Characterizations of Rayleigh Distribution based on Order Statistics and Record Values

M. Ahsanullah
Department of Management Sciences
Rider University
Lawrenceville, NJ 08648, USA
E-mail: ahsan@rider.edu

and

M. Shakil
Department of Mathematics
Miami Dade College, Hialeah Campus
Hialeah, Fl 33012, USA
E-mail: mshakil@mdc.edu

Abstract

In this paper, some new results on characterizations of Rayleigh distribution based on order statistics and record values have been established. The conditional expectation (also known as mean residual life) of a distribution plays important roles in modeling and analysis of life time data. Some results on characterizations based on mean residual life have also been established.


Keywords and Phrases: Characterization, Conditional Expectation, Order Statistics, Rayleigh Distribution, Record Values.
1. Introduction

Many researchers have studied the characterizations of probability distributions. For example, Su and Huang (2000) studied the characterizations based on conditional expectations. Recently, Nanda (2010) studied the characterizations through the expected values of failure rate and mean residual life functions of a nonnegative absolutely continuous random variable $X$. The problems of characterizations based on record values and order statistics started in late sixties by Tata (1969), and followed in seventies by Nagaraja (1977) and Ahsanullah (1979). For further development and various characterizations of probability distributions based on record values and order statistics, the interested readers are referred to Arnold et al. (1998), Rao and Shanbhag (1998), Ahsanullah (1991, 2004, 2009), Bairamov et al. (2005), Yanev et al. (2007), and Ahsanullah and Aliev (2008), among others. In this paper, some new results on characterizations of Rayleigh distribution based on order statistics and record values have been established.

Rayleigh Distribution: A random variable $X$ is said to have a Rayleigh distribution if its probability density function (pdf) $f$ is given by

$$f(x) = \begin{cases} 2ce^{-cx^2}, & x > 0, c > 0, \\ 0, & \text{otherwise}, \end{cases} \quad (2.1)$$

with the corresponding cumulative distribution function (cdf) $F$ is given by

$$F(x) = 1 - e^{-cx^2},$$

where $c > 0$ is known as the scale parameter of Rayleigh distribution. For detailed treatment on Rayleigh distribution, the interested readers are referred to Johnson et al. (1994).

Record Values: Suppose that $(X_n)_{n \geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random variables (rv’s) with cdf $F$. Let $Y_n = \max \{ \min \{ X_j | 1 \leq j \leq n \} \}$ for $n \geq 1$. We say $X_j$ is an upper (lower) record value of $\{X_n | n \geq 1\}$, if $Y_j > ( < ) Y_{j-1}, j > 1$. By definition $X_1$ is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = \min \{ j | j > U(n-1), X_j > X_{U(n-1)}, n > 1 \}$ and $U(1) = 1$. Many properties of the upper record value sequence can be expressed in terms of the cumulative hazard rate function $R(x) = -\ln \overline{F(x)}$, where $\overline{F(x)} = 1 - F(x)$, $0 < \overline{F(x)} < 1$. We will denote $X_{U(n)}$ by $X(n)$ and $F_n(x)$ as the cdf of $X(n)$ for $n \geq 1$. We have

$$F_n(x) = \int_{-\infty}^{x} \frac{R(u)^{n-1}}{R(n)} \, dF(u), \quad -\infty < x < \infty,$$
from which it is easy to see that
\[ F_n(x) = 1 - F(x) \sum_{j=0}^{n-1} \frac{(\theta(x))^j}{\Gamma(j+1)}, \]
that is,
\[ F_n(x) = F(x) \sum_{j=0}^{n-1} \frac{(\theta(x))^j}{\Gamma(j+1)}. \]  

(2.2)

We assume \( F(x) \) is absolutely continuous with respect to Lebesgue measure and denote \( f_n(x) \) as the pdf of \( X(n) \), where
\[ f_n(x) = \frac{(\theta(x))^n}{\Gamma(n)} f(x), \quad -\infty < x < \infty. \]  

(2.3)

From (2.2) and (2.3) it is easy to show that
\[ F_n(x) - F_{n-1}(x) = F(x) \frac{f_n(x)}{f(x)}. \]  

