A new iterative method for generalized equilibrium and fixed point problems of nonexpansive mappings

A. Razani$^{a,b}$ and M. Yazdi$^{a,*}$

$^a$ Department of Mathematics, Faculty of Science, I.Kh. International University, P.O. Box: 34149-16818, Qazvin, Iran.
$^b$ School of Mathematics, Institute for Research in Fundamental Sciences, P. O. Box 19395-5746, Tehran, Iran,
E-mail: razani@ikiu.ac.ir and msh_yazdi@ikiu.ac.ir

Abstract

In this paper, a new iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of infinitely many nonexpansive mappings in Hilbert spaces, is introduced. For this method, a strong convergence theorem is given. This improves and extends some recent results.

1 Introduction

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. A mapping $S$ of $C$ into itself is called nonexpansive, if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in C$. Also, a contraction on $C$ is a self-mapping $S$ of $C$ such that $\|S(x) - S(y)\| \leq k\|x - y\|$, for all $x, y \in C$, where $k \in (0, 1)$ is a constant. Moreover,

$^*$Corresponding author

2000 Mathematics Subject Classification: 47H10; 47H09.

Keywords: Equilibrium problem; Fixed point; Nonexpansive mapping; Iterative method; Variational inequality.

The first author would like to thank the Institute for School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Teheran, Iran, for supporting this research (Grant No. 89470126).
$F(S)$ denotes the fixed points set of $S$.
Let $\phi : C \times C \to \mathbb{R}$ be a bifunction of $C \times C$ into $\mathbb{R}$. We recall an equilibrium problem as follows:
The equilibrium problem for $\phi : C \times C \to \mathbb{R}$ is to find $u \in C$ such that
$$\phi(u, v) \geq 0, \quad \text{for all } v \in C. \quad (1.1)$$

The set of solutions of (1.1) is denoted by $EP(\phi)$. Set $\phi(u, v) = \langle Tu, v - u \rangle$, for all $u, v \in C$, where $T : C \to H$. Then, $w \in EP(\phi)$ if and only if $\langle Tw, v - w \rangle \geq 0$, for all $v \in C$, that is, $w$ is a solution of the variational inequality.

Combettes and Hirstoaga [4] introduced an iterative scheme for finding the best approximation to the initial data when $EP(\phi)$ is nonempty and proved a strong convergence theorem. The equilibrium problem (1.1) includes, as special cases, numerous problems in physics, optimization and economics. Some authors (such as [6, 7, 10, 11, 14, 15]) have proposed some useful methods for solving the equilibrium problem (1.1). We describe some of them as follows:

In 2007, Pliptieng and Punpaeng [11] introduced an iterative scheme for finding a common element of the set of the solutions (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$\left\{ \begin{array}{l}
\Phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in H, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)S u_n,
\end{array} \right. \quad (1.2)$$

where $\phi : H \times H \to \mathbb{R}$ is a bifunction, $A$ is a strongly positive bounded linear operator on $H$, $S$ is a nonexpansive mapping of $H$ into itself such that $F(S) \cap EP(\phi) \neq \emptyset$, $f$ is a contraction, $\gamma > 0$ is some constant, $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. Also, they proved the strong convergence of $\{x_n\}$, defined by (1.2) and showed $\lim_{n \to \infty} x_n$ is the unique solution of a certain variational inequality.

Jung [7] introduced the following composite iterative scheme by the viscosity approximation method for finding a common point of the set of solutions of (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space:

$$\left\{ \begin{array}{l}
\Phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C, \\
y_n = \alpha_n f(x_n) + (1 - \alpha_n)S u_n, \\
x_{n+1} = (1 - \beta_n)y_n + \beta_n S y_n,
\end{array} \right. \quad (1.3)$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction, $S$ is a nonexpansive mapping of $C$ into itself such that $F(S) \cap EP(\phi) \neq \emptyset$, $f$ is a contraction, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. He proved the sequence $\{x_n\}$, generated by (1.3), converges strongly to a point in $F(S) \cap EP(\phi)$ provided $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ satisfy

(C1) $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $0 < \liminf_{n \to \infty} r_n$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(C3) \( \lim_{n \to \infty} \beta_n = 0 \) and \( \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty \).

Jung [6] studied the following composite iterative scheme:

\[
\begin{align*}
& \Phi(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq 0, \quad \text{for all } y \in H, \\
& y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S u_n, \\
& x_{n+1} = (1 - \beta_n) y_n + \beta_n S y_n, \quad n \geq 1,
\end{align*}
\]

(1.4)

where \( \phi : H \times H \to \mathbb{R} \) is a bifunction, \( A \) is a strongly positive bounded linear operator on \( H \), \( S \) is a nonexpansive mapping of \( H \) into itself such that \( F(S) \cap EP(\phi) \neq \emptyset \), \( f \) is a contraction, \( \gamma > 0 \) is some constant, \( \{\alpha_n\}, \{\beta_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \). He proved, the sequence \( \{x_n\} \), generated by (1.4), converges strongly to a point in \( F(S) \cap EP(\phi) \) under the conditions (C1), (C2) and (C3).

