Notes on periodic solutions for a nonlinear discrete system involving the $p$-Laplacian

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Abstract In this paper, we first improve two inequalities, then by using critical point theory, improve an existence theorem of periodic solutions for a nonlinear discrete system involving the $p$-Laplacian, and present some estimates of periodic solutions.

Keywords $p$-Laplacian systems; Periodic solutions; Critical point; Sobolev’s inequality; Wirtinger’s inequality.

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1 Introduction and Main results

Let $\mathbb{R}$ denote the real number, $\mathbb{Z}$ the integers. Given $a < b$ in $\mathbb{Z}$, let $\mathbb{Z}[a, b] = \{a, a + 1, \cdots, b\}$. Let $T > 1$ and $N$ be fixed positive integers.

Consider the following nonlinear discrete system involving the $p$-Laplacian

$$\Delta[\Phi_p(\Delta x(t - 1))] + \nabla F(t, x(t)) = 0, \quad t \in \mathbb{Z},$$

(1.1)

where $p > 1, q > 1, 1/p + 1/q = 1$, $\Phi_p(u) = |u|^{p-2}u = \left(\sqrt{\sum_{i=1}^{N} u_i^2}\right)^{p-2} (u_1, u_2, \cdots, u_N)^\tau$, $u \in \mathbb{R}^N$, $\cdot^\tau$ stands for the transpose of a vector or a matrix, $F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$, $(t, x) \to F(t, x)$ is $T$–periodic in $t$ for all $x \in \mathbb{R}^N$ and continuously differentiable and convex in $x$ for every $t \in \mathbb{Z}$, $\nabla F(t, x)$ denotes the gradient of $F(t, x)$ in $x$, and $\Delta x(t) = x(t + 1) - x(t)$, $\Delta^2 x(t) = \Delta(\Delta x(t))$.

When $p = 2$, problem (1.1) becomes the second order discrete nonlinear system. By using the variational methods, some existence results for periodic solutions are obtained, such as [1], [5], [6], [9], [10] and [11]. When $p > 1$, recently, there are also some results,
see [2], [3], [4] and [7]. Especially, in [7], by using the dual least principle, the authors obtained the following result:

**Theorem 1.1.** Suppose $F$ satisfies the following conditions:

(A1) there exists $\beta : \mathbb{Z} \to \mathbb{R}^N$ such that for all $(t, y) \in \mathbb{Z} \times \mathbb{R}^N$,

$$F(t, y) \geq \left( \beta(t), |y|^\frac{p-2}{2} y \right) \quad \text{and} \quad \beta(t + T) = \beta(t);$$

(A2) there are constants $\alpha \in (0, T^{-1})$, and $\gamma : \mathbb{Z} \to \mathbb{R}$ such that for all $(t, y) \in \mathbb{Z} \times \mathbb{R}^N$,

$$F(t, y) \leq \alpha |y|^p + \gamma(t) \quad \text{and} \quad \gamma(t + T) = \gamma(t);$$

(A3) $\sum_{t=1}^{T} F(t, y) \to +\infty$, as $|y| \to \infty$, $y \in \mathbb{R}^N$.

Then, system (1.1) has at least one $T$-periodic solution.

In our paper, we will improve two discrete inequalities in [7] and [12]. Furthermore, we improve the condition (A2) and also obtain some estimates of periodic solution for system (1.1).

2 Preliminaries

In the following, we use $| \cdot |$ to denote the Euclidean norm in $\mathbb{R}^N$. Let

$$S = \{ u = (u_1, u_2)^\tau = \{ u(t) \} | u(t) = (u_1(t), u_2(t))^\tau \in \mathbb{R}^{2N},$$

$$u_i = \{ u_i(t) \}, u_i(t) \in \mathbb{R}^{N}, i = 1, 2, t \in \mathbb{Z} \}. $$

$E$ is defined as a subspace of $S$ by

$$E = \{ u = \{ u(t) \} \in S | u(t + T) = u(t), t \in \mathbb{Z} \}. $$

For $u = (u_1, u_2)^\tau \in E$, set

$$\| u_i \|_r = \left( \sum_{t=1}^{T} |u_i(t)|^r \right)^{1/r},$$

where $i = 1, 2, r > 1$. Then $E$ can be equipped with the norm as follows:

$$\| u \| = \| u_1 \|_p + \| u_2 \|_q$$

for $u = (u_1, u_2)^\tau \in E$. It is obvious that $E$ is a reflexive Banach space with dimension $2NT$, which can be identified with $\mathbb{R}^{2NT}$. Let

$$W = \left\{ u = (u_1, u_2)^\tau \in E | u_i(1) = u_i(2) = \cdots = u_i(T) = \frac{1}{T} \sum_{t=1}^{T} u_i(t), i = 1, 2 \right\}. $$
and

\[ Y = \left\{ u = (u_1, u_2) \in E \left| \sum_{i=1}^{T} u_i(t) = 0, i = 1, 2 \right. \right\}. \]

