Spacelike Maximal Surfaces with Constant Scalar Normal Curvature in a Normal Contact Lorentzian Manifold

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Abstract. If the scalar normal curvature of a spacelike maximal surface in a 5-dimensional normal contact Lorentzian manifold with constant φ-sectional curvature is constant, then the surface is totally geodesic or nonpositively curved.

1. Introduction

On some odd dimensional manifolds, the normal contact Riemannian metric structure (or Sasakian structure) can be defined. The study of manifolds with this structure has a long history.

If we change the Riemannian metric of the Sasakian structure to a Lorentzian one, we can define the normal contact Lorentzian structure. This definition was given at the starting time of the study of the Sasakian structure. But practical study of it has not been given sufficiently yet (cf. [5], [7]). In [3], [4], we study the fundamental properties of manifolds with the normal contact Lorentzian structure.

In this paper, we shall study the scalar normal curvature for spacelike maximal surfaces in a 5-dimensional normal contact Lorentzian manifold of constant φ-sectional curvature and prove.

Theorem. Let $M^5$ be a 5-dimensional normal contact Lorentzian manifold with constant φ-sectional curvature $k$ and $M^2$ a spacelike maximal surface with vector field $\xi$ normal to $M^2$. Assume that the scalar normal curvature $K_N$ of $M^2$ in $M^5$ is constant. Then $M^2$ is totally geodesic with Gauss curvature $K = \frac{k-\xi^2}{4}$ or a nonpositive curved surface.

2. Spacelike Submanifold

Let $\overline{M}$ be a $(2n+1)$-dimensional $(n \geq 2)$ manifold. The normal contact Lorentzian structure $(\phi, \xi, \eta, g)$ of $\overline{M}$ is given by a (1,1)-type skew-symmetric tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Lorentzian metric $g$ as
where $X$ is a vector field of $\bar{M}$ and $\nabla$ is the covariant derivative with respect to $g$ ([3], [4]). When the curvature tensor field of $K(X,Y)Z$ of $\bar{M}$ has the following form

\begin{equation}
4K(X,Y)Z = (k - 3)(g(Y,Z)X - g(X,Z)Y)
+ (k + 1)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(X,Z)\eta(Y)\xi)
- g(Y,Z)\eta(X)\xi + g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y - 2g(\phi X,Y)\phi Z,
\end{equation}

(2.2)

$\bar{M}$ is called a space of constant $\phi$-sectional curvature $k$.

Let $M$ be an $n$-dimensional submanifold of $\bar{M}$. By $\nabla$ we denote the covariant derivative of $M$ determined by the induced metric on $M$. Let $\mathfrak{X}(\bar{M})$ (resp. $\mathfrak{X}(M)$) be the Lie algebra of vector fields on $\bar{M}$ (resp. $M$) and $\mathfrak{X}^\perp(M)$ the set of all vector fields normal to $M$.

The Gauss-Weingarten formulas are given by

\begin{equation}
\nabla_X Y = \nabla_X Y + B(X,Y), \quad \nabla_X N = -A^N(X) + D_X N,
\end{equation}

(2.3)

$X, Y \in \mathfrak{X}(M), \quad N \in \mathfrak{X}^\perp(M),$

where $D$ is the normal connection [6]. $B$ is called the second fundamental form tensor and $A$ the shape operator, and they satisfy

\begin{equation}
g(A^N(X), Y) = g(B(X,Y), N).
\end{equation}

(2.4)

If the induced metric on $M$ is positive-definite, then $M$ is called a spacelike submanifold.

Let $M$ be a spacelike submanifold of $\bar{M}$ with the vector field $\xi$ normal to $M$, then from (2.1), (2.3) and (2.4), $M$ satisfies following properties.