For details on record values, see Ahsanullah (2004). Using Eq. (2.3), the pdf and cdf of the \( nth \) record value \( X(n) \) from Rayleigh \((c)\) distribution are, respectively, given by
\[ f_n(x) = \frac{2c^n x^{2n-1} e^{-cx^2}}{\Gamma(n)}, \quad n = 1, 2, 3, \ldots, \]  

(2.4)

and
\[ F_n(x) = \frac{\gamma(n, cx^2)}{\Gamma(n)}, \quad n = 1, 2, 3, \ldots, \]  

(2.5)

where \( x > 0 \), \( c > 0 \), and \( \gamma(\alpha, z) = \int_0^z t^{\alpha-1} e^{-t} dt \), \( \alpha > 0 \), denotes incomplete gamma function. The \( kth \) moment of the \( nth \) record value \( X(n) \) with the pdf (2.4) is given by
\[ E[X^k(n)] = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(n)} c^{-\frac{k}{2}}. \]  

(2.6)

In this paper, some new results on characterizations of Rayleigh distribution based on order statistics and record values have been established. The organization of this paper is as follows. Section 2 contains characterizations based on conditional expectations. In Section 3, some new results based on order statistics have been established. Section 4 contains characterizations based on record values.

2. Characterizations Based on Conditional Expectations
We first prove the following two lemmas.

Lemma 1

Case (i)

Let $X$ be an absolutely continuous (with respect to Lebesgue measure) random variable with cumulative distribution function $F(x)$ with $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. If for any non-negative number $s$, any $t > 0$ and $c > 0$,

$$\int_0^t \frac{(1-F(x))'}{(1-F(t))'} dx = \frac{1}{2} \sqrt{\frac{2}{cs}} (1 - \text{erf}(\sqrt{cs}t)) e^{ct^2},$$

where $\text{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2} dt$ denotes the error function, then $F(x) = 1 - e^{-cx^2}$, $x \geq 0$, $c > 0$.

Proof

We have

$$\int_0^t (1 - F(x))' dx = \frac{1}{2} \sqrt{\frac{2}{cs}} (1 - \text{erf}(\sqrt{cs}t)) e^{ct^2}. \quad (3.1)$$

Noting $\frac{d}{dt} \left( \frac{1}{2} \sqrt{\frac{2}{cs}} (1 - \text{erf}(\sqrt{cs}t)) e^{ct} \right) = -1 + t \sqrt{\pi cs} e^{ct} \left( 1 - \text{erf}(\sqrt{cs}t) \right)$, and differentiating both sides of (3.1) with respect to $t$, we obtain

$$- (1 - F(t))' = \frac{1}{2} \sqrt{\frac{2}{cs}} (1 - \text{erf}(\sqrt{cs}t)) e^{ct} f(t)$$

$$+ (1 - F(t))' \left( -1 + t \sqrt{\pi cs} e^{ct} \left( 1 - \text{erf}(\sqrt{cs}t) \right) \right).$$

On simplifying, we have from the above equation

$$\frac{f(t)}{1-F(t)} = 2ct \quad (3.2)$$

On integrating (3.2) with respect to $t$ and using the boundary conditions $F(0) = 0$ and $F(x) = 1$, we have

$$F(x) = 1 - e^{-cx^2}, \quad x \geq 0, c > 0.$$
Case (ii)

Let $X$ be an absolutely continuous (with respect to Lebesgue measure) random variable with cumulative distribution function $F(x)$ with $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. If for any non-negative number $s$, any $t > 0$, $k \geq 2$ and $c > 0$, 

$$\int_{t}^{\infty} (2k-1)x^{2k-2}(1-F(x))'\,dx$$

$$= \sum_{j=0}^{k-2} \frac{(2k-1)!}{(2x)^{2j}(2k-3-2j)!} t^{2k-3-2j} + \frac{(2k-1)!\sqrt{\pi}}{(2x)^{k}} (1-\text{erf}(\sqrt{cs})) e^{cs^2},$$

(3.3)

where $(2k-1)! = 1.3.5... (2k-1)$, then $F(x) = 1 - e^{-cx^2}$, $x \geq 0$, $c > 0$. 