Then \( E \) can be decomposed into the direct sum \( E = Y \oplus W. \) So, for any \( u \in E, \) \( u \) can be expressed in the form \( u = \tilde{u} + \bar{u}, \) where \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2)^T \in Y \) and \( \bar{u} = (\bar{u}_1, \bar{u}_2)^T \in W. \)

Obviously, \( u_i = \tilde{u}_i + \bar{u}_i, i = 1, 2. \)

For \( u = (u_1, u_2)^T \in Y, \) let

\[ \| \Delta u_i \|_r = \left( \sum_{i=1}^{T} |\Delta u_i(t)|^r \right)^{1/r}, \]

where \( i = 1, 2, r > 1. \) It is easy to verify that

\[ \| \Delta u \| = \| \Delta u_1 \|_q + \| \Delta u_2 \|_p \]

is also a norm on \( Y. \) Since \( Y \) is finite-dimensional, the norm \( \| \Delta u \| \) is equivalent to the norm \( \| u \| \) in \( E \) if \( u \in Y. \)

\( \Gamma_0(\mathbb{R}^N) \) denotes the set of all convex lower semi-continuous (l.s.c.) functions \( F : \mathbb{R}^N \to (-\infty, +\infty) \) whose effective domain \( D(F) = \{ u \in \mathbb{R}^N : F(u) < +\infty \} \) is nonempty. Let \( H : \mathbb{Z} \times \mathbb{R}^{2N} \to \mathbb{R}, (t, u) \to H(t, u) \) be a smooth Hamiltonian such that for each \( t \in \mathbb{Z}[1, T], \)

\( H(t, \cdot) \in \Gamma_0(\mathbb{R}^{2N}) \) is strictly convex and \( H(t, u)/|u| \to +\infty, \) if \( |u| \to \infty. \) The Fenchel transform \( H^*(t, \cdot) \) of \( H(t, \cdot) \) is defined by

\[ H^*(t, v) = \sup_{u \in \mathbb{R}^N} \{ (v, u) - H(t, u) \} \tag{2.1} \]

or

\[ \begin{cases} H^*(t, v) = (v, u) - H(t, u), \\ v = \nabla H(t, u), \quad \text{or} \quad u = \nabla H^*(t, v). \tag{2.2} \end{cases} \]

If for \( u = (u_1, u_2), u_1, u_2 \in \mathbb{R}^N, \) \( H(t, u) \) can be split into parts \( H(t, u) = H_1(t, u_1) + H_2(t, u_2), \) then by (2.2), \( H^*(t, v) = H_1^*(t, v_1) + H_2^*(t, v_2), v = (v_1, v_2), v_1, v_2 \in \mathbb{R}^N. \) We denote by \( J \) the symplectic matrix. Then \( J^2 = -I \) and \( (Ju, v) = -(u, Jv) \) for all \( u, v \in \mathbb{R}^{2N}. \) It is clear that \( (J\dot{v}, v) = (\dot{v}_2, v_1) - (\dot{v}_1, v_2), \) where \( v = (v_1, v_2)^T \in \mathbb{R}^N \times \mathbb{R}^N, i = 1, 2. \)

Let \( u_1(t) = x(t), u_2(t) = \alpha^{-1} \Phi_p(\Delta u_1(t - 1)), t \in \mathbb{Z}. \) Then problem (1.1) is equivalent to the non-autonomous system

\[ \begin{cases} \Delta u_2(t) + \alpha^{-1} \nabla F(t, u_1(t)) = 0, & t \in \mathbb{Z}, \\ -\Delta u_1(t - 1) + \alpha^{q-1} \Phi_q(u_2(t)) = 0, \end{cases} \tag{2.3} \]

that is

\[ J\Delta u(t) + \nabla H(t, Lu(t)) = 0, \quad t \in \mathbb{Z}, \tag{2.4} \]
where \( Lu(t) = (u_1(t), u_2(t + 1))^\tau, \) \( L^{-1}u(t) = (u_1(t), u_2(t - 1))^\tau, \) \( u = (u_1, u_2)^\tau, H(t, u) = H_1(t, u_1) + H_2(t, u_2) \) and

\[
H_1(t, u_1) = \frac{1}{\alpha} F(t, u_1), \quad H_2(t, u_2) = \frac{\alpha^{q-1}}{q} |u_2|^q. \tag{2.5}
\]

The dual action is defined on \( E \) by

\[
I(v) = \frac{1}{2} \sum_{t=1}^{T} (L(J \Delta v(t - 1)), v(t)) + \sum_{t=1}^{T} \left[ H_1^*(t, \Delta v_1(t)) + H_2^*(t, \Delta v_2(t)) \right],
\]

where \( v = (v_1, v_2)^\tau \in E. \) Since \( I(v) = I(\bar{v} + \bar{v}) = I(\bar{v}) \) for \( v = \bar{v} + \bar{v} \in E, \) in order to find the \( T \)-periodic solution of (1.1), it suffices to find the critical point of \( I \) on subspace \( Y \) of \( E. \) The above knowledge and statement come from [7], [8] and [11].