Wang et al. [15] introduced the following composite iterative scheme:

\[
\begin{align*}
& \Phi(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq 0, \quad \text{for all } y \in H, \\
& y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S u_n, \\
& x_{n+1} = (1 - \beta_n) y_n + \beta_n S y_n, \quad n \geq 1,
\end{align*}
\]

(1.5)

where \( \phi : H \times H \to \mathbb{R} \) is a bifunction, \( A \) is a strongly positive bounded linear operator on \( H \), \( \{S_n\} \) is a countable family of nonexpansive mappings of \( H \) into itself such that \( \bigcap_{n=1}^{\infty} F(S_n) \cap EP(\phi) \neq \emptyset \), \( f \) is a contraction, \( \gamma > 0 \) is some constant, \( x_1 \in H, \{\alpha_n\}, \{\beta_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \). They proved, under any of the following conditions:

(H1) \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \);

(H2) \( \alpha_n \in (0, 1] \) for every \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \frac{\alpha_n}{\sigma_{n+1}} = 1 \);

(H3) \( |\alpha_{n+1} - \alpha_n| < \sigma(\alpha_{n+1}) + \sigma_n \) and \( \sum_{n=1}^{\infty} \sigma_n < \infty \),

on the sequence \( \{x_n\} \) (generated by (1.5)) converges strongly to a point in \( \bigcap_{n=1}^{\infty} F(S_n) \cap EP(\phi) \neq \emptyset \).

Recently, Razani and Yazdi [14] study the convergence of a new version of composite iterative scheme (1.5).

In this paper, we prove a strong convergence theorem, concerning a new iterative scheme, for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space. In order to do this, we recall some definitions as follows:

A generalized equilibrium problem is to find \( z \in C \) such that

\( \phi(z, y) + \langle Az, y - z \rangle \geq 0, \quad \text{for all } y \in C, \)

(1.6)

where \( \phi : C \times C \to \mathbb{R} \) is a bifunction and \( A : C \to H \) is a monotone map. The set of such \( z \in C \) is denoted by \( EP \), i.e.,

\[ EP = \{ z \in C : \phi(z, y) + \langle Az, y - z \rangle \geq 0, \quad \text{for all } y \in C \}. \]
In the case of $A \equiv 0$, $EP$ is denoted by $EP(\phi)$. Numerous problems in physics, variational inequalities, optimization, minimax problems, the Nash equilibrium problem in noncooperative games and economics reduce to finding a solution of (1.6) (see [8], for instance).

A mapping $A : C \rightarrow H$ is called $\alpha$-inverse-strongly monotone [3], if there exists a positive real number $\alpha$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \text{for all } x, y \in C.$$ 

**Remark 1.1.** If $A : C \rightarrow H$ is $\alpha$-inverse-strongly monotone map, then it is $\frac{1}{\alpha}$-Lipschitzian mapping.

Let $B$ be a bounded operator on $C$. $B$ is strongly positive; that is, there exists a constant $\gamma > 0$ such that $\langle Bx, x \rangle \geq \gamma \|x\|^2$, for all $x \in C$. A typical problem is that of minimizing a quadratic function over the set of the fixed points of nonexpansive mapping on a real Hilbert space:

$$\min_{x \in F(S)} \frac{1}{2} \langle Bx, x \rangle - \langle x, b \rangle,$$

where $b$ is a given point in $H$.

**Remark 1.2.** Iterative method for nonexpansive mappings have been applied to solve convex minimization problems (see [12, 13]).

In this paper, a new iterative method (motivated by the above results) is introduced as follows:

$$\begin{align*}
\Phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle + \langle Ax_n, y - u_n \rangle &\geq 0, \quad \text{for all } y \in C, \\
y_n &= \alpha_n \gamma f(x_n) + (I - \alpha_n B)S_n u_n, \\
x_{n+1} &= (1 - \beta_n)y_n + \beta_n S_n y_n, \quad n \geq 1,
\end{align*}$$

(1.7)

where $\phi : C \times C \rightarrow \mathbb{R}$ is a bifunction, $A$ is an $\alpha$-inverse-strongly monotone, $B$ is a strongly positive bounded linear operator on $C$, $\{S_n\}$ is a countable family of nonexpansive mappings of $C$ into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap EP \neq \emptyset$, $f$ is a contraction, $x_1 \in C$, $\left\{\alpha_n\right\}, \left\{\beta_n\right\} \subset [0, 1]$ and $\{r_n\} \subset [a,b] \subset (0,2\alpha)$. Then, under any of three conditions $(H_1)$, $(H_2)$ and $(H_3)$ on the sequence $\left\{\alpha_n\right\}$, the sequence $\{x_n\}$, generated by (1.7), converges strongly to a point in $\bigcap_{n=1}^{\infty} F(S_n) \cap EP$.

2 Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle ., . \rangle$ and the norm $\| . \|$. Weak and strong convergence are denoted by notation $\rightharpoonup$ and $\rightarrow$, respectively. In a real Hilbert space $H$,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2,$$
for all $x, y \in H$ and $\lambda \in \mathbb{R}$. Let $C$ be a nonempty closed convex subset of $H$. Then, for any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \text{for all } y \in C.$$  

$P_C$ is called the metric projection of $H$ onto $C$. It is known $P_C$ is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C(x) \iff \langle x - z, z - y \rangle \geq 0, \quad \text{for all } y \in C.$$  

Now, we collect some lemmas which will be used in the main result.

**Lemma 2.1.** Let $H$ be a real Hilbert space. Then for all $x, y \in H$,

(I) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$;

(II) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x \rangle$.

**Lemma 2.2.** [5] Let $H$ be a real Hilbert space, $C$ a closed convex subset of $H$ and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in $C$ weakly converging to $x$ and if $\{(I - T)x_n\}$ converges to $y$, then $(I - T)x = y$.