Proof

We have from (3.3)

$$\int_{t}^{\infty} (2k-1)x^{2k-2}(1-F(x))'\,dx$$

$$= (1 - F(t))' \left[ \sum_{j=0}^{k-2} \frac{(2k-1)!}{(2x)^{2j}(2k-3-2j)!} t^{2k-3-2j} + \frac{(2k-1)!\sqrt{\pi}}{(2x)^{k}} (1-\text{erf}(\sqrt{cs})) e^{cs^2} \right].$$

(3.4)

Noting

$$\frac{(2k-1)!\sqrt{\pi}}{(2x)^{k}} \frac{d}{dt} (1-\text{erf}(\sqrt{cs} t)) e^{cs^2} = \frac{(2k-1)!}{(2x)^{k}} (-1 + e^{cs^2} t\sqrt{\pi cs} (1-\text{erf}(t\sqrt{cs}))),$$

and

$$\frac{d}{dt} \left[ \sum_{j=0}^{k-3} \frac{(2k-1)!}{(2x)^{2j}(2k-3-2j)!} t^{2k-3-2j} \right] = \frac{d}{dt} \left[ \sum_{j=0}^{k-3} \frac{(2k-1)!}{(2x)^{2j}(2k-3-2j)!} t^{2k-3-2j} \right]$$

$$= \left[ \frac{(2k-1)!}{(2x)^{k}} + \sum_{j=0}^{k-3} (2k-3-2j) \frac{(2k-1)!}{(2x)^{2j}(2k-3-2j)!} t^{2k-3-2j-1} \right],$$

we have, on simplification,

$$\frac{d}{dt} \left[ \sum_{j=0}^{k-3} \frac{(2k-1)!}{(2x)^{2j}(2k-3-2j)!} t^{2k-3-2j} \right] + \frac{(2k-1)!\sqrt{\pi}}{(2x)^{k}} (1-\text{erf}(\sqrt{cs} t)) e^{cs^2}$$

$$= \sum_{j=0}^{k-3} (2k-3-2j) \frac{(2k-1)!}{(2x)^{2j}(2k-3-2j)!} t^{2k-3-2j-1} + \frac{(2k-1)!\sqrt{\pi}}{(2x)^{k}} e^{cs^2} t\sqrt{\pi cs} (1-\text{erf}(t\sqrt{cs})).$$
Thus differentiating both sides of (3.4) with respect to $t$ and using the above equation, we obtain

$$-(2k-1)t^{2k-2} \left(1 - F(t)\right)^x$$

$$= \left(1 - F(t)\right)^x \left[\sum_{j=0}^{k-3} (2k-3-2j) \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!} t^{2k-3-2j-1} + \frac{(2k-1)!}{(2cs)^t} e^{ct} t^{\sqrt{\pi cs}} (1 - \text{erf}(t^{\sqrt{cs}}))\right]$$

$$- s(1 - F(t))^{x-1} f(t) \left[\sum_{j=0}^{k-3} (2k-3-2j) \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!} t^{2k-3-2j-1} + \frac{(2k-1)!}{(2cs)^t} e^{ct} t^{\sqrt{\pi cs}} (1 - \text{erf}(t^{\sqrt{cs}}))\right]$$

from which we have

$$s(f(t))^{-1} f(t) \left[\sum_{j=0}^{k-3} (2k-3-2j) \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!} t^{2k-3-2j-1} + \frac{(2k-1)!}{(2cs)^t} e^{ct} t^{\sqrt{\pi cs}} (1 - \text{erf}(t^{\sqrt{cs}}))\right]$$

$$= (2k-1)t^{2k-2} + \sum_{j=0}^{k-3} (2k-3-2j) \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!} t^{2k-3-2j-1} + \frac{(2k-1)!}{(2cs)^t} e^{ct} t^{\sqrt{\pi cs}} (1 - \text{erf}(t^{\sqrt{cs}})).$$