**Lemma 2.1.** Let \( u = (u_1, u_2) \in Y. \) Then

\[
\max_{t \in \mathbb{Z}[1,T]} |u_i(t)| \leq \min \left\{ \frac{(T - 1)^{(p+1)/p}}{T}, \left( \frac{(T + 1)^{p+1} - 2}{T^p(p+1)} \right)^{1/p}, \left( \sum_{s=1}^{T} |\Delta u_i(s)|^q \right)^{1/q}, \right. \quad i = 1, 2, \tag{2.6}
\]

\[
\max_{t \in \mathbb{Z}[1,T]} |u_i(t)| \leq \min \left\{ \frac{(T - 1)^{(q+1)/q}}{T}, \left( \frac{(T + 1)^{q+1} - 2}{T^q(q+1)} \right)^{1/q}, \left( \sum_{s=1}^{T} |\Delta u_i(s)|^p \right)^{1/p}, \right. \quad i = 1, 2, \tag{2.7}
\]

and

\[
\sum_{t=1}^{T} |u_i(t)|^q \leq \min \left\{ \frac{(T - 1)^{2q-1}}{T^{q-1}}, \frac{T^{q-1}\Theta(p, q)}{(p+1)^q/p} \right\} \sum_{s=1}^{T} |\Delta u_i(s)|^q, \quad i = 1, 2, \tag{2.8}
\]

\[
\sum_{t=1}^{T} |u_i(t)|^p \leq \min \left\{ \frac{(T - 1)^{2p-1}}{T^{p-1}}, \frac{T^{p-1}\Theta(q, p)}{(q+1)^p/q} \right\} \sum_{s=1}^{T} |\Delta u_i(s)|^p, \quad i = 1, 2. \tag{2.9}
\]

where

\[
\Theta(p, q) = \sum_{t=1}^{T} \left[ \left( \frac{t}{T} \right)^{p+1} + \left( 1 - \frac{t}{T} + \frac{1}{T} \right)^{p+1} - \frac{2}{T^{p+1}} \right]^{p/q},
\]

\[
\Theta(q, p) = \sum_{t=1}^{T} \left[ \left( \frac{t}{T} \right)^{q+1} + \left( 1 - \frac{t}{T} + \frac{1}{T} \right)^{q+1} - \frac{2}{T^{q+1}} \right]^{q/p}.
\]

**Proof.** Fix \( t \in \mathbb{Z}[1,T]. \) For every \( \tau \in \mathbb{Z}[1, t-1], \) we have

\[
u_1(t) = u_1(\tau) + \sum_{s=\tau}^{t-1} \Delta u_1(s) \tag{2.10}
\]

and for every \( \tau \in \mathbb{Z}[t,T], \)

\[
u_1(t) = u_1(\tau) - \sum_{s=t}^{\tau-1} \Delta u_1(s). \tag{2.11}
\]
Summing \((2.10)\) over \(\mathbb{Z}[1, t - 1]\) and \((2.11)\) over \(\mathbb{Z}[t, T]\), we have
\[
(t - 1)u_1(t) = \sum_{\tau=1}^{t-1} u_1(\tau) + \sum_{\tau=1}^{t-1} \sum_{s=\tau}^{t-1} \Delta u_1(s) = \sum_{\tau=1}^{t-1} u_1(\tau) + \sum_{s=1}^{t-1} s \Delta u_1(s) \tag{2.12}
\]
and
\[
(T - t + 1)u_1(t) = \sum_{\tau=t}^{T} u_1(\tau) - \sum_{\tau=t}^{T-1} \sum_{s=t}^{\tau-1} \Delta u_1(s) = \sum_{\tau=t}^{T-1} u_1(\tau) - \sum_{s=t}^{T-1} (T - s) \Delta u_1(s). \tag{2.13}
\]
Set
\[
\phi(s) = \begin{cases} 
s, & 1 \leq s \leq t - 1, \\
T - s, & t \leq s \leq T.
\end{cases}
\]
Since \(\sum_{\tau=1}^{T} u_1(\tau) = 0\), combining \((2.12)\) with \((2.13)\) and using the Hölder inequality, we obtain
\[
|T|u_1(t)| = \left| \sum_{s=1}^{t-1} s \Delta u_1(s) - \sum_{s=t}^{T-1} (T - s) \Delta u_1(s) \right|
\leq \sum_{s=1}^{t-1} s |\Delta u_1(s)| + \sum_{s=t}^{T-1} (T - s)|\Delta u_1(s)|
= \sum_{s=1}^{T} \phi(s)|\Delta u_1(s)|
= \sum_{s=1}^{T} \phi(s)|\Delta u_1(s)|
\leq \left( \sum_{s=1}^{T} |\phi(s)|^p \right)^{1/p} \left( \sum_{s=1}^{T} |\Delta u_1(s)|^q \right)^{1/q}
= \left( \sum_{s=1}^{t-1} s^p + \sum_{s=t}^{T-1} (T - s)^p \right)^{1/p} \left( \sum_{s=1}^{T} |\Delta u_1(s)|^q \right)^{1/q}. \tag{2.14}
\]
Since
\[
\sum_{s=1}^{t-1} s^p < \frac{t^{p+1} - 1}{p + 1}, \quad \sum_{s=t}^{T-1} (T - s)^p = \sum_{k=1}^{T-t} k^p < \frac{(T - t + 1)^{p+1} - 1}{p + 1}, \tag{2.15}
\]
and
\[
\sum_{s=1}^{t-1} s^p + \sum_{s=t}^{T-1} (T - s)^p \leq \sum_{s=1}^{T-1} (T - 1)^p = (T - 1)^{p+1}, \tag{2.16}
\]
it follows from \((2.14)\) that \((2.6)\) with \(i = 1\) holds. On the other hand, from \((2.14),(2.15)\)
and (2.16), we have