**Lemma 2.3.** [2] Let $C$ be a nonempty closed convex subset of $H$ and $\phi : C \times C \to \mathbb{R}$ a bifunction satisfying the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;

(A2) $\phi$ is monotone, i.e., $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \phi(tz + (1 - t)x, y) \leq \phi(x, y);$$

(A4) for each $x \in C, y \mapsto \phi(x, y)$ is convex and weakly lower semicontinuous.

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$\Phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \text{for all } y \in C.$$  

**Lemma 2.4.** [4] Assume $\phi : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r x = \{z \in C : \Phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \text{for all } y \in C\},$$

for all $x \in H$. Then, the following hold:
(I) $T_r$ is single-valued;

(II) $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$,
$$
\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;
$$

(III) $F(T_r) = \text{EP}(\phi)$;

(IV) $\text{EP}(\phi)$ is closed and convex.

Lemma 2.5. [9] Assume $B$ is a strongly positive bounded linear operator on a Hilbert space $H$ with coefficient $\gamma > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho \gamma$.

Lemma 2.6. [1] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that
$$
a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n v_n + \mu_n,
$$
where $\{\gamma_n\}$ is a sequence in $[0, 1]$, $\{\mu_n\}$ is a sequence of nonnegative real numbers and $\{v_n\}$ is a sequence in $\mathbb{R}$ such that

(I) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(II) $\lim \sup_{n \to \infty} v_n \leq 0$;

(III) $\sum_{n=1}^{\infty} \mu_n < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.7. [1] Let $C$ be a nonempty closed convex subset of $H$. Suppose
$$
\sum_{n=1}^{\infty} \sup \{\|T_{n+1} z - T_n z\| : z \in C\} < \infty.
$$

Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of $C$. Moreover, let $T$ be a mapping of $C$ into itself defined by $Ty = \lim_{n \to \infty} T_n y$, for all $y \in C$. Then $\lim_{n \to \infty} \sup \{\|T z - T_n z\| : z \in C\} = 0$.

3 Main result

In this section, we prove a strong convergence theorem, concerning the iterative scheme (1.7), for finding a common element of the set of solutions of the generalized equilibrium problem (1.6) and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space. Before this, three lemmas are proved as follows:
Lemma 3.1. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Assume $f$ is a contraction of $C$ into itself with coefficient $k$, $B$ is a strongly positive bounded linear operator on $C$ with coefficient $\gamma > 0$ such that $0 < \gamma < \frac{\tau}{k}$ and $\|B\| \leq 1$. Then $P_C(I - B + \gamma f)$ is a contraction.

Proof. Let $Q = P_C$. Then
\[
\|Q(I - B + \gamma f)(x) - Q(I - B + \gamma f)(y)\| \leq \|(I - B + \gamma f)(x) - (I - B + \gamma f)(y)\|
\leq \|(I - B)(x) - (I - B)(y)\| + \gamma\|f(x) - f(y)\| \\
\leq (1 - \tau)\|x - y\| + \gamma k\|x - y\| \\
= (1 - (\gamma - \gamma k))\|x - y\|,
\]
for all $x, y \in C$. Therefore, $Q(I - B + \gamma f)$ is a contraction of $C$ into itself. \qed

Lemma 3.2. Suppose $C$ is a nonempty closed convex subset of a real Hilbert space $H$, $A$ is an $\alpha$-inverse-strongly monotone on $C$ and $0 < r < 2\alpha$. Then $I - rA$ is nonexpansive.

Proof. For $x, y \in C$,
\[
\|(I - rA)x - (I - rA)y\|^2 = \|x - y - r(Ax - Ay)\|^2 \\
= \|x - y\|^2 - 2r\langle x - y, Ax - Ay \rangle + r^2\|Ax - Ay\|^2 \\
\leq \|x - y\|^2 - 2\alpha r\|Ax - Ay\|^2 + r^2\|Ax - Ay\|^2 \\
= \|x - y\|^2 + r(2\alpha - 2\alpha)\|Ax - Ay\|^2 \\
\leq \|x - y\|^2.
\]
Thus $I - rA$ is nonexpansive. \qed

Lemma 3.3. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A_1) - (A_4)$ (of Lemma 2.3) and $A$ be an $\alpha$-inverse-strongly monotone map. Suppose $\{x_n\}$ is a bounded sequence in $C$ and $\{r_n\} \subset [a, b] \subset (0, 2\alpha)$ is a real sequence. If $u_n = T_{r_n}(x_n - r_nAx_n)$, then
\[
\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + |r_n - r_{n+1}|M_1,
\]
where $M_1 = \sup\{\|Ax_n\| + \frac{1}{\alpha}\|u_{n+1} - k_{n+1}\| : n \in \mathbb{N}\}$.

