The Linear Arboricity of the Schrijver Graph

\[ SG(2k + 2, k) \]

Bing Xue\(^1,2\) and Liancui Zuo\(^1\)∗

1. College of Mathematical Science, Tianjin Normal University, Tianjin, 300387, P. R. China
2. School of Mathematics, Shandong University, Jinan, 250100, P. R. China

Abstract

The linear arboricity \(la(G)\) of a graph \(G\) is the minimum number of linear forests which partition the edge set \(E(G)\) of \(G\). The vertex linear arboricity \(vla(G)\) of a graph \(G\) is the minimum number of subsets into which the vertex set \(V(G)\) can be partitioned so that every subset induces a linear forest. The Schrijver graph \(SG(n, k)\) is the graph whose vertex set consists of all 2-stable \(k\)-subsets of the set \([n] = \{0, 1, \ldots, n-1\}\) and two vertices \(A\) and \(B\) are adjacent if and only if \(A \cap B = \emptyset\).

In this paper, it is proved that \(la(SG(2k + 2, k)) = \lceil (k + 2)/2 \rceil\) for \(k \geq 3\) and \(vla(SG(2k + 2, k)) = va(SG(2k + 2, k)) = 2\) for \(k \geq 2\).

Keywords: Linear forest; Linear arboricity; Vertex Linear arboricity; Schrijver graph

1 Introduction

Throughout this paper, all graphs considered are finite, undirected and simple. For a real number \(x\), \([x]\) is the least integer not less than \(x\), and \(\lfloor x\rfloor\) is the most integer not more than \(x\). For a graph \(G\), we use \(V(G)\), \(E(G)\), \(\Delta(G)\) to denote the vertex set, the edge set and the maximum degree, respectively. \(N_G(v)\) denotes the set of vertices adjacent to the vertex \(v\) in \(G\). \(G[W]\) denotes the subgraph induced by \(W \subseteq V(G)\) (or \(W \subseteq E(G)\)) in \(G\). For disjoint subsets \(S\) and \(S'\) of \(V(G)\), we denote the set of edges with one end in \(S\) and the other in \(S'\) by \([S, S']\), which is called an edge cut if \(S' = \overline{S}\), where \(\overline{S} = V(G) \setminus S\) is the subset obtained by removing all vertices of \(S\) from \(V(G)\). Let \(G \setminus H\) be the graph \(G - E(H)\) that is obtained by taking away all edges of \(H\) from \(G\). A \(k\)-path is a path with length \(k\).

∗The corresponding author: zuolc@yahoo.com.cn
Supported in part by Tianjin normal university(NO.5RL066)
A linear forest is a graph in which each component is a path. The linear arboricity $la(G)$ of a graph as defined by Harary [11] is the minimum number of linear forests which partition the edge set $E(G)$ of $G$. Akiyama et al. [1] conjectured that $la(G) = \lceil (\Delta(G) + 1)/2 \rceil$ for any regular graph $G$, and proved that the conjecture is true for complete graphs and graphs with $\Delta = 3, 4$ [1, 2]. Enomoto and Péroche [7] proved that the conjecture is true for graphs with $\Delta = 5, 6, 8$. Guldan [10] proved that the conjecture is true for graphs with $\Delta = 10$. It is obvious that $la(G) \geq \lceil \Delta(G)/2 \rceil$ for every graph $G$ and $la(G) \geq \lceil (\Delta(G) + 1)/2 \rceil$ for every regular graph $G$. So the conjecture is equivalent to the following conjecture.

**Linear Arboricity Conjecture (LAC)** [1] For any graph $G$,

$$\lceil \Delta(G)/2 \rceil \leq la(G) \leq \lceil (\Delta(G) + 1)/2 \rceil.$$ 

Akiyama et al. [1] determined the linear arboricity of complete bipartite graphs and trees. Martinov [12] determined the linear arboricity of extremal locally-tree-like graphs which have a minimal number of edges according to the number of vertices. Martinova [13] determined the linear arboricity of maximal outerplanar graphs. Wu [19] determined the linear arboricity of series-parallel graphs, moreover, Wu [20] proved the conjecture is true for a planar graph $G$ with $\Delta(G) \neq 7$, and the case $\Delta(G) = 7$ was also settled in Wu [21]. Tan et al. [18] determined the linear arboricity of planar graphs with maximum degree at least five.