\[
T^q \sum_{t=1}^{T} |u_1(t)|^q \leq \left( \sum_{s=1}^{T} \left| \Delta u_1(s) \right|^q \right)^{\frac{1}{q}} \sum_{t=1}^{T} \left( \sum_{s=1}^{t-1} s^p + \sum_{s=t}^{T} (T-s)^p \right)^{\frac{q}{p}} T(T-1)^{2q-1}
\]

\[
\leq \left( \sum_{s=1}^{T} \left| \Delta u_1(s) \right|^q \right)^{\frac{1}{q}} \min \left\{ \sum_{t=1}^{T} \left( \frac{t^{p+1} - 1}{p+1} + \frac{(T-t+1)^{p+1} - 1}{p+1} \right) ^{\frac{q}{p}}, T(T-1)^{2q-1} \right\}
\]

\[
= \left( \sum_{s=1}^{T} \left| \Delta u_1(s) \right|^q \right)^{\frac{1}{q}} \cdot \min \left\{ \frac{T^{2q-1}}{(p+1)^{q/p}} \sum_{t=1}^{T} \left[ \left( \frac{t}{T} \right)^{p+1} + \left( 1 - \frac{t}{T} + \frac{1}{T} \right)^{p+1} - \frac{2}{T^{p+1}} \right] ^{\frac{q}{p}}, T(T-1)^{2q-1} \right\}
\]

\[
= \min \left\{ \frac{T^{2q-1}}{(p+1)^{q/p}}, T(T-1)^{2q-1} \right\} \left( \sum_{s=1}^{T} \left| \Delta u_1(s) \right|^q \right)^{\frac{1}{q}}
\]

It follows that (2.8) with \( i = 1 \) holds. Similarly, we can prove other inequalities also hold. Thus the proof is complete.

**Remark 2.1.** Since

\[
\min \left\{ \frac{(T-1)^{(p+1)/p}}{T}, \left( \frac{(T+1)^{p+2} - 2}{T^{p+1}} \right)^{1/p} \right\} \leq \frac{(T-1)^{(p+1)/p}}{T} < \frac{T^{(p+1)/p}}{T} = T^{1/p}
\]

and

\[
\min \left\{ \frac{(T-1)^{(q+1)/q}}{T}, \left( \frac{(T+1)^{(q+1)} - 2}{T^{q+1}} \right)^{1/q} \right\} \leq \frac{(T-1)^{(q+1)/q}}{T} < \frac{T^{(q+1)/q}}{T} = T^{1/q},
\]

(2.6) and (2.7) improve (2.9) and (2.10) in [7] which shows that for \( u = (u_1, u_2) \in Y \) and \( t \in \mathbb{Z}[1,T] \),

\[
|u_i(t)| \leq T^{1/p} \| \Delta u_i \|_{L^q}, \quad |u_i(t)| \leq T^{1/q} \| \Delta u_i \|_{L^p}, \quad i = 1, 2,
\]

respectively. Moreover, Lemma 2.1 also improves Lemma 2.2 in [12].

**Lemma 2.2.** For every \( u = (u_1, u_2)^T \in E \),

\[
\sum_{t=1}^{T} (L(J\Delta u(t-1)), u(t)) \geq -\frac{C}{q} \| \Delta u_1 \|_{L^q}^q - \frac{C}{p} \| \Delta u_2 \|_{L^p}^p
\]

(2.17)

and

\[
\sum_{t=1}^{T} (L^{-1}(J\Delta u(t)), u(t)) \geq -\frac{C}{p} \| \Delta u_1 \|_{L^p}^p - \frac{C}{q} \| \Delta u_2 \|_{L^q}^q,
\]

(2.18)
where
\[ C = C(p, q) + C(q, p), \quad C^q(p, q) = \min \left\{ \frac{(T - 1)^{2q-1}}{T^{q-1}}, \frac{T^{q-1} \Theta(p, q)}{(p + 1)^{q/p}} \right\}, \]
and
\[ C^p(q, p) = \min \left\{ \frac{(T - 1)^{2p-1}}{T^{p-1}}, \frac{T^{p-1} \Theta(q, p)}{(q + 1)^{p/q}} \right\}. \]