Proof. Let $p \in EP$. Then $\phi(p, y) + \langle Ap, y - p \rangle \geq 0$, for all $y \in C$. So
\[
\phi(p, y) + \frac{1}{r_n}(p - (p - r_nAp), y - p) \geq 0,
\]
for all $y \in C$. Therefore, by Lemma 3.2,
\[
\|u_n - p\| = \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)p\| \leq \|x_n - p\|, \quad n \geq 1.
\]
Therefore, \( \{u_n\} \) is a bounded sequence. Set \( k_n = x_n - r_n Ax_n \), we have \( u_n = T_{r_n}k_n \) and \( u_{n+1} = T_{r_{n+1}}k_{n+1} \). So
\[
\phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - k_n \rangle \geq 0, \quad \text{for all } y \in C, \tag{3.9}
\]
and
\[
\phi(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - k_{n+1} \rangle \geq 0, \quad \text{for all } y \in C. \tag{3.10}
\]
Set \( y = u_{n+1} \) in (3.9) and \( y = u_n \) in (3.10), then by adding these two last inequalities and using condition (\( A_2 \)), we have
\[
\langle u_{n+1} - u_n, \frac{u_n - k_n}{r_n} - \frac{u_{n+1} - k_{n+1}}{r_{n+1}} \rangle \geq 0,
\]
and hence
\[
\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - k_n - \frac{r_n}{r_{n+1}}(u_{n+1} - k_{n+1}) \rangle \geq 0.
\]
This implies
\[
\|u_{n+1} - u_n\|^2 \leq \langle u_{n+1} - u_n, k_{n+1} - k_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - k_{n+1}) \rangle \\
\leq \|u_{n+1} - u_n\|^2 \{\|k_{n+1} - k_n\| + \frac{1}{a} |r_n - r_{n+1}| \|u_{n+1} - k_{n+1}\| \}.
\]
Therefore
\[
\|u_{n+1} - u_n\| \leq \|k_{n+1} - k_n\| + \frac{1}{a} |r_n - r_{n+1}| \|u_{n+1} - k_{n+1}\| \\
= \|x_{n+1} - r_{n+1} Ax_{n+1} - (x_n - r_n Ax_n)\| \\
+ \frac{1}{a} |r_n - r_{n+1}| \|u_{n+1} - k_{n+1}\| \\
\leq \|x_{n+1} - r_{n+1} Ax_{n+1} - (x_n - r_n Ax_n)\| + |r_n - r_{n+1}| \|Ax_n\| \\
+ \frac{1}{a} |r_n - r_{n+1}| \|u_{n+1} - k_{n+1}\| \\
\leq \|x_{n+1} - x_n\| + |r_n - r_{n+1}| M_1,
\]
where \( M_1 = \sup\{\|Ax_n\| + \frac{1}{a} \|u_{n+1} - k_{n+1}\| : n \in \mathbb{N} \}. \)

\[ \square \]

\textbf{Theorem 3.4.} Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( \phi : C \times C \to \mathbb{R} \) be a bifunction satisfying the conditions (\( A_1 \)) - (\( A_4 \)) (of Lemma 2.3) and \( \{S_n\}_{n=1}^{\infty} \) be an infinite family of nonexpansive self-mappings on \( C \) satisfying \( F := \bigcap_{n=1}^{\infty} F(S_n) \cap EP \neq \emptyset \). Let \( f \) be a contraction of \( C \) into itself with coefficient \( k \), \( B \) be a strongly positive bounded linear operator on \( C \) with coefficient \( \gamma > 0 \) such that \( \|B\| \leq 1 \) and \( A \) be an \( \alpha \)-inverse-strongly monotone on \( C \). Assume \( 0 < \gamma < \frac{\alpha}{k} \).

Suppose \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \( [0, 1] \) and \( \{r_n\} \subset [a, b] \subset (0, 2\alpha) \) is a real sequence satisfying the following conditions:

(\( B_1 \)) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);
(B₂) \( \lim_{n \to \infty} \beta_n = 0 \) and \( \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty; \)

(B₃) \( \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \)

Suppose \( \sum_{n=1}^{\infty} \sup\{|S_{n+1}z - S_nz| : z \in K\} < \infty \) for any bounded subset \( K \) of \( C. \)

Let \( S \) be a mapping of \( C \) into itself defined by \( S = \lim_{n \to \infty} S_n z \) for all \( z \in C \) and \( F(S) = \bigcap_{n=1}^{\infty} F(S_n). \) If any of three conditions \((H_1) - (H_3)\) satisfies, then the sequences \( \{x_n\} \) and \( \{u_n\} \) defined by \((1.7)\) converge strongly to \( q \in F, \) where \( q = P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP(I-B+\gamma f)}(q), \) which solves the following variational inequality:

\[
\langle (B - \gamma f)q, q - x \rangle \leq 0, \text{ for all } x \in F.
\]

Proof. Let \( Q = P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP} Q(I - B + \gamma f) \) is a contraction of \( C \) into itself by Lemma 3.1. So, there exists a unique element \( q \in C \) such that \( q = Q(I-B+\gamma f)(q) = P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP}(I-B+\gamma f)(q). \) By using conditions \((B_1) \) and \((B_2), \) we may assume, without loss of generality, \( \alpha_n \leq (1-\beta_n)\|B\|^{-1}. \) Since \( B \) is strongly positive bounded linear operator on \( C, \)

\[
\|B\| = \sup\{|\langle Bx, x \rangle| : x \in C, \|x\| = 1\}.
\]

Observe

\[
\langle (1 - \beta_n)I - \alpha_n B)x, x \rangle = 1 - \beta_n - \alpha_n \langle Bx, x \rangle
\]

\[
\geq 1 - \beta_n - \alpha_n \|B\|\|
\]

\[
\geq 0.
\]

Thus \((1-\beta_n)I - \alpha_n B\) is positive, and

\[
\| (1 - \beta_n)I - \alpha_n B \| = \sup\{|\langle (1 - \beta_n)I - \alpha_n B)x, x \rangle : x \in C, \|x\| = 1 \}
\]

\[
= \sup\{1 - \beta_n - \alpha_n \langle Bx, x \rangle : x \in C, \|x\| = 1 \}
\]

\[
\leq 1 - \beta_n - \alpha_n \gamma.
\]

