defined in a recurrence relationship

$$\begin{aligned} u_0(x,t) &= f(x) + L_t^{-1}(\phi(x,t)), \\ u_{n+1}(x,t) &= -L_t^{-1}(A_n(x,t)), \quad n \geq 0. \end{aligned} \tag{2.6}$$

The scheme (2.6) is obviously equivalent to

$$\begin{aligned} u_0(x,t) &= f(x) + L_t^{-1}(\phi(x,t)), \\ u_1(x,t) &= -L_t^{-1}(A_0(x,t)), \\ u_2(x,t) &= -L_t^{-1}(A_1(x,t)), \\ &\vdots \end{aligned} \tag{2.7}$$

As a result of (2.7), the terms of $u_0, u_1, u_2, \cdots,$ are easily calculated. With these components evaluated, the solution $u(x,t)$ of (2.1) follows immediately in a decomposition series form upon using (2.3). However, as mentioned in an earlier section, the series form obtained for $u(x,t)$ mostly yields the exact solution in a closed form as will be seen in later section. It is formally justified by [1-3] that few components of the series usually evaluate the higher accuracy level of approximation.

Adomian and Rach [5] and Wazwaz [10] have investigated the phenomena of the self-canceling “noise” terms whose sum vanishes in the limit. An important observation was made that “noise” terms appear for nonhomogenous cases only. Further, it was formally justified that if terms in u_0 are canceled by terms in u_1, even though u_1

includes further terms, then the remaining non canceled terms in u_0 may constitute the exact solution of the equation.

In the next section, we describe some examples that illustrate the above outlined framework of the decomposition method and the noise terms phenomenon.

3. Examples

Example 1. We consider the nonlinear differential equation discussed in [11]. The problem is given by

$$u_t + u_x^2 = 0\,, \quad u(x,\,0) = -x^2. \tag{3.1}$$

Following the scheme (2.7) gives

$$u_0 = -x^2, \tag{3.2}$$

and

$$u_1 = -L_t^{-1}(A_0) = -L_t^{-1}\left[(u_{0_x})^2\right] = -4x^2t, \tag{3.3}$$

$$u_2 = -L_t^{-1}(A_1) = -L_t^{-1}\left[2u_{0_x}u_{1_x}\right] = -16x^2t^2, \tag{3.4}$$

$$u_3 = -L_t^{-1}(A_2) = -L_t^{-1}\left[(u_{1x})^2 + 2u_{0_x}u_{2_x}\right] = -64x^2t^3, \tag{3.5}$$

and so on for other components.

Substituting (3.2)-(3.5) into (2.5), the solution $u(x,t)$ of (3.1) in a series form

$$u(x,t) = -x^2 - 4x^2t - 16x^2t^2 - 64x^2t^3 - \cdots \tag{3.6}$$

follows immediately. After some tedious algebra factoring, (3.6) can be rewritten as

$$u(x,t) = x^2\left[-1 - 4t - (4t)^2 - (4t)^3 - \cdots\right]. \tag{3.7}$$

It can be easily observed that (3.7) is equivalent to the exact solution

$$u(x,t) = \frac{x^2}{4t-1}\,.$$

This can be verified through substitution.

Example 2. Next we will solve the equation of the form

$$u_t + u_x^2 = 0\,, \quad u(x,\,0) = ax$$

Using (2.6) to determine the individual terms of the decomposition, we find

$$u_0 = ax,$$

and

$$u_1 = -L_t^{-1}(A_0) = -L_t^{-1}\left[(u_{0_x})^2\right] = -a^2 t. \tag{3.8}$$

As a result of (3.8), $u_k = 0$, for $k \geq 2$. However, the exact solution

$$u(x,t) = ax - a^2 t\,,$$

is obtained by adding u_0 and u_1. This result can be verified through substitution.

Example 3. Finally to illustrate the technique discussed above for the nonhomogenous equation, we consider an equation of the form

$$u_t + u_x^2 = 1 + \cosh^2 x, \quad u(x,\,0) = \sinh x \tag{3.9}$$

Using (2.6) to determine the individual terms of the decomposition, we find

$$u_0 = \sinh x + t + t\cosh^2 x\,, \tag{3.10}$$

and

$$\begin{aligned} u_1 &= -L_t^{-1}(A_0) = -L_t^{-1}\left[(u_{0_x})^2\right] \\ &= -t\ \cosh^2 x - 2t^2 \cosh^2 x\ \sinh x - \frac{4}{3} t^3 \cosh^2 x\ \sinh^2 x, \end{aligned} \tag{3.11}$$

and so on for other components. By canceling the noise terms $t\cosh^2 x$ and $-\,t\cosh^2 x$ in (3.10) and (3.11), the remaining non canceled terms in u_0 provides the exact solution of (3.9) that will provide the exact solution given by

$$u(x,t) = t + sh\,x\,.$$

In closing, the Adomian decomposition methodology is very powerful and efficient in finding exact solutions for wide classes of problems. The convergence can be made faster if the noise terms appear for the nonhomogeneous case as discussed in [5,10]. The method avoids the difficulties and massive computational work. These schemes showed superiority over existing techniques, particularly by determining the analytic solution and by minimizing the size of computations and work.

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