Proof. For \( u = (u_1, u_2) \in E \), we write \( u_i = \tilde{u}_i + \bar{u}_i \), where \( \tilde{u}_i = 1/T \sum_{t=1}^{T} u_i(t), i = 1, 2 \). Since \( \sum_{t=1}^{T} \tilde{u}_i(t) = 0 \) and \( \Delta u_i(t) = \Delta \tilde{u}_i(t), i = 1, 2 \), then by (2.8), (2.9), Hölder’s inequality and Young’s inequality, we have
\[
\sum_{t=1}^{T} (L(J\Delta u(t - 1)), u(t)) = \sum_{t=1}^{T} [(\Delta u_2(t - 1), u_1(t)) - (\Delta u_1(t), u_2(t))]
= \sum_{t=1}^{T} [(\Delta \tilde{u}_2(t - 1), \tilde{u}_1(t)) - (\Delta \tilde{u}_1(t), \tilde{u}_2(t))]
\geq -C(p, q) \| \Delta \tilde{u}_2 \|_p \| \Delta \tilde{u}_1 \|_q - C(q, p) \| \Delta \tilde{u}_2 \|_p \| \Delta \tilde{u}_1 \|_q
= -C \| \Delta u_2 \|_p \| \Delta u_1 \|_q
\geq -\frac{C}{q} \| \Delta u_1 \|_q^q - \frac{C}{p} \| \Delta u_2 \|_p^p.
\]
Similarly to the above process, (2.18) also holds for \( u = (u_1, u_2) \in E \). \qed

Remark 2.2. Note that
\[
C = C(p, q) + C(q, p) \leq \left( \frac{(T - 1)^{2q-1}}{T^{q-1}} \right)^{1/q} + \left( \frac{(T - 1)^{2p-1}}{T^{p-1}} \right)^{1/p} < 2T. \tag{2.19}
\]
So our Lemma 2.2 improves Lemma 2.3 in [7].

Lemma 2.3. ([8], Proposition 1.4) Let \( G \in C^1(\mathbb{R}^N, \mathbb{R}) \) be a convex function. Then, for all \( x, y \in \mathbb{R}^N \), we have
\[
G(x) \geq G(y) + (\nabla G(y), x - y).
\]

3 Main results and proofs

Theorem 3.1. Suppose \( F \) satisfies \((A_1), (A_3)\) and the following conditions:
\( (A_2)^f \) there are constants \( \alpha \in (0, 2/C) \), and \( \gamma : \mathbb{Z} \rightarrow \mathbb{R} \) such that for all \( (t, y) \in \mathbb{Z} \times \mathbb{R}^N \),
\[
F(t, y) \leq \frac{\alpha}{p} |y|^p + \gamma(t) \quad \text{and} \quad \gamma(t + T) = \gamma(t);
\]
Then, system (2.3) has at least one solution \( u \in E \) such that

\[
v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = -J \left[ u(t) - \frac{1}{T} \sum_{s=1}^{T} u(s) \right] = \begin{pmatrix} -u_2(t) + \frac{1}{T} \sum_{s=1}^{T} u_2(s) \\ u_1(t) - \frac{1}{T} \sum_{s=1}^{T} u_1(s) \end{pmatrix}
\]

minimizes the dual action \( I \), that is to say, system (1.1) has at least one solution \( x = u_1 \).

**Proof.** The proof is the same as in [7]. We only need to replace Lemma 2.3 in [7] with our Lemma 2.2 in the proof. In order to make the paper self-contained, we present a brief outline of the proof. More details can be seen in [7].

Step 1. We consider the existence of one \( T \)-periodic solution for a perturbed problem. Note that \( \alpha < 2/C \). So there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \),

\[
\alpha(1 + \varepsilon)^{p-1} < 2/C, \quad \alpha(1 + \varepsilon)^{q-1} < 2/C.
\]

Consider the following perturbed problem:

\[
\begin{align*}
\Delta u_2(t) + \varepsilon \alpha^{p-1} \phi_p(u_1(t)) + \nabla H_1(t, u_1(t)) &= 0, & t \in \mathbb{Z}, \\
-\Delta u_1(t-1) + \varepsilon \alpha^{q-1} \phi_q(u_2(t)) + \nabla H_2(t, u_2(t)) &= 0, \\
u_1(t + T) &= u_1(t), u_2(t + T) &= u_2(t).
\end{align*}
\]

(3.1)

In order to obtain the solution of the perturbed problem, consider the following perturbed dual action functional

\[
I_\varepsilon(v) = \frac{1}{2} \sum_{t=1}^{T} (L(J\Delta v(t - 1)), v(t)) + \sum_{t=1}^{T} H_\varepsilon^*(t, \Delta v(t)),
\]

where

\[
H_\varepsilon(t, \Delta v) = \varepsilon \alpha^{p-1} \frac{|u_1|^p}{p} + H_1(t, u_1) + \varepsilon \alpha^{q-1} \frac{|u_2|^q}{q} + H_2(t, u_2).
\]