In the above equation, noting that

$$(2k-1)t^{2k-2} + \sum_{j=0}^{k-3} (2k-3-2j) \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!} t^{2k-3-2j-1}$$

$$= \frac{2cs(2k-1)!}{2cs(2k-3)!!} t^{2k-2} + \sum_{j=0}^{k-3} \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!!} t^{2k-3-2j-1}$$

$$= \frac{2cs(2k-1)!}{2cs(2k-3)!!} t^{2k-2} + 2cst \sum_{j=0}^{k-3} \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!!} t^{2k-3-2j}$$

$$= \frac{2cs(2k-1)}{2cs(2k-3)!!} t^{2k-2} + \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!!} t^{2k-3-2j}$$

$$= 2cst \sum_{j=0}^{k-3} \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!!} t^{2k-3-2j}$$

we obtain, on simplification, the following equation

$$s(f(t))^{-1} f(t) \left[\sum_{j=0}^{k-3} (2k-3-2j) \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!} t^{2k-3-2j-1} + \frac{(2k-1)!}{(2cs)^t} e^{ct} t^{\sqrt{\pi cs}} (1 - \text{erf}(t^{\sqrt{cs}}))\right]$$

$$= 2cst \left[\sum_{j=0}^{k-3} \frac{(2k-1)!}{(2cs)^{j+1}(2k-3-2j)!!} t^{2k-3-2j} + \frac{(2k-1)!}{(2cs)^t} e^{ct} t^{\sqrt{\pi cs}} (1 - \text{erf}(t^{\sqrt{cs}}))\right].$$

Hence
\[ \frac{f(t)}{1-F(t)} = 2ct. \]

Thus on integrating the above equation with respect to \( t \) and using the boundary conditions \( F(0) = 0 \) and \( F(\infty) = 1 \), we have

\[ F(x) = 1 - e^{-cx^2}, \quad x \geq 0, \ c > 0. \]

This completes the proof of Lemma 1.

**Lemma 2**

Let \( X \) be an absolutely continuous (with respect to Lebesgue measure) random variable with cumulative distribution function \( F(x) \) with \( F(0) = 0 \) and \( F(x) > 0 \) for all \( x > 0 \). If for any non-negative number \( s \), any \( t > 0 \), \( k \geq 2 \) and \( c > 0 \),

\[ \int_0^\infty 2kx^{2k-1}(1-F(x))^s \, dx = \sum_{j=1}^{k} \frac{k^{(j)}(2j-1)}{(cx)^j}, \tag{3.5} \]

where \( k^{(j)} = k(k-1)...(k-j+1) \), then \( F(x) = 1 - e^{-cx^2}, \ x \geq 0, \ c > 0. \)

**Proof**

We have from (3.5)

\[ \int_0^\infty 2kx^{2k-1}(1-F(x))^s \, dx = (1-F(t))^s (\sum_{j=1}^{k} \frac{k^{(j)}2j-1}{(cx)^j}). \tag{3.6} \]

Differentiating both sides of (3.6) with respect to \( t \), we obtain

\[ -2kt^{2k-1}(1-F(t))^s = -s(1-F(t))^{s-1} f(t) \sum_{j=1}^{k} \frac{k^{(j)}2j-1}{(cx)^j} \]

\[ + (1-F(t))^s \sum_{j=1}^{k} 2(k-j)k^{(j)}t^{2j-2}j^{-1} \frac{1}{(cx)^j}, \]

that is,

\[ s (1-F(t))^{s-1} f(t) \sum_{j=1}^{k} \frac{k^{(j)}2j-1}{(cx)^j} \]

\[ = (1-F(t))^s \sum_{j=1}^{k} 2(k-j)k^{(j)}t^{2j-2}j^{-1} \frac{1}{(cx)^j} + 2kt^{2k-1}(1-F(t))^s \]
\[ = (1 - F(t))^2 t \left( \sum_{j=1}^{k+1} \frac{(k-j)k^{2j-2}}{(2^j)^{2j}} \right) + k t^{2k-2} \]

\[ = (1 - F(t))^2 c s t \left( \sum_{j=1}^{k} \frac{k^{2j-2}}{(2^j)^{2j}} \right). \]

Simplifying the above equation, we have

\[ \frac{f(t)}{1-F(t)} = 2ct. \]

Thus on integrating the above equation with respect to \( t \) and using the boundary conditions \( F(0) = 0 \) and \( F(\infty) = 1 \), we have

\[ F(x) = 1 - e^{-cx^2}, \quad x \geq 0, \quad c > 0. \]

This completes the proof of Lemma 2.