We proceed with the following steps:

Step 1. First, we claim, \( \{x_n\} \) and \( \{u_n\} \) are bounded. Let \( p \in F. \) From the definition of \( T_r, u_n = T_r(I - r_n A)x_n. \) Then, from \((1.7), (3.8)\) and Lemma 2.5,

\[
\|x_{n+1} - p\| = \| (1 - \beta_n)(y_n - p) + \beta_n (S_n y_n - p) \|
\]

\[
\leq \|y_n - p\|
\]

\[
= \| \alpha_n (\gamma f(x_n) - Bp) + (I - \alpha_n) (S_n u_n - p) \|
\]

\[
\leq (1 - \alpha_n \gamma) \|x_n - p\| + \| \alpha_n \gamma (f(x_n) - f(p)) \| + \alpha_n \| f(p) - Bp \|
\]

\[
\leq (1 - \alpha_n \gamma) \|x_n - p\| + \alpha_n \gamma k \|x_n - p\| + \alpha_n \| f(p) - Bp \|
\]

\[
\leq (1 - \alpha_n (\gamma - k)) \|x_n - p\| + \alpha_n (\gamma - k) \| \frac{f(p) - Bp}{\gamma - k} \|
\]

\[
\leq \max\{\|x_n - p\|, \frac{1}{\gamma - k}\| f(p) - Bp \|\}, \text{ for all } n \geq 1.
\]

By induction,

\[
\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{1}{\gamma - k}\| f(p) - Bp \|\}, \text{ for all } n \geq 1.
\]
Hence, \( \{x_n\} \) is bounded, so are \( \{u_n\}, \{y_n\}, \{f(x_n)\}, \{BS_nu_n\} \) and \( \{S_ny_n\} \). Without loss of generality, we may assume \( \{x_n\}, \{u_n\}, \{y_n\}, \{f(x_n)\}, \{BS_nu_n\}, \{S_ny_n\} \subset K \), where \( K \) is a bounded subset of \( C \).

Step 2. We claim, \( \lim_{n \to \infty} ||x_{n+1} - x_n|| = 0 \). Since \( K \) is bounded, \( \{S_ny_n - y_n\}, \{f(x_n)\}, \{BS_nu_n\} \) are bounded. Set

\[
M = \sup\{||S_ny_n - y_n||, ||f(x_n)||, ||BS_nu_n|| : n \in \mathbb{N}\}.
\]

By the definition of \( \{x_n\} \),

\[
||x_{n+2} - x_{n+1}|| = \|(1 - \beta_{n+1})y_{n+1} + \beta_{n+1}S_{n+1}y_{n+1} - (1 - \beta_n)y_n - \beta_nS_ny_n\|
\]

\[
= \|(1 - \beta_{n+1})y_{n+1} + \beta_{n+1}S_{n+1}y_{n+1} - (1 - \beta_{n+1})y_n - \beta_nS_ny_n + (1 - \beta_{n+1})y_n - \beta_nS_ny_n + \beta_{n+1}S_ny_n\|
\]

\[
= \|(1 - \beta_{n+1})(y_{n+1} - y_n) + \beta_{n+1}(S_{n+1}y_{n+1} - S_ny_n) + (\beta_{n+1} - \beta_n)(S_ny_n - y_n)\|
\]

\[
\leq (1 - \beta_{n+1})||y_{n+1} - y_n|| + \beta_{n+1}||S_{n+1}y_{n+1} - S_ny_n|| + |\beta_{n+1} - \beta_n|M
\]

\[
\leq (1 - \beta_{n+1})||y_{n+1} - y_n|| + \beta_{n+1}||S_{n+1}y_{n+1} - S_ny_n|| + |\beta_{n+1} - \beta_n|M
\]

\[
\leq ||y_{n+1} - y_n|| + ||S_{n+1}y_{n+1} - S_ny_{n+1}|| + |\beta_{n+1} - \beta_n|M.
\]

(3.11)

for all \( n \in \mathbb{N} \). From (1.7),

\[
||y_{n+1} - y_n|| = ||\alpha_{n+1}\beta f(x_{n+1}) + (I - \alpha_{n+1}B)S_{n+1}u_{n+1} - \alpha_n\beta f(x_n) - (I - \alpha_nB)S_nu_n||
\]

\[
= \||(I - \alpha_{n+1}B)(S_{n+1}u_{n+1} - S_nu_n) - (\alpha_{n+1} - \alpha_n)BS_nu_n + \alpha_{n+1}\beta f(x_{n+1}) - f(x_n) + (\alpha_{n+1} - \alpha_n)\beta f(x_{n+1})||
\]

\[
\leq (1 - \alpha_{n+1}\gamma)\beta ||S_{n+1}u_{n+1} - S_nu_n|| + |\alpha_{n+1} - \alpha_n||||BS_nu_n||
\]

\[
+ |\alpha_{n+1} - \alpha_n|\gamma ||f(x_{n+1}) - f(x_n)||
\]

\[
\leq (1 - \alpha_{n+1}\gamma)\beta ||S_{n+1}u_{n+1} - S_nu_n|| + |\alpha_{n+1} - \alpha_n|M + |\alpha_{n+1} - \alpha_n|\gamma M
\]

\[
\leq (1 - \alpha_{n+1}\gamma)\beta ||u_{n+1} - u_n|| + |\alpha_{n+1} - \alpha_n|M + |\alpha_{n+1} - \alpha_n|\gamma M
\]

\[
+ |\alpha_{n+1} - \alpha_n|\gamma ||x_{n+1} - x_n|| + ||S_{n+1}u_n - S_nu_n||,
\]