By (A1), (A2)', Lemma 2.1 in [7] and Lemma 2.2, one can obtain that

\[
I_\varepsilon(v) \geq -\frac{C}{2q} \| \Delta v_1 \|^q_q - \frac{C}{2p} \| \Delta v_2 \|^p_p + \frac{(1 + \varepsilon)^{-(q-1)} \alpha^{-1}}{q} \| \Delta v_1 \|^q_q
\]

\[
+ \frac{(1 + \varepsilon)^{-(p-1)} \alpha^{-1}}{p} \| \Delta v_2 \|^p_p - \frac{1}{\alpha} \sum_{s=1}^{T} \gamma(t).
\]

(3.2)

Since \( (1 + \varepsilon)^{-(q-1)} \alpha^{-1} > C/2 \) and \( (1 + \varepsilon)^{-(p-1)} \alpha^{-1} > C/2 \), \( I_\varepsilon \) is bounded from below and coercive in subspace \( Y \). By Lemma 2.2 in [7], we know that \( I_\varepsilon \) is continuously differentiable in \( Y \). Then by Theorem 1.1 in [8], \( I_\varepsilon \) attains its minimum at some point \( v_\varepsilon \in Y \). Then by Lemma 2.2 in [7],

\[
u_\varepsilon(t) = L^{-1}(\nabla H_\varepsilon^*(t, \Delta v_\varepsilon(t))), u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})^T, v_\varepsilon = (v_{1\varepsilon}, v_{2\varepsilon})^T
\]
is a solution of the perturbed problem (3.1).

Step 2. We prove that \( u_\varepsilon \) is bounded in \( E \). By (A3), we can get a \( y_0 \in E \) such that \( \sum_{t=1}^{T} y_0(t) \). Then
\[
I_\varepsilon(v_\varepsilon) \leq I_\varepsilon(y_0) \leq \frac{1}{2} \sum_{t=1}^{T} (L(J\Delta y_0(t - 1)), y_0(t)) + \sum_{t=1}^{T} H^*(t, \Delta y_0(t)) < +\infty.
\]

Note that \( \Delta u_\varepsilon(t) = J\Delta v_\varepsilon(t) \). So (3.2), (2.8) and (2.9) imply that there exists a constant \( K_1 \) such that
\[
\|\tilde{u}_{1\varepsilon}\|_p \leq K_1 \text{ and } \|\tilde{u}_{2\varepsilon}\|_q \leq K_1.
\]

By virtue of the convexity of \( H_i(t, \cdot)(i = 1, 2) \), (3.3), (A2)' and (A3), we can obtain that there exists a constant \( K_2 \) such that
\[
|\bar{u}_{1\varepsilon}| \leq K_2 \text{ and } |\bar{u}_{2\varepsilon}| \leq K_2.
\]

So
\[
\|u_\varepsilon\| = \|u_{1\varepsilon}\|_p + \|u_{2\varepsilon}\|_q \leq \|\tilde{u}_{1\varepsilon}\|_p + |\tilde{u}_{1\varepsilon}|_p + \|\tilde{u}_{2\varepsilon}\|_q + |\tilde{u}_{2\varepsilon}|_q \leq 2K_1 + K_2(T^{1/p} + T^{1/q}),
\]
which shows that \( u_\varepsilon \) is bounded in \( E \).

Step 3. We prove the existence of a \( T \)-periodic solution for system (1.1). Note that \( u_\varepsilon \) is bounded in \( E \) and \( E \) is dimensional. Then there exists a sequence \( \{\varepsilon_n\} \subset (0, \varepsilon_0) \) and some point \( u = (u_1, u_2)^T \in E \) such that
\[
\varepsilon_n \to 0, \quad u_{\varepsilon_n} \to u \text{ as } n \to \infty.
\]

Let \( n \to \infty \) in (3.1). Then it is easy to obtain that \( u_1 \) is a \( T \)-periodic solution of system (1.1). Moreover, since \( \Delta v_{\varepsilon_n}(t) = -J\Delta u_{\varepsilon_n}(t) \), we have \( v_{\varepsilon_n}(t) = -J(u_{\varepsilon_n}(t) - \bar{u}_{\varepsilon_n}) \). Let \( n \to \infty \). Then
\[
v_{\varepsilon_n}(t) \to -J(u(t) - \bar{u}) := v(t).
\]

Step 4. We prove that \( v = (v_1, v_2)^T \in E \) minimizes the dual action \( I \). Since \( \Delta v_{\varepsilon_n}(t) = \nabla H_{\varepsilon_n}(t, Lu_{\varepsilon_n}(t)) \),
\[
\Delta v_{1\varepsilon_n}(t) = \nabla H_{1\varepsilon_n}(t, u_{1\varepsilon_n}(t)), \quad \Delta v_{2\varepsilon_n}(t - 1) = \nabla H_{2\varepsilon_n}(t, u_{2\varepsilon_n}(t)).
\]