**Proposition.** Let $M$ be an $m$-dimensional spacelike submanifold in a normal contact Lorentzian manifold $\bar{M}^{2n+1}$ with structure $(\phi, \xi, \eta, g)$. Then

(i) The dimension $m$ of $M$ satisfies $m \leq n$.
(ii) The shape operator of $\xi$ direction is identically zero.
(iii) If $X \in \mathfrak{X}(M)$ then $\phi X \in \mathfrak{X}^\perp(M)$.
(iv) If $m = n$, then $A^{\phi Y}(Y) = A^{\phi X}(X)$, for $X, Y \in \mathfrak{X}(M)$. 

\begin{align*}
\phi^2 X &= -X + \eta(X)\xi, \quad \phi \xi = 0, \\
g(\xi, \xi) &= -1, \quad \eta(X) = -g(X, \xi), \\
\left(\nabla_X \eta\right)Y &= g(\phi X, Y), \quad \nabla_X \xi = \phi X, \\
\left(\nabla_X \phi\right)Y &= -\eta(Y)X - g(X,Y)\xi.
\end{align*}

(2.1)
3. Local Formulas

We consider a spacelike surface $M^2$ in a 5-dimensional normal contact Lorentzian manifold $\overline{M}$. Let $\{e_1, e_2, e_3, e_4, \xi\}$ be an orthonormal frame field on $\overline{M}$ so that

$$e_1, e_2 \in X(M), \quad e_1^* = e_3, \quad e_2^* = e_4.$$

We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, \cdots \leq 5, \quad 1 \leq i, j, \cdots \leq 2, \quad 3 \leq i^*, j^*, \cdots \leq 4, \quad 3 \leq \alpha, \beta, \cdots \leq 5.$$

Let $\{w^1, w^2, w^3, w^4, w^5\}$ be the field of dual frames. Then the structure equations of $\overline{M}$ are given by

$$dw^A = -\sum e_B w^A_B \wedge w_B^A, \quad w_A^A + w_A^B = 0,$$

$$dw_B^A = -\sum e_C w_B^C \wedge w_B^D + \Phi_B^A,$$

$$\Phi_B^A = \frac{1}{2} e_C e_D \sum K_{BCD}^A w^C \wedge w^D,$$

$$K_{BCD}^A + K_{BCD}^A = 0.$$

Restricting these forms to $M^2$, we have $w^a = 0$. Since $0 = dw^a = -\sum w^a_i \wedge w^i$, by Cartan’s lemma we may write

$$w^a_i = \sum h^a_i \wedge w^i, \quad h^a_i = h^a_{ji}.$$

From these formulas we obtain

$$dw^i = -\sum w^j_i \wedge w^j, \quad w^j_i + w^i_j = 0,$$

$$\omega^i_j = \frac{1}{2} \sum R^i_{jk} \wedge w^j,$$

$$R^i_{jk} = K^i_{jk} + \sum e_\alpha \left( h^i_\alpha h^j_\beta - h^j_\alpha h^i_\beta \right),$$

$$dw^\alpha = -\sum \varepsilon_j w^\alpha_j \wedge w^i + \Omega^\alpha_i,$$

$$\Omega^\alpha_i = \frac{1}{2} \sum R^\alpha_{jk} w^k \wedge w^i,$$

$$R^\alpha_{jk} = K^\alpha_{jk} + \sum \left( h^\alpha_k h^\beta_j - h^\beta_k h^\alpha_j \right).$$

An immersion is said to be maximal if $\sum h^a_i = 0$ for all $\alpha$. 
We define $h_{ij}^a$ and $h_{ijkl}^a$ by
\[ \sum h_{ij}^a w^k = dh_{ij}^a - \sum h_{ij}^a w_j^i - \sum h_{ij}^a w_i^j + \sum \varepsilon_i h_{ij}^b w_{ij}^k, \quad (3.1) \]
\[ \sum h_{ijkl}^a w^{ie} = dh_{ijkl}^a - \sum h_{ijkl}^a w_{ij}^k - \sum h_{ijkl}^a w_{ik}^j - \sum h_{ijkl}^a w_{ij}^k + \sum \varepsilon_i h_{ijkl}^b w_{ijkl}^a. \]

The Laplacian $\Delta h_{ij}^a$ is given by
\[ \Delta h_{ij}^a = \sum h_{ijkl}^a. \]

When $M^2$ is maximal in $\overline{M}^5$, that is $\sum h_{ik}^a = 0$ for all $a$, $\Delta h_{ij}^a$ can be written as
\[ \Delta h_{ij}^a = \sum \left( 2K_{ijkl}^a h_{kj}^b - K_{ijkl}^b h_{kj}^i + 2K_{ijkl}^a h_{kl}^b \right) + \sum \left( K_{mkij}^a h_{mj}^b + K_{mkij}^b h_{mj}^a + 2K_{ijkl}^a h_{mk}^b \right) + \sum \left( 2h_{iak}^a h_{kj}^b - h_{ik}^a h_{km}^b h_{kj}^i - h_{ik}^a h_{kim}^b h_{kj}^i - h_{ik}^a h_{kim}^b h_{kj}^i \right) \quad (3.2) \]

