Multilinear Singular Integral Operators on Triebel-Lizorkin and Lebesgue Spaces

Liu Lanzhe

Department of Mathematics
Hunan University, Changsha 410082
P.R. of China
E-mail: lanzheliu@163.com

Abstract: The boundedness for the multilinear operators associated to some singular integral operators with non-smooth kernels on Triebel-Lizorkin and Lebesgue spaces is obtained.

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1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1-4]). From [8][10], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce some multilinear operators associated to certain singular integral operators with non-smooth kernels and prove the boundedness properties for the multilinear operators on the Triebel-Lizorkin and Lebesgue spaces.

2. Preliminaries and Theorems

In this paper, we will study a class of multilinear operators associated to some singular integral operators with non-smooth kernels as follows.
**Definition 1.** A family of operators $D_t$, $t > 0$ is said to be an “approximations to the identity” if, for every $t > 0$, $D_t$ can be represented by the kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{\mathbb{R}^n} a_t(x, y)f(y)dy$$

for every $f \in L^p(\mathbb{R}^n)$ with $p \geq 1$ and every $x \in \mathbb{R}^n$, and $a_t(x, y)$ satisfies:

$$|a_t(x, y)| \leq h_t(x, y) = Ct^{-n/2}s(|x - y|^2/t),$$

where $s$ is a positive, bounded and decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\epsilon}s(r^2) = 0$$

for some $\epsilon > 0$.

Let $m_j$ be positive integers $(j = 1, \cdots, l)$, $m_1 + \cdots + m_l = m$ and $b_j$ be functions on $\mathbb{R}^n$ with $b_j \in C^{m_j}$ for $1 \leq j \leq l$. Set

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \leq m_1} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha.$$

We have

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha - \sum_{|\alpha| = m} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha$$

$$= R_{m_j}(b_j; x, y) - \sum_{|\alpha| = m} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha.$$

**Definition 2.** A linear operator $T$ is called the singular integral operators with non-smooth kernels if $T$ is bounded on $L^2(\mathbb{R}^n)$ and associated with a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$

for every continuous function $f$ with compact support, and for almost all $x$ not in the support of $f$. Moreover, the following conditions hold:

(1) There exists an ”approximations to the identity” $\{B_t, t > 0\}$ such that $TB_t$ has associated kernel $k_t(x, y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x - y| > c_1 t^{1/2}} |K(x, y) - k_t(x, y)|dx \leq c_2$$

for all $y \in \mathbb{R}^n$. 
(2) There exists an "approximations to the identity" \{A_t, t > 0\} such that \(A_tT\) has associated kernel \(K_t(x, y)\) which satisfies

\[ |K_t(x, y)| \leq c_4 t^{-n/2} \text{ if } |x - y| \leq c_3 t^{1/2}, \]

and

\[ |K(x, y) - K_t(x, y)| \leq c_4 t^{\delta/2} |x - y|^{-n-\delta} \text{ if } |x - y| \geq c_3 t^{1/2}, \]

for some \(c_3, c_4 > 0, \delta > 0\).

Let \(m_j\) be positive integers \((j = 1, \cdots, l)\), \(m_1 + \cdots + m_l = m\) and \(b_j\) be functions on \(\mathbb{R}^n\) with \(b_j \in C^{m_j}\) for \(1 \leq j \leq l\). The multilinear operator associated to \(T\) is defined by

\[ T^b(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^l R_{m_j+1}(b_j; x, y) \frac{K(x, y) f(y) dy}{|x - y|^{m_j}}. \]

Note that when \(m = 0\), \(T^b\) is just the multilinear commutator of \(T\) and \(b\) (see [5][10-11]), while when \(m > 0\), it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-4]). The purpose of this paper is to study the boundedness properties for the multilinear operator \(T^b\) on Triebel-Lizorkin and Lebesgue spaces. In Section 4, some applications of Theorems in this paper are given.