Let \( n \to \infty \). Then (3.5) and (2.4) imply that
\[
\Delta v_1(t) = \nabla H_1(t, u_1(t)), \quad \Delta v_2(t - 1) = \nabla H_2(t, u_2(t)).
\]
As $H^*_v(t, v) \leq H^*(t, v)$, we obtain that

$$I_{t_n}(v_{t_n}) \leq I_{t_n}(h) \leq I(h).$$

for all $h \in E$. Let $n \to \infty$. By (3.6) and Lemma 2.1 in [7], we can get $I(v) \leq I(h)$ for all $h \in E$. Thus the proof is complete.

\[ \square \]

**Remark 3.1.** By (2.19), it is easy to obtain that $2/C > 2/(2T) = 1/T$. So Theorem 3.1 improves Theorem 1.1 since the range of $\alpha$ is larger.

Next, we consider the estimate of solutions for system (1.1).

**Theorem 3.2.** Assume that there exists $\alpha \in (0, C^{-1})$, $\beta, \gamma \in [0, +\infty)$, $\delta \in (0, +\infty)$ such that

$$\delta |y|^{p/2} - \beta \leq F(t, y) \leq \frac{\alpha p}{p} |y|^p + \gamma,$$

(3.7)

for all $t \in \mathbb{Z}$ and $y \in \mathbb{R}^N$. Then each solution $x = u_1$ of system (1.1) satisfies

$$\sum_{t=1}^{T} |x(t)|^{p/2} \leq \frac{(\gamma + \beta) T}{\delta} + \frac{\alpha^q C(q, p) B^{1/p} D^{1/q}}{\delta},$$

(3.8)

$$\|\Delta x\|_p \leq \frac{p T (\gamma + \beta) 1 - C \alpha}{\alpha^q - C \alpha^{q+1}},$$

(3.9)

where

$$B = \frac{p T (\gamma + \beta)}{\alpha^q - C \alpha^{q+1}}, \quad D = \frac{q T (\gamma + \beta)}{\alpha^{1-q/p} - C \alpha}.$$ Proof. By (3.7), for all $u = (u_1, u_2) \in \mathbb{R}^N \times \mathbb{R}^N$, we have

$$\frac{\delta}{\alpha} |u_1|^{p/2} - \frac{\beta}{\alpha} + \frac{\alpha^{q-1}}{q} |u_2|^q \leq H(t, u) = \frac{1}{\alpha} F(t, u_1) + \frac{\alpha^{q-1}}{q} |u_2|^q \leq \frac{1}{\alpha} |u_1|^p + \frac{\gamma}{\alpha} + \frac{\alpha^{q-1}}{q} |u_2|^q.$$ Then, we have

$$(u, v) - H(t, u) \geq (u, v) - \frac{\alpha^{q-1}}{p} |u_1|^p - \frac{\gamma}{\alpha} - \frac{\alpha^{q-1}}{q} |u_2|^q, \quad \forall \ u \in \mathbb{R}^N \times \mathbb{R}^N.$$ Since

$$(u, v) - \frac{\alpha^{q-1}}{p} |u_1|^p - \frac{\gamma}{\alpha} - \frac{\alpha^{q-1}}{q} |u_2|^q = (u_1, v_1) + (u_2, v_2) - \frac{\alpha^{q-1}}{p} |u_1|^p - \frac{\gamma}{\alpha} - \frac{\alpha^{q-1}}{q} |u_2|^q \leq |u_1||v_1| - \frac{\alpha^{q-1}}{p} |u_1|^p - \frac{\gamma}{\alpha} + |u_2||v_2| - \frac{\alpha^{q-1}}{q} |u_2|^q \leq \sup_{u_1 \in \mathbb{R}^N} \left\{ |u_1||v_1| - \frac{\alpha^{q-1}}{p} |u_1|^p - \frac{\gamma}{\alpha} \right\} + \sup_{u_2 \in \mathbb{R}^N} \left\{ |u_2||v_2| - \frac{\alpha^{q-1}}{q} |u_2|^q \right\} = \frac{|v_1|^q}{q \alpha} - \frac{\gamma}{\alpha} + \frac{1}{p \alpha} |v_2|^p, \quad \forall \ u \in \mathbb{R}^N \times \mathbb{R}^N.$$
Hence, by (2.1), we have
\[
H^*(t,v) \geq \frac{|v_1|^q}{q\alpha} - \frac{\gamma}{\alpha} + \frac{1}{p\alpha}|v_2|^p.
\] (3.11)

When \( v = \nabla H(t,u) \), by (2.2) and (3.10), we get
\[
H^*(t,v) = (u,v) - H(t,u) \leq (u,v) + \frac{\beta}{\alpha}.
\] (3.12)

Then
\[
\frac{|v_1|^q}{q\alpha} - \frac{\gamma}{\alpha} + \frac{1}{p\alpha}|v_2|^p \leq (u,v) + \frac{\beta}{\alpha}.
\] (3.13)

Note that \( v = \nabla H(t,u) = \nabla H_1(t,u_1) \nabla H_2(t,u_2) \).