(3.12)

for all \( n \in \mathbb{N} \). On the other hand, \( u_n = T_{r_n}(x_n - r_nAx_n) \) (by Lemma 2.4). From Lemma 3.3,

\[
||u_{n+1} - u_n|| \leq ||x_{n+1} - x_n|| + |r_n - r_{n+1}|M_1,
\]

(3.13)

where \( M_1 = \sup\{||Ax_n|| + \frac{1}{n}||u_{n+1} - k_{n+1}|| : n \in \mathbb{N}\} \). Substituting (3.13) in (3.12),
Therefore, by Lemma 2.6, \( \lim_{n \to \infty} \|y_{n+1} - y_n\| = 0 \). Substituting (3.14) in (3.11), we have

\[
\|x_{n+2} - x_{n+1}\| \leq [1 - \alpha_{n+1}(\tau - \gamma k)]\|x_{n+1} - x_n\| + M_2(|r_n - r_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\}.
\]

(3.15)

Now, we show that under any of three conditions \((H_1)-(H_3)\), \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\) as follows:

Let \((H_1)\) holds. Set \(\mu_n = M_2(|r_n - r_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\}\), then

\[
\sum_{n=1}^{\infty} \mu_n = M_2 \sum_{n=1}^{\infty} (|r_n - r_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\} < \infty.
\]

Therefore, by Lemma 2.6, \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\).

If \((H_2)\) holds, then from (3.15),

\[
\|x_{n+2} - x_{n+1}\| \leq [1 - \alpha_{n+1}(\tau - \gamma k)]\|x_{n+1} - x_n\| + M_2M_1|1 - \frac{\alpha_n}{\alpha_{n+1}}| + M_2(|r_{n+1} - r_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\}.
\]

Set \(\mu_n = M_2(|r_{n+1} - r_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\}\), then

\[
\sum_{n=1}^{\infty} \mu_n = M_2 \sum_{n=1}^{\infty} (|r_{n+1} - r_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\} < \infty.
\]

Therefore, by Lemma 2.6, \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\).

If \((H_3)\) holds, then from (3.15),

\[
\|x_{n+2} - x_{n+1}\| \leq [1 - \alpha_{n+1}(\tau - \gamma k)]\|x_{n+1} - x_n\| + M_2o(\alpha_{n+1}) + M_2(|r_n + |r_{n+1} - r_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\}.
\]

Set \(\mu_n = M_2(|r_n + |r_{n+1} - r_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\}\), then

\[
\sum_{n=1}^{\infty} \mu_n = M_2 \sum_{n=1}^{\infty} (\sigma_n + |r_{n+1} - r_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S.nz\| : z \in K\} < \infty.
\]
Therefore, by Lemma 2.6, \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \)

Step 3. We claim, \( \lim_{n \to \infty} \| x_n - y_n \| = 0. \) Indeed, from \((B_2)\) and \((1.7), \)

\[
\lim_{n \to \infty} \| x_{n+1} - y_n \| = \lim_{n \to \infty} \beta_n \| y_n - S_n y_n \| = 0.
\]

So, from step 2 and \( \| x_n - y_n \| \leq \| x_{n+1} - x_n \| + \| x_{n+1} - y_n \|, \) we get

\[
\lim_{n \to \infty} \| x_n - y_n \| = 0.
\]

Step 4. We claim \( \lim_{n \to \infty} \| x_n - u_n \| = 0, \lim_{n \to \infty} \| y_n - u_n \| = 0 \) and \( \lim_{n \to \infty} \| S u_n - u_n \| = 0. \) To this end, let \( p \in F. \) Then

\[
\| u_n - p \|^2 = \| T_n(x_n - r_n A x_n) - T_n(p - r_n A p) \|^2 \\
\leq \| x_n - r_n A x_n - p + r_n A p \|^2 \\
= \| x_n - p \|^2 + r_n^2 \| A x_n - A p \|^2 - 2 r_n \langle x_n - p, A x_n - A p \rangle \\
\leq \| x_n - p \|^2 + r_n (r_n - 2 \alpha) \| A x_n - A p \|^2.
\]

Therefore

\[
\| x_{n+1} - p \|^2 \leq (1 - \beta_n) \| y_n - p \|^2 + \beta_n \| S_n y_n - p \|^2 \\
\leq \| y_n - p \|^2 + \| \alpha_n \gamma f(x_n) + (1 - \alpha_n B) S_n u_n - p \|^2 \\
= \| \alpha_n \gamma f(x_n) - B S_n u_n \|^2 + \| S_n u_n - p \|^2 \\
+ 2 \alpha_n \langle \gamma f(x_n) - B S_n u_n, S_n u_n - p \rangle \\
\leq \alpha_n^2 \| \gamma f(x_n) - B S_n u_n \|^2 + \| u_n - p \|^2 \\
+ 2 \alpha_n \langle \gamma f(x_n) - B S_n u_n, S_n u_n - p \rangle \\
\leq \alpha_n^2 \| \gamma f(x_n) - B S_n u_n \|^2 + \| x_n - p \|^2 + r_n (r_n - 2 \alpha) \| A x_n - A p \|^2 \\
+ 2 \alpha_n \| \gamma f(x_n) - B S_n u_n \| \| u_n - p \|.
\]