The vertex linear arboricity $vla(G)$ of a graph is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that every subset induces a linear forest. The vertex arboricity $va(G)$ of a graph $G$ can be defined similarly. Matsumoto [14] proved that for any finite graph $G$, $vla(G) \leq \lceil \Delta(G)/2 \rceil$, moreover, if $\Delta(G)$ is even, then $vla(G) = \lceil \Delta(G)/2 \rceil$ if and only if $G$ is the complete graph of order $\Delta(G) + 1$ or a cycle. Goddard [9] and Poh [15] proved that $vla(G) \leq 3$ for a planar graph $G$. Akiyama [3] proved $vla(G) \leq 2$ if $G$ is an outerplanar graph. Alavi [4] proved that $vla(G) + vla(G^c) \leq 1 + \lceil n/2 \rceil$ for any graph $G$ of order $n$, where $G^c$ is the complement of $G$. Zuo [22, 23] determined the vertex linear arboricity of distance graphs and a class of integer distance graphs with special distance sets, respectively. Raspaud and Wang [16] discussed the vertex arboricity of planar graphs, and Borodin and Ivanova [5] proved that planar graphs without 4-cycles adjacent to 3-cycles are list vertex 2-arborable.

The following result is obvious.
Lemma 1.1. If $G = G_1 \cup G_2 \cup \cdots \cup G_n$, then $la(G) \leq la(G_1) + la(G_2) + \cdots + la(G_n)$. In particular, $la(G) = \max\{la(G_1), la(G_2), \ldots, la(G_n)\}$, where $G_i (i = 1, 2, \ldots, n)$ are connected components of $G$.

The Kneser graph $KG(n, k)$ is the graph whose vertex set consists of all $k$-subsets of an $n$-set, and two vertices are adjacent if and only if they are disjoint.

A subset $S$ of $[n] = \{0, 1, \ldots, n-1\}$ is said to be 2-stable if $2 \leq |x - y| \leq n - 2$ for any two distinct elements $x$ and $y$, i.e., $S$ does not contain two consecutive numbers in the cyclic ordering of $[n]$.

Definition 1.2. ([17]) The Schrijver graph $SG(n, k)$ is defined as follows. Its vertices are those $k$-element subsets of the set $[n] = \{0, 1, \ldots, n-1\}$ that do not contain cyclically consecutive elements $i, i+1$ or $n-1, 0$. Two such vertices are adjacent if they represent disjoint $k$-subsets.

Equivalently, the Schrijver graph $SG(n, k)$ is the graph whose vertex set consists of all 2-stable $k$-subsets of the set $[n] = \{0, 1, \ldots, n-1\}$ and two vertices $A$ and $B$ are adjacent if and only if $A \cap B = \phi$.

Clearly, the Schrijver graph $SG(n, k)$ is the subgraph of $KG(n, k)$ induced by all vertices that are 2-stable subsets.

The structure of Schrijver graph $SG(2k+2, k)$ was studied in [6]. Now we recall some results that will be used here.

The vertex set of the Schrijver graph $SG(n, k)$ has cardinality $\frac{n}{k} \binom{n-k-1}{k-1}$. In particular, $SG(2k+2, k)$ has $(k+1)^2$ vertices.

For $0 \leq i \leq 2k+1$, let $v(0, i) = \{i, i+2, \ldots, i+2k-2\}$, in which each element is taken modulo $2k+2$. We make the convention that all indices and elements are taken modulo $2k+2$ in the following except special instruction. We also regard $v(0, i)$ as a sequence with the elements ordered in the above manner.