Then by (2.2) and (3.13), we have
\[
\left| \frac{1}{\alpha} \nabla F(t,u_1) \right|^q - \frac{\gamma}{\alpha} + \frac{1}{p\alpha} |\alpha^{-1}\alpha^{-1/2}u_2|^p \leq (u, \nabla H(t,u)) + \frac{\beta}{\alpha},
\]
that is
\[
\frac{\alpha^{-1}(1+q)}{q} |\nabla F(t,u_1)|^q - \frac{\gamma}{\alpha} + \frac{\alpha^{-1}}{p} |u_2|^q \leq (u, \nabla H(t,u)) + \frac{\beta}{\alpha}.
\]

For each solution \( u \in E \) of system (1.1), by (2.3) and (2.4), we know
\[
\nabla F(t,u_1(t)) = -\alpha \Delta u_2(t)
\]
and
\[
L \nabla H(t,u(t)) = \nabla H(t,Lu(t)) = -J\Delta u(t).
\]

Hence
\[
\frac{1}{q\alpha} |\Delta u_2(t)|^q - \frac{\gamma}{\alpha} + \frac{\alpha^{-1}}{p} |u_2(t)|^q \leq (u(t), -L^{-1}(J\Delta u(t))) + \frac{\beta}{\alpha}.
\]

Summing the above inequality over \( \mathbb{Z}[1,T] \) and using Lemma 2.2 and (2.3), we obtain
\[
\frac{1}{q\alpha} \| \Delta u_2 \|^q_{q} - \frac{\gamma T}{\alpha} + \frac{\alpha^{-1}}{p} \| u_2 \|^p_p \leq -\sum_{t=1}^{T} (u(t), L^{-1}(J\Delta u(t))) + \frac{\beta T}{\alpha}
\]
\[
\leq \frac{C}{q} \| \Delta u_2 \|^q_{q} + \frac{C}{p} \| \Delta u_1 \|^p_p + \frac{\beta T}{\alpha}
\]
\[
= \frac{C}{q} \| \Delta u_2 \|^q_{q} + \frac{C\alpha^q}{p} \| \Phi_q(u_2) \|^p_p + \frac{\beta T}{\alpha}
\]
\[
= \frac{C}{q} \| \Delta u_2 \|^q_{q} + \frac{C\alpha^q}{p} \| u_2 \|^q_q + \frac{\beta T}{\alpha}.
\]
So
\[
\left( \frac{1}{q} - \frac{C}{q} \right) \| \Delta u_2 \|_q^q + \left( \frac{\alpha^{q-1} - C\alpha^q}{p} \right) \| u_2 \|_q^q \leq \frac{T(\beta + \gamma)}{\alpha}.
\]
Since \( \alpha \in (0, C^{-1}) \), we have
\[
\| u_2 \|_q^q \leq \frac{pT(\gamma + \beta)}{\alpha^q - C\alpha^{q+1}} = B, \quad \| \Delta u_2 \|_q^q \leq \frac{qT(\gamma + \beta)}{1 - C\alpha} = D.
\] (3.14)
Hence,
\[
\| \Delta u_1 \|_p^p = \alpha^q \| \Phi_q(u_2) \|_p^p = \alpha^q \| u_2 \|_q^q \leq B\alpha^q.
\] (3.15)
It follows that (3.9) holds. Since \( F \) is continuously differentiable and convex in \( x \), then by Lemma 2.3, (3.7), (2.3), Lemma 2.2, Hölder’s inequality, (3.14) and (3.15), we have
\[
\delta \sum_{t=1}^T |u_1(t)|^{p/2} - \beta T \leq \sum_{t=1}^T F(t, u_1(t)) \leq \sum_{t=1}^T \left[ F(t, 0) + (\nabla F(t, u_1(t)), u_1(t)) \right] \leq \gamma T - \sum_{t=1}^T (\alpha \Delta u_2(t), u_1(t)) = \gamma T - \sum_{t=1}^T (\alpha \Delta u_2(t), \tilde{u}_1(t)) \leq \gamma T + \alpha \left( \sum_{t=1}^T |\tilde{u}_1(t)|^p \right)^{1/p} \left( \sum_{t=1}^T |\Delta u_2(t)|^q \right)^{1/q} \leq \gamma T + \alpha C(q, p) \||\Delta u_1||_p \| \Delta u_2 \|_q \leq \gamma T + \alpha^q C(q, p) B^{1/p} D^{1/q}.
\]
So, we get
\[
\sum_{t=1}^T |u_1(t)|^{p/2} \leq \frac{(\gamma + \beta)T}{\delta} + \frac{\alpha^q C(q, p) B^{1/p} D^{1/q}}{\delta}.
\]
It follows that (3.8) holds. The proof is complete. \( \square \)

References