This implies

\[
r_n (2 \alpha - r_n) \| A x_n - A p \|^2 \leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + \alpha_n^2 \| \gamma f(x_n) - B S_n u_n \|^2 \\
+ 2 \alpha_n \| \gamma f(x_n) - B S_n u_n \| \| u_n - p \| \\
\leq \| x_n - p \|^2 + \| x_{n+1} - p \| \| x_{n+1} - x_n \| \\
+ \alpha_n^2 \| \gamma f(x_n) - B S_n u_n \|^2 \\
+ 2 \alpha_n \| \gamma f(x_n) - B S_n u_n \| \| u_n - p \|.
\]

From \( \lim_{n \to \infty} \alpha_n = 0, \) \( r_n \in [a, b] \subset (0, 2 \alpha) \) and \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0, \)

\[
\lim_{n \to \infty} \| A x_n - A p \| = 0.
\]
Also, from \((II)\) in Lemma 2.4,

\[
\|u_n - p\|^2 = \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(p - r_n Ap)\|^2 \\
\leq \left(\|x_n - r_n Ax_n - (p - r_n Ap)\| + \|u_n - p\|\right)^2 \\
= \frac{1}{2}\left(\|x_n - r_n Ax_n - (p - r_n Ap)\|^2 + \|u_n - p\|^2 \right) \\
\leq \frac{1}{2}\left(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(Ax_n - Ap)\|^2\right) \\
= \frac{1}{2}\left(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n(x_n - u_n, Ax_n - Ap) - r_n^2\|Ax_n - Ap\|^2\right).
\]

This implies

\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n(x_n - u_n, Ax_n - Ap) - r_n^2\|Ax_n - Ap\|^2. \\
\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n(x_n - u_n, Ax_n - Ap) \\
\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\||Ax_n - Ap|.
\]

By the same argument in \((3.16)\),

\[
\|x_{n+1} - p\|^2 \leq \alpha_n^2\|\gamma f(x_n) - BS_n u_n\|^2 + \|u_n - p\|^2 \\
+2\alpha_n\|\gamma f(x_n) - BS_n u_n, S_n u_n - p\| \\
\leq \alpha_n^2\|\gamma f(x_n) - BS_n u_n\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 \\
+2r_n\|x_n - u_n\||Ax_n - Ap| + 2\alpha_n\|\gamma f(x_n) - BS_n u_n\|\|u_n - p||. \\
\]

Therefore

\[
\|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\|\gamma f(x_n) - BS_n u_n\|^2 \\
+2r_n\|x_n - u_n\||Ax_n - Ap| + 2\alpha_n\|\gamma f(x_n) - BS_n u_n\|\|u_n - p|| \\
\leq \left(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\right)\|x_{n+1} - x_n\|^2 + \alpha_n^2\|\gamma f(x_n) - BS_n u_n\|^2 \\
+2r_n\|x_n - u_n\||Ax_n - Ap| + 2\alpha_n\|\gamma f(x_n) - BS_n u_n\|\|u_n - p||. \\
\]

Then \(\lim_{n \to \infty} \alpha_n = 0\), \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\) and \((3.17)\) show

\[
\lim_{n \to \infty} \|x_n - u_n\| = 0.
\]

Moreover, from \(\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\\) and step 3,

\[
\lim_{n \to \infty} \|y_n - u_n\| = 0.
\]

Since

\[
\|S_n u_n - u_n\| \leq \|S_n u_n - y_n\| + \|y_n - u_n\| \\
\leq \alpha_n\|\gamma f(x_n) - BS_n u_n\| + \|y_n - u_n\|, \\
\]

we have \(\lim_{n \to \infty} \|S_n u_n - u_n\| = 0\). Observe

\[
\|S u_n - u_n\| \leq \|S u_n - S_n u_n\| + \|S_n u_n - u_n\| \\
\leq \sup\{\|Sz - S_n z\| : z \in K\} + \|S_n u_n - u_n\|.
\]
From Lemma 2.7, \( \lim_{n \to \infty} \|Su_n - u_n\| = 0 \).