**Theorem 1**

Let \( X \) be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with \( F(0) = 0, \quad F(x) > 0 \) for all \( x > 0 \) and finite \( E(X^{2n}) \), for some fixed \( n \geq 1 \). Then \( X \) has a Rayleigh distribution with \( F(x) = 1 - e^{-cx^2}, \quad x \geq 0, \quad c > 0, \) iff \( E(X^{2n} | X > t) = \sum_{i=0}^{n} \frac{n^{(i)} t^{2(n-i)}}{c^{(i)}} \), where \( n^{(i)} = n(n-1)...(n-i+1), \quad n^{(0)} = 1. \)

**Proof**

It is easy to show that if

\[ F(x) = 1 - e^{-cx^2}, \quad x \geq 0, \quad c > 0, \]

then

\[ E(X^{2n} | X > t) = \sum_{i=0}^{n} \frac{n^{(i)} t^{2(n-i)}}{c^{(i)}}. \]

We will prove here the "only if" condition. Suppose

\[ E(X^{2n} | X > t) = \sum_{i=0}^{n} \frac{n^{(i)} t^{2(n-i)}}{c^{(i)}}, \]

then we have
\[ E(X^{2n} | X > t) = \int_{-F(t)}^{+\infty} x^2 f(x) dx = t^{2n} + \int_{-F(t)}^{+\infty} t^{2n-2} (1-F(x)) dx. \] (3.7)

Hence, using Lemma 2 in Eq. (3.7), the "only if" condition easily follows.

**Theorem 2**

Let \( X \) be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with \( F(0) = 0 \) and \( F(x) > 0 \) for all \( x > 0 \) and finite \( E(X^{2n+1}) \), for some fixed \( n \geq 1 \). Then \( X \) has a Rayleigh distribution with \( F(x) = 1 - e^{-cx^2}, \; x \geq 0, \; c > 0, \) iff

\[ E(X^{2n-1} | X > t) = \sum_{j=0}^{n-1} \frac{(2n-1)!}{(2n-1-2j)!} \cdot t^{2n-1-2j} \]

\[ + \frac{(2n-1)!}{(2c)^j} \sqrt{c\pi} (1 - \text{erf} (\sqrt{c} t)) e^{ct^2}, \]

where \((2n-1)! = 1.3... (2n-1), \; n \geq 1.\)

**Proof**

It is easy to show that if

\[ F(x) = 1 - e^{-cx^2}, \; x \geq 0, \; c > 0, \]

then

\[ E(X^{2n-1} | X > t) = \sum_{j=0}^{n-1} \frac{(2n-1)!}{(2n-1-2j)!} \cdot t^{2n-1-2j} \]

\[ + \frac{(2n-1)!}{(2c)^j} \sqrt{c\pi} (1 - \text{erf} (\sqrt{c} t)) e^{ct^2}. \]

We will prove here the "only if" condition. Suppose

\[ E(X^{2n-1} | X > t) = \sum_{j=0}^{n-1} \frac{(2n-1)!}{(2n-1-2j)!} \cdot t^{2n-1-2j} \]

\[ + \frac{(2n-1)!}{(2c)^j} \sqrt{c\pi} (1 - \text{erf} (\sqrt{c} t)) e^{ct^2}. \]

Then we have
\[
E(X^{2n+1} | X > t) = \frac{\int_0^t x^{2n+1} f(x) \, dx}{1-F(t)} + \frac{\int_0^t (2n+1)x^{2n} (1-F(x)) \, dx}{1-F(t)}. \tag{3.8}
\]

Using Lemma 1 in Eq. (3.8), the "only if" condition easily follows.

3. Characterizations Based on Order Statistics

In this section, we establish some results based on order statistics.