A sequence is called a $k$-sequence if it has $k$ elements. Let $m = \lfloor k/2 \rfloor$. For $1 \leq j \leq m$, let $A_j$ be the $k$-sequence in which the $(k-j+1)$-th entry is equal to 2, and the other $k-1$ entries are equal to 1. Clearly, $A_j$ can be viewed as a row vector with $k$ components, and $v(0, i)$ and $A_j$ can be added to $v(0, i) + A_j$. In fact, when a $k$-set $A$, regarded as a row vector, and a $k$-sequence $B$ are added to get $A + B$, we just add the two sequences entrywise to get a $k$-sequence if all the sums are distinct. For the sake of convenience, in addition operation, one can view $v(0, i)$ as a row vector with $k$ components $(i, i+2, \ldots, i+2k-2)$ in $R^k$ over real number field $R$, in which each element is taken modulo $2k+2$. 

3
Now for $0 \leq i \leq 2k + 1$ and $1 \leq j \leq m$, let

$$v(j, i) = v(j - 1, i) + A_j$$

be the recursion formula, where $v(0, i) = (i, i + 2, \ldots, i + 2k - 2)$ and the addition is taken modulo $2k + 2$.

Let $V_0 = \{v(0, i) \mid i = 0, 1, \ldots, 2k + 1\}$, and $V_j = \{v(j, i) \mid i = 0, 1, \ldots, 2k + 1\}$ for $1 \leq j \leq m$. We need the following lemmas for the proof of our main results.

**Lemma 1.3.** ([6]) For $0 \leq j \leq m - 1$, $|V_j| = 2k + 2$, and

$$|V_m| = \begin{cases} 2k + 2, & \text{if } k \text{ is odd,} \\ k + 1, & \text{otherwise.} \end{cases}$$

Note that $|V_m| = k + 1$ when $k$ is even. Thus, in this case, the index $i$ of $v(m, i)$ is taken modulo $k + 1$ for even $k$ henceforth.

**Lemma 1.4.** ([6]) For each $v(0, i) \in V_0$, and $v(j, i) \in V_j$, we have

$$N_G(v(0, i)) = \{v(0, i + p) \mid p = 1, 3, \ldots, 2k + 1\} \cup \{v(1, i)\},$$

$$N_G(v(j, i)) = \{v(j, i - 1), v(j, i + 1), v(j - 1, i), v(j + 1, i)\}$$

for $1 \leq j \leq m - 1$,

$$N_G(v(m, i)) = \{v(m, i - 1), v(m, i + 1), v(m, i + k + 1), v(m - 1, i)\}$$

for $k$ is odd, and

$$N_G(v(m, i)) = \{v(m, i - 1), v(m, i + 1), v(m - 1, i), v(m - 1, i + k + 1)\}$$

for $k$ is even.

By Lemma 1.4, $\Delta(G) = k + 2$ for $k \geq 3$, and the following two results are obtained immediately.

**Corollary 1.5.** [6] $G[V_0]$ is a complete bipartite graph with two partite subsets

$$X = \{v(0, 0), v(0, 2), \ldots, v(0, 2k)\}$$

and

$$Y = \{v(0, 1), v(0, 3), \ldots, v(0, 2k + 1)\}.$$

**Corollary 1.6.** ([6]) $G[V_j]$ is a cycle with length $2k + 2$ for $1 \leq j \leq m - 1$, $G[V_m]$ is a cycle with length $k + 1$ for even $k$, and $G[V_m]$ is a 3-regular graph for odd $k$. 

4
2 The linear arboricity of $SG(2k + 2, k)$

Let $G(X, Y)$ be a balanced bipartite graph with partite sets $X = \{x_i \mid i \in \mathbb{Z}_n\}$ and $Y = \{y_i \mid i \in \mathbb{Z}_n\}$. In [8], it was defined that the bipartite difference $\alpha$ of an edge $x_py_q$ in $G(X, Y)$ by the value $(q - p)(\text{mod } n)$, i.e., $\alpha = (q - p)(\text{mod } n)$. It is obvious that an edge subset in $G(X, Y)$ containing the edges with the same bipartite difference must be a matching. In particular, this edge subset is also a perfect matching if $G(X, Y)$ is $K_{n,n}$.