The scalar normal curvature $K_N$ of $M^2$ is defined by
\[ K_N = \sum \varepsilon_{i} \varepsilon_{j} S_{bij}^a S_{bij}^a, \quad S_{bij}^a = \sum \left( h_{ik}^a h_{jk}^b - h_{jk}^a h_{ik}^b \right). \]

The covariant derivative of $S_{bij}^a$ is defined by
\[ S_{bij}^a = \sum \left( h_{ik}^a h_{jk}^b + h_{ik}^b h_{jk}^a - h_{ik}^a h_{jk}^b + h_{ij}^a h_{jk}^b \right). \]

Then for the Laplacian of $K_N$, we have the following formula
\[ \frac{1}{2} \Delta K_N = \sum \varepsilon_{i} \varepsilon_{j} \left( S_{bij}^a \right)^2 + \sum S_{bij}^a \left( h_{ik}^a h_{jk}^b - h_{ik}^b h_{jk}^a \right) \quad + \sum 4 \varepsilon_{i} \varepsilon_{j} S_{bij}^a \left( \Delta h_{ij}^a \right) h_{ij}^b. \]

4. Proof of Theorem

Let $M^2$ be a spacelike maximal surface in a normal contact Lorentzian manifold $\overline{M}^5$ of constant $\phi$-sectional curvature $k$. Then, from Proposition, we obtain
\[ (h_i^a) = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad (h_j^a) = \begin{bmatrix} 0 & -a \\ -a & 0 \end{bmatrix}, \quad (h_k^a) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (4.1) \]
From (3.2), it follows that

\[
\begin{align*}
\Delta h_{11}^v &= \frac{3k-5}{4}a - 6a^3, & \Delta h_{11}^v &= \Delta h_{22}^v = 0, \\
\Delta h_{12}^v &= \frac{3k-5}{4}a + 6a^3, & \Delta h_{12}^v &= \Delta h_{22}^v = 0, \\
\Delta h_{12}^v &= \Delta h_{21}^v = -\frac{3k-5}{4}a + 6a^3, \\
\Delta h_{11}^v &= \Delta h_{12}^v = \Delta h_{21}^v = \Delta h_{22}^v = 0,
\end{align*}
\]

by virtue of (2.2).

From (3.1), we have

\[
\begin{align*}
h_{111}^v &= -h_{121}^v = -h_{212}^v = -h_{222}^v = -a, \\
h_{112}^v &= -h_{122}^v = -h_{212}^v = h_{222}^v = a, \\
\end{align*}
\]

Since

\[
S_{111}^v = S_{122}^v = 0, \quad S_{112}^v = -S_{121}^v = 2a^2,
\]

by virtue of (4.1), we obtain

\[
\sum (S_{ijk}^v)^2 = \sum (S_{ijk}^v)^2 + 2a \sum (S_{ijk}^v)^2 = 4 \left( (2a^2)_{11}^2 + (2a^2)_{12}^2 \right) - 32a^4,
\]

\[
\sum S_{ijk}^v (h_{jkl}^v h_{jkl}^v - h_{jkl}^v h_{jkl}^v) = 2 \left( (2a^2)_{11}^2 + (2a^2)_{12}^2 \right),
\]

\[
\sum S_{ijk}^v (\Delta h_{jkl}^v) h_{jkl}^v = 8a^4 \left( \frac{3k-5}{4} - 6a^2 \right),
\]

so that

\[
\frac{1}{2} \Delta K_N = 8 \left( (2a^2)_{11}^2 + (2a^2)_{12}^2 \right) + 32a^4 \left( \frac{3k-9}{4} - 6a^2 \right).
\]
If we assume $K_N = \text{constant}$, then since $a$ is continuous, this equation reduces to $a = 0$ or $a^2 \geq \frac{k-1}{8}$ everywhere. On the other hand, the Gauss curvature $K$ of $M^2$ is given by $K = \frac{k-1}{4} - 2a^2$. Hence if $a^2 \geq \frac{k-1}{8}$ then $K \leq 0$. This completes the proof.

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**References**


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