**Theorem 3**

Let \( X \) be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with \( F(0) = 0 \) and \( F(x) > 0 \) for all \( x > 0 \) and finite \( E(X^2) \). Then \( X \) has the Rayleigh distribution with \( F(x) = 1 - e^{-cx^2}, \ x \geq 0 \), iff

\[
E(X_{i,n}^{2m} | X_{i-1,n} = t) = \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left( \frac{1}{e^{(x-1)x^2}} \right)^j t^{2(m-j)}, \text{ for some fixed } n \geq 1, \ m \geq 1, \text{ where } X_{i,n} \text{ is the } i\text{th order statistics in a sample of size } n. 
\]

**Proof**

Suppose that \( F(x) = 1 - e^{-cx^2}, \ c > 0, \ x > 0 \). Then, it can easily be seen, after integration, that

\[
E(X_{i,n}^{2m} | X_{i-1,n} = t) = \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left( \frac{1}{e^{(x-1)x^2}} \right)^j t^{2(m-j)}. 
\]

Suppose that

\[
E(X_{i,n}^{2m} | X_{i-1,n} = t) = \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left( \frac{1}{e^{(x-1)x^2}} \right)^j t^{2(m-j)}. 
\]

Since

\[
E(X_{i,n}^{2m} | X_{i-1,n} = t) = t^{2m} + \frac{\int_0^t x^{2m-1} (1-F(x)) e^{-cx^2} \, dx}{(1-F(t))e^{-ct}}, 
\]

therefore, we have
\[ \int_{-\infty}^{\infty} 2mx^{2m-1} (1 - F(x))^{n-i+1} \, dx = [(1 - F(t))^{n-i+1}] \sum_{j=1}^{m} \frac{m! (n-j)!}{(n+2m+1)j!} t^{2j} \] 

The result easily follows from Lemma 2.

**Remark:** Let \( X \) be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with \( F(0) = 0 \) and \( F(x) > 0 \) for all \( x > 0 \) and finite \( E(X^2) \). Then \( X \) has a Rayleigh distribution with \( F(x) = 1 - e^{-x^2} \), \( c > 0, x \geq 0 \), iff \( E(X_{i,n}^2 \mid X_{i-1,n} = t) = t^2 + \frac{1}{(n-i)c} \), where \( X_{i,n} \) is the \( ith \) order statistics in a sample of size \( n \).

**Proof**

The proof easily follows by taking \( m = 1 \) in Theorem 3.

**4. Characterization Based on Record Values**

In this section, some results based on record values have been established.

**4.1.** We first prove the following theorem based on conditional expectation.

**Theorem 4**

Let \( X \) be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with \( F(0) = 0 \) and \( F(x) > 0 \) for all \( x > 0 \), Assume that \( E(X_{U_{(n+1)}}) \) is finite. Then \( X \) has a Rayleigh distribution with \( F(x) = 1 - e^{-x^2} \), \( x \geq 0 \), iff

\[ E(X_{U_{(n+1)}} \mid X_{U_{(n)}} = t) = t + \sqrt{2}e^{t^2} - \sqrt{2}e^{t^2} \text{erf}(\sqrt{2}t), \]

for some fixed \( n \geq 1 \).

**Proof**

Since

\[ E(X_{U_{(n+1)}} \mid X_{U_{(n)}} = t) = t + \int_{0}^{t} \frac{(1-F(x)) \, dx}{1-F(t)}, \]

the proof follows from Theorem 2.

**Remarks:** Since the conditional pdf of \( (X_{U_{(n+1)}} \mid X_{U_{(n)}} = t) \) is the same as the conditional pdf of \( (X \mid X > t) \), the characterizations (see Ahsanullah (2004)) using \( E(X_{U_{(n+1)}}^2 \mid X_{U_{(n)}} = t) \) and \( E(X_{U_{(n+1)}}^{2m-1} \mid X_{U_{(n)}} = t) \) are same as given in Theorems 1 and 2.
4.2. It is well known that if $X$ has a Rayleigh distribution with $F(x) = 1 - e^{-cx^2}$, $c > 0$, $x \geq 0$, then $Y = X^2$ has an exponential distribution with $F_Y(y) = 1 - e^{-cy}$. Further, applying Theorem 8.4.1, page 256, Ahsanullah (2004), it is easy to see that, if $X$ be a nonnegative random variable with the Rayleigh distribution, the we have

$$X_{U(a)}^2 = X_1^2 + \ldots + X_n^2, n \geq 1,$$

where $X_{U(a)}$ is the $n$th upper record and $X_1, \ldots, X_n$ are independent copies of $X$. Also see Ahsanullah (2009). Thus, by the definition of the Erlang distribution, it follows that $X_{U(a)}^2$ is distributed as Erlang which is defined as follows: A random variable is said to be distributed as Erlang if its pdf is given by

$$f_{c,n}(x) = \frac{1}{\Gamma(n)} c^n x^{n-1} e^{-cx},$$

where $c > 0$, $x \geq 0$, and $n > 0$ is an integer.