Step 5. We claim, \( \limsup_{n \to \infty} (\gamma f(q) - Bq, y_n - q) \leq 0 \), where
\[
q = P_{\bigcap_{n=1}^\infty F(S_n)} \cap EP(I - B + \gamma f)(q).
\]
To show this, choose a subsequence \( \{u_{n_i}\} \) of \( \{u_n\} \) such that
\[
\limsup_{n \to \infty} \langle (B - \gamma f)q, q - u_n \rangle = \lim_{i \to \infty} \langle (B - \gamma f)q, q - u_{n_i} \rangle.
\]
Since \( \{u_{n_i}\} \) is bounded in \( C \), without loss of generality, we assume \( u_{n_i} \to z \in C \).
From \( \lim_{n \to \infty} \|Su_n - u_n\| = 0 \) and Lemma 2.2, \( z \in EP \). Now, we show \( z \in EP \).
By \( u_n = T_{r_n}(x_n - r_nAx_n) \), one can write
\[
\phi(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C.
\]
From (A2),
\[
\langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \phi(y, u_n), \quad \text{for all } y \in C.
\]
Replacing \( n \) by \( n_i \), we have
\[
\langle Ax_{n_i}, y - u_{n_i} \rangle + \frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq \phi(y, u_{n_i}), \quad \text{for all } y \in C. \tag{3.18}
\]
Set \( y_t = ty + (1 - t)z \), for all \( t \in (0, 1] \) and \( y \in C \). Then \( y_t \in C \). So, from (3.18),
\[
\langle y_t - u_{n_i}, Ay_t \rangle \geq \langle y_t - u_{n_i}, Ay_{n_i} \rangle - \langle Ax_{n_i}, y_t - u_{n_i} \rangle - \langle y_t - u_{n_i}, u_{n_i} - x_{n_i} \rangle + \phi(y_t, u_{n_i})
\]
\[
= \langle y_t - u_{n_i}, Ay_t - Au_{n_i} \rangle + \langle y_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle - \langle y_t - u_{n_i}, u_{n_i} - x_{n_i} \rangle + \phi(y_t, u_{n_i}).
\]
Since \( \lim_{i \to \infty} \|u_{n_i} - x_{n_i}\| = 0 \), we have \( \lim_{i \to \infty} \|Au_{n_i} - Ax_{n_i}\| = 0 \). Further, from the monotonicity of \( A \), \( \langle y_t - u_{n_i}, Ay_t - Au_{n_i} \rangle \geq 0 \). So, from (A1),
\[
\langle y_t - z, Ay_t \rangle \geq \phi(y_t, z), \tag{3.19}
\]
as \( i \to \infty \). From (A1), (A2) and (3.19),
\[
0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1 - t)\phi(y_t, z)
\]
\[
\leq t\phi(y_t, y) + (1 - t)\langle y_t - z, Ay_t \rangle
\]
\[
= t\phi(y_t, y) + (1 - t)\langle y - z, Ay_t \rangle,
\]
hence
\[
0 \leq \phi(y_t, y) + (1 - t)\langle y - z, Ay_t \rangle.
\]
Letting $t \to 0$, we get
\[ 0 \leq \phi(z, y) + \langle y - z, Az \rangle, \quad \text{for all } y \in C. \]

This implies $z \in EP$. Since $q = \bigcap_{n=1}^{\infty} F(S_{n}) \cap EP(I - B + \gamma f)(q)$,
\[
\limsup_{n \to \infty} \langle (B - \gamma f)(q), q - y_{n} \rangle = \lim_{n \to \infty} \langle (B - \gamma f)(q), q - y_{n} \rangle \\
= \lim_{n \to \infty} \langle (B - \gamma f)(q), q - u_{n} \rangle \\
= \lim_{n \to \infty} \langle (B - \gamma f)(q), q - z \rangle \leq 0.
\]

Step 6. We claim $\{u_{n}\}$ and $\{x_{n}\}$ converges strongly to $q$. From (1.7),
\[
\|x_{n+1} - q\|^2 = \| (1 - \beta_{n})(y_{n} - q) + \beta_{n}(S_{n}y_{n} - q)\|^2 \\
\leq \|y_{n} - q\|^2 = \| \alpha_{n} \gamma f(x_{n}) + (I - \alpha_{n}B)S_{n}u_{n} - q\|^2 \\
= \| \alpha_{n}(\gamma f(x_{n}) - Bq) + (I - \alpha_{n}B)(S_{n}u_{n} - q)\|^2 \\
\leq \| (I - \alpha_{n}B)(S_{n}u_{n} - q)\|^2 + 2\alpha_{n} \gamma f(x_{n}) - \gamma f(q), y_{n} - q) \\
\leq (1 - \alpha_{n} \gamma)^{2}\|u_{n} - q\|^2 + 2\alpha_{n} \gamma f(x_{n}) - \gamma f(q), y_{n} - q) \\
\leq (1 - \alpha_{n} \gamma)^{2}\|x_{n} - q\|^2 + 2\alpha_{n} \gamma f(x_{n}) - \gamma f(q), y_{n} - q) \\
\leq (1 - 2\alpha_{n} \gamma)^{2}\|x_{n} - q\|^2 + (\alpha_{n} \gamma)^{2}\|x_{n} - q\|^2 \\
+ 2\alpha_{n} \gamma f(x_{n}) - \gamma f(q), y_{n} - q) \\
\leq (1 - 2\alpha_{n} \gamma)^{2}\|x_{n} - q\|^2 + 2\alpha_{n} \gamma f(x_{n}) - \gamma f(q), y_{n} - q) \\
\leq \frac{\gamma_{n} M_{3}}{2(\gamma - \gamma)}\|y_{n} - x_{n}\| + \frac{1}{\gamma - \gamma} \gamma f(q), y_{n} - q) \\
= (1 - \delta_{n})\|x_{n} - q\|^2 + \delta_{n} \theta_{n},
\]
where $M_{3} = \sup\{\|x_{n} - q\| : n \geq 1\}$, $\delta_{n} = 2\alpha_{n} \gamma$ and $\theta_{n} = (\alpha_{n} \gamma)^{2} M_{3}^{2} + \frac{\gamma_{n} M_{3}}{2(\gamma - \gamma)}\|y_{n} - x_{n}\| + \frac{1}{\gamma - \gamma} \gamma f(q), y_{n} - q)$. It is easy to see that $\lim_{n \to \infty} \delta_{n} = 0$, $\sum_{n=1}^{\infty} \delta_{n} = \infty$ and $\limsup_{n \to \infty} \theta_{n} \leq 0$. Hence, by Lemma 2.6, $\{x_{n}\}$ converges strongly to $q$. Consequently, $\{u_{n}\}$ converges strongly to $q$. This completes the proof. 

**Remark 3.5.** Theorem 3.4 is a generalization of [15, Theorem 3.1].

To see this, Set $A = 0$ in Theorem 3.4, and assume $r_{n} \geq a > 0$ (it is not necessary to assume $\{r_{n}\} \subset [a, b] \subset (0, 2\alpha))$.

**References**