Let $M_\alpha$ be the edge set consisting of edges with bipartite difference $\alpha$. The following lemmas give a decomposition of $K_{n,n}$.

**Lemma 2.1.** Let $K_{n,n}$ be a balanced complete bipartite graph with partite sets $X = \{x_i \mid i = 0, 1, \ldots, n - 1\}$ and $Y = \{y_i \mid i = 0, 1, \ldots, n - 1\}$, then $K_{n,n}$ can be decomposed into the union of $n/2$ Hamiltonian paths and a matching for even $n$, and decomposed into the union of $(n - 1)/2$ Hamilton paths and a linear forest for odd $n$.

**Proof.** If $n$ is even, then $K_{n,n}$ can be decomposed into the union of $n/2$ Hamiltonian cycles $M_\alpha \cup M_{\alpha+1}(\alpha = 0, 2, \ldots, n - 2)$. Next, we take away one edge $x_{\alpha/2}y_{n-\alpha/2-1}$ from each $M_\alpha \cup M_{\alpha+1}(\alpha = 0, 2, \ldots, n - 2)$. Then

$$H_{\alpha/2} = M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\}(\alpha = 0, 2, \ldots, n - 2)$$

are $n/2$ Hamiltonian paths of $K_{n,n}$, and $M = \{x_{\alpha/2}y_{n-\alpha/2-1} \mid \alpha = 0, 2, \ldots, n - 2\}$ is a matching.

Similarly, for odd $n$, each $M_\alpha \cup M_{\alpha+1}(\alpha = 0, 2, \ldots, n - 3)$ generates a Hamiltonian cycle. Therefore $M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\}$ is a Hamiltonian path. Let

$$H_{\alpha/2} = M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\}(\alpha = 0, 2, \ldots, n - 3).$$

Moreover, it is clear that $M = M_{n-1} \cup \{x_{\alpha/2}y_{n-\alpha/2-1} \mid \alpha = 0, 2, \ldots, n - 3\}$ forms a linear forest.  

Therefore, $la(K_{n,n}) \leq \lceil (n + 1)/2 \rceil$ and $la(K_{n,n} \setminus M) \leq \lceil (n - 1)/2 \rceil$.

**Lemma 2.2.** [1] The linear arboricity of every $3$-regular graph is $2$.

**Lemma 2.3.** [2] The linear arboricity of every $4$-regular graph is $3$.

Now we give the main result of this paper.
Theorem 2.4. Let $G = SG(2k + 2, k)$ $(k \geq 2)$ be a Schrijver graph, then

$$la(G) = \begin{cases} 
3, & \text{for } k = 2, \\
\lceil (k + 2)/2 \rceil, & \text{for } k \geq 3.
\end{cases}$$

Proof. For $k = 2$, $SG(2k + 2, k) = SG(6, 2)$ is a 4-regular graph, and the result holds by Lemma 2.3. So, in this section, suppose that $k \geq 3$ hereafter.

It is obvious that $la(G) \geq \lceil (k + 2)/2 \rceil$ since $\Delta(G) = k + 2$. So it suffices to show that $la(G) \leq \lceil (k + 2)/2 \rceil$.

By Lemma 1.4,

$$[V_j, V_{j+1}] = \{v(j, i)v(j + 1, i) \mid i = 0, 1, \ldots, 2k + 1\}$$

for $0 \leq j \leq m - 2$,

$$[V_{m-1}, V_m] = \{v(m - 1, i)v(m, i) \mid i = 0, 1, \ldots, 2k + 1\}$$

which is a matching if $k$ is odd, and

$$G[[V_{m-1}, V_m]] = \{v(m - 1, i)v(m, i)v(m - 1, i + k + 1) \mid i = 0, 1, \ldots, k\}$$

if $k$ is even, in which each component is a 2-path.