**Theorem 5**

Let $X$ be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ with $F(0) = 0$ and $F(x) > 0$ for all $x > 0$ and finite $E(X^2)$. Then $X$ has a Rayleigh distribution with $F(x) = 1 - e^{-cx^2}$, $c > 0$, $x \geq 0$, iff $X_{U(a)}^2$ is distributed as Erlang, for some fixed $n \geq 1$.

**Proof**

The "if condition" is known. We will prove here the "only if" condition. If $X_{U(a)}^2$ is distributed as Erlang, then

$$\int_0^x \frac{1}{\Gamma(n)} (R(u))^{n-1} f(u) du = \int_0^x \frac{1}{\Gamma(n)} (c)^n u^{n-1} e^{-cu} du \quad \text{for all} \quad x \geq 0.$$

We can rewrite the above equality as

$$\int_0^R(x) \frac{1}{\Gamma(n)} t^{n-1} e^{-t} dt = \int_0^{u^2} \frac{1}{\Gamma(n)} t^{n-1} e^{-t} dt \quad \text{for all} \quad x \geq 0.$$

Thus, we have

$$R(x) = cx^2$$

for almost all $x$, $x \geq 0$.

That is,
\[ F(x) = 1 - e^{-cx^2}, \quad c > 0, \quad x \geq 0. \]

**Theorem 6**

Let \( X \) be a nonnegative random variable with absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with \( F(0) = 0 \) and \( F(x) > 0 \), for all \( x > 0 \).

Assume that \( E(X_{U(n+1)}) \) is finite. Then \( X \) has a Rayleigh distribution with \( F(x) = 1 - e^{-x^2} \), \( x \geq 0 \), iff

\[ E(X_{U(n+1)} | X_{U(n)} = t) = t + \sqrt{\frac{c}{2}}e^{t^2} - \sqrt{\frac{c}{2}}e^{t^2} \text{erf}(\sqrt{2t}), \quad \text{for some fixed } n \geq 1. \]

**Proof**

Since

\[ E(X_{U(n+1)} | X_{U(n)} = t) = t + \int_{t}^{\infty} \frac{(1-F(x))dx}{1-F(t)}, \]

the proof follows from Theorem 2.

**Remarks:** Since the conditional pdf of \( (X_{U(n+1)} | X_{U(n)} = t) \) is \( f_{n+1|n}(x | t) = \frac{f(x)}{1-F(t)}, \quad x > t \), which is exactly equal to the conditional pdf of \( (X | X > t) \), that is, \( f_{n+1|n}(x | t) = \frac{f(x)}{1-F(t)} \), \( x > t \), see, for example, Ahsanullah (2004), the following results related to the characterizations using \( E(X_{U(n+1)}^2 | X_{U(n)} = t) \) and \( E(X_{U(n+1)}^{2m-1} | X_{U(n)} = t) \) are same as given in Theorems 1 and 2 above, that is,

(i) \[ E(X_{U(n+1)}^m | X_{U(n)} = t) = \sum_{j=0}^{m} m^{(j)} \frac{(\frac{1}{c})}{2} t^{2m-2j}, \]

where \( m^{(j)} = m(m-1)\cdots(m-j+1), \quad m^{(0)} = 1 \), and

(ii) \[ E(X_{U(n+1)}^{2m-1} | X_{U(n)} = t) = \sum_{j=0}^{m-1} \frac{(2m-1)!}{(2m-1-2j)!} \frac{(\frac{1}{c})}{2} t^{2m-2j} + \frac{(2m-1)!}{(2c)^{m}} \text{erf}(\sqrt{ct})^{2}. \]

The proofs of the above remarks (i) and (ii) are similar to the proofs of Theorems 1 and 2, respectively.
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