**Definition 3.** Let \(0 < \beta < 1\) and \(1 \leq p < \infty\). The Triebel-Lizorkin space associated with the "approximations to the identity" \(\{A_t, t > 0\}\) is defined by

\[ \dot{F}^{\beta, \infty}_{p,A}(\mathbb{R}^n) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : ||f||_{\dot{F}^{\beta, \infty}_{p,A}} < \infty \}, \]

where

\[ ||f||_{\dot{F}^{\beta, \infty}_{p,A}} = \left\| \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - A_{t_Q} f(x)| dx \right\|_{L^p}, \]

and the supremum is taken over all cubes \(Q\) of \(\mathbb{R}^n\) with sides parallel to the axes, \(t_Q = l(Q)^2\) and \(l(Q)\) denotes the side length of \(Q\).
Now, let us introduce some notations. Throughout this paper, $Q$ will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For a locally integrable function $f$, the sharp function of $f$ is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [7][12])

$$f^\#(x) \approx \sup_{Q \ni x, c \in Q} \inf \left\{ \frac{1}{|Q|} \int_Q |f(y) - c|dy \right\}.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $1 \leq p < \infty$ and $0 \leq \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|^{1-\frac{p\delta}{n}}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

For $\beta > 0$, the Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is the space of functions $f$ such that (see [10])

$$||f||_{Lip_\beta} = \sup_{x, h \in \mathbb{R}^n, h \neq 0} |f(x + h) - f(x)|/|h|^\beta < \infty.$$

The sharp maximal function $M_A(f)$ associated with the ”approximations to the identity” $\{A_t, t > 0\}$ is defined by(see [6][9])

$$M_A^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)|dy,$$

where $t_Q = (l(Q))^2$ and $l(Q)$ denotes the side length of $Q$.

Now we can state our theorem as following.

**Theorem 1.** Suppose $T$ is a singular integral operator with non-smooth kernel as Definition 2. Let $0 < \beta < \min(1, \epsilon, \delta/l)$, $\{A_t, t > 0\}$ be the ”approximations to the identity” and $D^\alpha b_j \in Lip_\beta(\mathbb{R}^n)$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then $T^b$ is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}_{p,A}^{\delta,\beta,\infty}(\mathbb{R}^n)$ for any $1 < p < \infty$. 

Theorem 2. Suppose $T$ is a singular integral operator with non-smooth kernel as Definition 2. Let $0 < \beta < 1$ and $D^{\alpha}b_j \in Lip_\beta(\mathbb{R}^n)$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then $T^b$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for any $1 < p < n/l\beta$ and $1/p - 1/q = l\beta/n$.

3. Proof of Theorems

To prove the theorem, we need the following lemmas.

Lemma 1. (see [1]) Suppose that $0 \leq \eta < n$, $1 \leq r < p < n/\eta$ and $1/q = 1/p - \eta/n$. Then

$$||M_{\eta,r}(f)||_{L^q} \leq C||f||_{L^p}.$$

Lemma 2. (see [4]). Let $A$ be a function on $\mathbb{R}^n$ and $D^{\alpha}A \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{|Q(x, y)|} \int_{\tilde{Q}(x,y)} |D^{\alpha}A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

Lemma 3. (see [6][9]) For any $\gamma > 0$, there exists a constant $C > 0$ independent of $\gamma$ such that

$$||\{|x \in \mathbb{R}^n : M(f)(x) > \gamma\lambda\}| \leq C\gamma||\{|x \in \mathbb{R}^n : M(f)(x) > \lambda\}|$$

for $\lambda > 0$, where $D$ is a fixed constant which only depends on $n$. So that

$$||M(f)||_{L^p} \leq C||M_A(f)||_{L^p}$$

for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Lemma 4. (see [9]) Let $1 < p < \infty$ and $T$ a singular integral operator with non-smooth kernels as Definition 2. Then $T$ is bounded on $L^p(\mathbb{R}^n)$.

Lemma 5. Let $\{A_t, t > 0\}$ be an “approximations to the identity”, $0 < \beta < 1$ and $b \in Lip_\beta(\mathbb{R}^n)$. Then for
every $f \in L^p(\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$, we have

(a) $\sup_{Q \ni \tilde{x}} \frac{1}{|Q|} \int_Q |A_{t_\sigma}((b - b_Q)f)(x)| \, dx \leq C||b||_{\text{Lip}_\beta} M_{\beta,1}(f)(\tilde{x})$;

(b) $\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |A_{t_\sigma}((b - b_Q)f)(x)| \, dx \leq C||b||_{\text{Lip}_\beta} M(f)(\tilde{x})$ if $\beta < \epsilon$;

(c) $\sup_{Q \ni \tilde{x}} \frac{1}{|Q|} \int_Q |A_{t_\sigma}(f)(x)| \, dx \leq CM(f)(\tilde{x})$.