By Corollary 1.5, $G[V_0]$ is a balanced complete bipartite graph with two partite subsets

$$X = \{v(0, 0), v(0, 2), \ldots, v(0, 2k)\}$$

and

$$Y = \{v(0, 1), v(0, 3), \ldots, v(0, 2k + 1)\}.$$ 

Let $v(0, 2i) = x_i$ and $v(0, 2i + 1) = y_i$. Then $G[V_0] = K_{k+1,k+1}$ is a balanced complete bipartite graph with two partite subsets

$$X = \{x_i \mid i = 0, 1, \ldots, k\} \text{ and } Y = \{y_i \mid i = 0, 1, \ldots, k\}.$$ 

Case 1. $k \geq 3$ is odd.

It is not difficult to see that

$$G[\bigcup_{j=0}^{m-1}[V_j, V_{j+1}]] = \{v(0, i)v(1, i) \ldots v(m, i) \mid i = 0, 1, \ldots, 2k + 1\}$$

is a linear forest in which each component is an $m$-path.
Let \( B = \cup_{j=0}^{m-1}[V_j, V_{j+1}] \) and \( S_j = V_0 \cup V_1 \cup \cdots \cup V_j \). By Lemma 1.4, it is not difficult to see that every \([V_j, V_{j+1}] = [S_j, S_{j+1}]\) \((0 \leq j \leq m - 1)\) is an edge cut of \( G \). Hence \( G \setminus B \) is a graph whose components are \( G[V_0], G[V_1], \ldots, G[V_m] \). Next, we will take away a matching from \( G[V_0] \). By Lemma 2.1, \( G[V_0] = K_{k+1,k+1} \) can be decomposed into the union of \((k+1)/2\) Hamiltonian path and a matching \( M = \{x_{\alpha/2}y_{k-\alpha/2} \mid \alpha = 0, 2, \ldots, k-1\} \). Then \( M \cup B \) forms a linear forest. Moreover, we have \( G = (G[V_0] \setminus M) \cup G[V_1] \cup \cdots \cup G[V_m] \cup (M \cup B) \). Thus by Lemma 1.1, Corollary 1.6 and Lemma 2.2, \( la(G) \leq la((G[V_0] \setminus M) \cup G[V_1] \cup \cdots \cup G[V_m])) + 1 = la(G[V_0] \setminus M) + 1 \leq (k+1)/2 + 1 = \lceil (k+2)/2 \rceil \).

**Case 2.** \( k \geq 4 \) is even.

Let

\[
B' = G[\cup_{j=0}^{m-2}[V_j, V_{j+1}]] = \{v(0,i)v(1,i) \cdots v(m-1,i) \mid i = 0, 1, 3, \ldots, 2k+1\},
\]

then

\[
G = G[V_0] \cup G[V_1] \cup \cdots \cup G[V_m] \cup B' \cup G[[V_{m-1}, V_m]].
\]

In the following, we first decompose \( G[V_j](j = 0, 1, \ldots, m) \) and \( B' \). Let

\[
P_i = v(0,i)v(1,i) \cdots v(m-1,i) \text{ for } 0 \leq i \leq 2k+1.
\]

By Lemma 2.1, \( G[V_0] = K_{k+1,k+1} \) can be decomposed into the union of \( k \) Hamiltonian paths

\[
H_{\alpha/2} = M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{k-\alpha/2}\}(\alpha = 0, 2, 4, \ldots, k-2)
\]

and a linear forest

\[
M = M_k \cup \{x_{\alpha/2}y_{k-\alpha/2} \mid \alpha = 0, 2, 4, \ldots, k-2\}.
\]

Hence \( G[V_0] = H_0 \cup H_1 \cup \cdots \cup H_{k/2-1} \cup M \). For \( 1 \leq j \leq k/2 - 1 \), let \( G[V_j] = P_{j,1} \cup P_{j,2} \), where

\[
P_{j,1} = v(j,0)v(j,1) \cdots v(j, 2k) \text{ and } P_{j,2} = v(j,0)v(j, 2k+1)v(j, 2k).
\]

For \( j = m \), let \( G[V_m] = P_{m,1} \cup P_{m,2} \), where

\[
P_{m,1} = v(m,0)v(m,1) \cdots v(m, k-1) \text{ and } P_{m,2} = v(m,0)v(m, k)v(m, k-1).
\]
**Subcase 2.1.** \( k \equiv 0(\text{mod } 4) \).

Let \( P_0 = M_0 \cup M'_0 \), where

\[
M_0 = \{ v(2t, 0) v(2t + 1, 0) \mid t = 0, 1, \ldots, k/4 - 1 \},
\]

and

\[
M'_0 = \{ v(2t + 1, 0) v(2t + 2, 0) \mid t = 0, 1, \ldots, k/4 - 2 \}.
\]

And let \( P_{2k} = M_{2k} \cup M'_{2k} \), where

\[
M_{2k} = \{ v(2t, 2k) v(2t + 1, 2k) \mid t = 0, 1, \ldots, k/4 - 1 \},
\]

and

\[
M'_{2k} = \{ v(2t + 1, 2k) v(2t + 2, 2k) \mid t = 0, 1, \ldots, k/4 - 2 \}.
\]

Then

\[
H_0 \cup M_0 \cup M'_{2k} \cup P_{2k+1} \cup (\bigcup_{j=1}^{m} P_{j,1}) \cup \{ v(m - 1, 2k) v(m, k - 1), v(m - 1, 2k + 1) v(m, k) \}
\]

forms a Hamiltonian path of \( G \). Let

\[
T = [V_{m-1}, V_m] \setminus \{ v(m - 1, 2k + 1) v(m, k), v(m - 1, 2k) v(m, k - 1) \}.
\]

Then

\[
H_1 \cup P_2 \cup P_{2k-1} \cup (\bigcup_{j=1}^{m-1} P_{j,2}) \cup T
\]

forms a linear forest. For \( 2 \leq j \leq m - 1 \), each \( H_j \cup P_{2j} \cup P_{2k-2j+1} \) forms a linear forest. Finally,

\[
M \cup M'_0 \cup M_{2k} \cup P_{m,2} \cup (\bigcup_{j=0}^{m} P_{2j+1}) \cup (\bigcup_{j=0}^{m-1} P_{2j+k})
\]

forms a linear forest. Thus, the edge set \( E(G) \) is partitioned into \((k + 2)/2\) linear forest.

Hence \( la(G) \leq (k + 2)/2 \).

**Subcase 2.2.** \( k \equiv 2(\text{mod } 4) \).

Similar to Subcase 2.1, let \( P_0 = N_0 \cup N'_0 \), where

\[
N_0 = \{ v(2t, 0) v(2t + 1, 0) \mid t = 0, 1, \ldots, (k - 6)/4 \},
\]

and

\[
N'_0 = \{ v(2t + 1, 0) v(2t + 2, 0) \mid t = 0, 1, \ldots, (k - 6)/4 \}.
\]
Let $P_{2k} = N_{2k} \cup N'_{2k}$, where

$$N_{2k} = \{v(2t, 2k)v(2t + 1, 2k) \mid t = 0, 1, \ldots, (k - 6)/4\},$$

and

$$N'_{2k} = \{v(2t + 1, 2k)v(2t + 2, 2k) \mid t = 0, 1, \ldots, (k - 6)/4\}.$$

Then it is not difficult to see that

$$H_0 \cup N_0 \cup N'_{2k} \cup P_{2k+1} \cup \left( \bigcup_{j=1}^{m} P_{j,1} \right) \cup \{v(m - 1, 0)v(m, 0), v(m - 1, 2k + 1)v(m, k)\},$$

forms a Hamiltonian path of $G$. Let

$$T' = [V_{m-1}, V_m] \setminus \{v(m - 1, 0)v(m, 0), v(m - 1, 2k + 1)v(m, k)\}.$$

Clearly,

$$H_1 \cup P_2 \cup P_{2k-1} \cup \left( \bigcup_{j=1}^{m-1} P_{j,2} \right) \cup T'$$

forms a linear forest, and for $2 \leq j \leq m - 1$, each $H_j \cup P_{2j} \cup P_{2k-2j+1}$ forms a linear forest. Finally, it is not difficult to verify that

$$M \cup N'_0 \cup N_{2k} \cup P_{m,2} \cup \left( \bigcup_{j=0}^{m} P_{2j+1} \right) \cup \left( \bigcup_{j=0}^{m-1} P_{2j+k} \right)$$

forms a linear forest. Thus, the edge set $E(G)$ is partitioned into $(k + 2)/2$ linear forests. Hence we have $la(G) \leq (k + 2)/2$, too.

Up to now, we have shown that $la(G) \leq \lceil (k + 2)/2 \rceil$, and then the theorem holds. $\square$

Therefore, the linear arboricity conjecture holds for Schrijver graph $SG(2k + 2, k)$ for $k \geq 2$.

## 3 The vertex linear arboricity and vertex arboricity of Schrijver graph $SG(2k + 2, k)$

In this section, we discuss the vertex linear arboricity and the vertex arboricity of the Schrijver graph.

**Theorem 3.1.** $vla(G) = 2$ for the Schrijver graph $G = SG(2k + 2, k)$ ($k \geq 2$).
**Proof.** The proof will be split into three cases. The main idea is to partition the vertex set \( V(G) \) into two subsets such that every subset induces a linear forest.

**Case 1.** \( k \geq 3 \) is odd.

Let
\[
Q = \{v(j,i) \mid 0 \leq j \leq m, \ i = 0,2,\ldots,2k\},
\]
and
\[
R = \{v(j,i) \mid 0 \leq j \leq m, \ i = 1,3,\ldots,2k+1\}.
\]
By Lemma 1.4,
\[
G[Q] = \{v(0,i)v(1,i)\cdots v(m,i)v(m,i+k+1)v(m-1,i+k+1)\cdots v(0,i+k+1)
| i = 0,2,\ldots,k-1\}
\]
and
\[
G[R] = \{v(0,i)v(1,i)\cdots v(m,i)v(m,i+k+1)v(m-1,i+k+1)\cdots v(0,i+k+1)
| i = 1,3,\ldots,k\}
\]
are two linear forests in which every component is a \( k \)-path.

**Case 2.** \( k \geq 4 \) is even.

Let
\[
Q' = \{v(j,i) \mid 0 \leq j \leq m-1, \ i = 0,2,\ldots,2k\} \cup \{v(m,i) \mid i = 0,2,\ldots,k\},
\]
and
\[
R' = \{v(j,i) \mid 0 \leq j \leq m-1, \ i = 1,3,\ldots,2k+1\} \cup \{v(m,i) \mid i = 1,3,\ldots,k-1\}.
\]
By Lemma 1.4,
\[
G[Q'] = \{v(0,0)v(1,0)\cdots v(m,0)v(m,k)v(m-1,k)\cdots v(0,k)
| i = 2,4,\ldots,k-2\}
\]
\[
\cup\{v(0,i)v(1,i)\cdots v(m,i) \mid i = k+2,k+4,\ldots,2k\}
\]
and
\[
G[R'] = \{v(0,i)v(1,i)\cdots v(m,i) \mid i = 1,3,\ldots,k-1\}
\]
\[
\cup\{v(0,i)v(1,i)\cdots v(m-1,i) \mid i = k+1,k+3,\ldots,2k+1\}
\]
are two linear forests.

**Case 3.** \( k = 2 \).

One can partition the vertex set \( V(G) \) into two subsets
\{v(0,0), v(0,2), v(0,4), v(1,0), v(1,2)\} and \{v(0,1), v(0,3), v(0,5), v(1,1)\}.

It is easy to verify that every subset induces a linear forest.

The following result follows from the fact that the Schrijver graph \(G = SG(2k + 2,k)\) contains a cycle for \(k \geq 2\).

**Corollary 3.2.** \(va(G) = 2\) for the Schrijver graph \(G = SG(2k + 2,k)\) \((k \geq 2)\).

**References**


[5] O.V. Borodin, A.O. Ivanova, Planar graphs without 4-cycles adjacent to 3-cycles are list vertex 2-arborable, J. Graph Theory 62 (2009), 234-240.