**Proof.**

(a). Write
\[
\frac{1}{|Q|} \int_Q |A_{t_\sigma}((b - b_Q)f)(x)| \, dx \\
\leq \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} h_{t_\sigma}(x, y)| (b(y) - b_Q)f(y) | \, dy \, dx \\
= \frac{1}{|Q|} \int_Q \int_{2Q} h_{t_\sigma}(x, y)| (b(y) - b_Q)f(y) | \, dy \, dx \\
+ \sum_{k=1}^{\infty} \frac{1}{|Q|} \int_Q \int_{2^{k+1}Q \setminus 2^k Q} h_{t_\sigma}(x, y)| (b(y) - b_Q)f(y) | \, dy \, dx \\
= I_1 + I_2.
\]

We have, by the Hölder’s inequality,
\[
I_1 \leq \frac{C}{|Q|^{2Q}} \int_Q \int_{2Q} | (b(y) - b_Q)f(y) | \, dy \, dx \\
\leq C \frac{1}{|2Q|} \int_{2Q} | (b(y) - b_Q)f(y) | \, dy \\
\leq C||b||_{\text{Lip}_\beta} |Q|^\frac{\beta}{n} \frac{1}{|2Q|} \int_{2Q} |f(y) | \, dy \\
\leq C||b||_{\text{Lip}_\beta} \frac{1}{|2Q|^{1-\frac{\beta}{n}}} \int_{2Q} |f(y) | \, dy \\
\leq C||b||_{\text{Lip}_\beta} M_{\beta,1}(f)(\tilde{x}).
\]
For $I_2$, notice for $x \in Q$ and $y \in 2^{k+1}Q \setminus 2^kQ$, then $|x - y| \geq 2^{k-1}t_Q^{1/2}$ and $h_{t_Q}(x, y) \leq C t_Q^{-n/2} s(2^{2(k-1)})$. Thus

$$I_2 \leq C \sum_{k=1}^{\infty} 2^{kn} s(2^{2(k-1)}) \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b(y) - b_Q)f(y)|dy$$

$$\leq C \sum_{k=1}^{\infty} 2^{kn} s(2^{2(k-1)}) \|b\|_{Lip_b} |2^{k+1}Q|^{\frac{\beta}{n}} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|dy$$

$$\leq C \sum_{k=1}^{\infty} 2^{(k-1)n} s(2^{2(k-1)}) \|b\|_{Lip_b} \frac{1}{|2^{k+1}Q|^{1-\frac{\beta}{n}}} \int_{2^{k+1}Q} |f(y)|dy$$

$$\leq C \sum_{k=1}^{\infty} 2^{kn} s(2^{2(k-1)}) \|b\|_{Lip_b} M_{\beta, 1}(f)(x)$$

$$\leq C \|b\|_{Lip_b} M_{\beta, 1}(f)(\tilde{x}),$$

where the last inequality follows from

$$\sum_{k=1}^{\infty} 2^{kn} s(2^{2(k-1)}) \leq C \sum_{k=1}^{\infty} 2^{(k-1)n} s(2^{2(k-1)}) \leq C \sum_{k=1}^{\infty} 2^{-k\epsilon} < \infty$$

for $\epsilon > 0$.

(b).

$$\frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |A_{t_Q}((b - b_Q)f)(x)|dx$$

$$\leq \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q \int_{2Q} h_{t_Q}(x, y)|b(y) - b_Q|f(y)dydx$$

$$+ \sum_{k=1}^{\infty} \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q \int_{2^{k+1}Q \setminus 2^kQ} h_{t_Q}(x, y)|b(y) - b_Q|f(y)dydx$$

$$\leq C \|b\|_{Lip_b} \frac{1}{|2Q|} \int_{2Q} |f(y)|dy$$

$$+ C \sum_{k=1}^{\infty} 2^{(n+\beta)k} s(2^{2(k-1)}) \|b\|_{Lip_b} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|dy$$

$$\leq C \|b\|_{Lip_b} M(f)(\tilde{x}),$$

where the last inequality follows from

$$\sum_{k=1}^{\infty} 2^{(n+\beta)k} s(2^{2(k-1)}) \leq C \sum_{k=1}^{\infty} 2^{3k} s(2^{2(k-1)}) \leq C \sum_{k=1}^{\infty} 2^{k(\beta - \epsilon)} < \infty$$

for $\beta < \epsilon$. This completes the proof.
Proof of Theorem 1. Following [9], we first prove the sharp estimate for $T^b$ as following:

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |T^b(f)(x) - A_t \alpha \alpha T^b(f)(x)|dx \leq CM_\alpha(f)(\tilde{x})$$

for any cube $Q$ and $1 < r < p < \infty$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha| = m_j} \frac{1}{\alpha} (D^\alpha b_j)_{\tilde{\tilde{Q}}} x^\alpha$, then, by [4], $R_m(b_j; x, y) = R_m(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{\tilde{b}}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$T^b(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y) \frac{K(x, y) f(y) dy}{|x-y|^m}$$

$$= \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y) \frac{K(x, y) f_1(y) dy}{|x-y|^m}$$

$$- \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} R_{m_2}(\tilde{b}_2; x, y) (x-y)^{\alpha_1} D^\alpha_1 \tilde{\tilde{b}}_1(y) K(x, y) f_1(y) dy$$

$$- \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} R_{m_1}(\tilde{b}_1; x, y) (x-y)^{\alpha_2} D^\alpha_2 \tilde{\tilde{b}}_2(y) K(x, y) f_1(y) dy$$

$$+ \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} (x-y)^{\alpha_1 + \alpha_2} D^\alpha_1 \tilde{\tilde{b}}_1(y) D^\alpha_2 \tilde{\tilde{b}}_2(y) K(x, y) f_1(y) dy$$

$$+ \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y) K(x, y) f_2(y) dy$$

$$= T \left( \prod_{j=1}^2 \frac{R_{m_j+1}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_1 \right)$$

$$- T \left( \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2; x, \cdot) (x-\cdot)^{\alpha_1} D^\alpha_1 \tilde{\tilde{b}}_1}{|x-\cdot|^m} f_1 \right)$$

$$- T \left( \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1; x, \cdot) (x-\cdot)^{\alpha_2} D^\alpha_2 \tilde{\tilde{b}}_2}{|x-\cdot|^m} f_1 \right)$$

$$+ T \left( \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{\alpha_1! \alpha_2!} \frac{(x-\cdot)^{\alpha_1 + \alpha_2} D^\alpha_1 \tilde{\tilde{b}}_1 D^\alpha_2 \tilde{\tilde{b}}_2}{|x-\cdot|^m} f_1 \right)$$

$$+ T \left( \prod_{j=1}^2 \frac{R_{m_j+1}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_2 \right)$$
\[ A_{tQ} T^b(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) K_{tQ}(x, y) f_1(y) dy \]
\[ - \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x - y)^{\alpha_1} D^{\alpha_1} \tilde{b}_1(y)}{|x - y|^m} K_{tQ}(x, y) f_1(y) dy \]
\[ - \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x - y)^{\alpha_2} D^{\alpha_2} \tilde{b}_2(y)}{|x - y|^m} K_{tQ}(x, y) f_1(y) dy \]
\[ + \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} \frac{(x - y)^{\alpha_1 + \alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x - y|^m} K_{tQ}(x, y) f_1(y) dy \]
\[ + \int_{\mathbb{R}^n} \prod_{j=1}^{2} R_{m_j+1}(\tilde{b}_j; x, y) K_{tQ}(x, y) f_2(y) dy \]
\[ = A_{tQ} T \left( \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, \cdot) \right) \]
\[ - A_{tQ} T \left( \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1!} \right. \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x - \cdot)^{\alpha_1} D^{\alpha_1} \tilde{b}_1}{|x - \cdot|^m} f_1(\cdot) \]
\[ - A_{tQ} T \left( \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2!} \right. \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x - \cdot)^{\alpha_2} D^{\alpha_2} \tilde{b}_2}{|x - \cdot|^m} f_1(\cdot) \]
\[ + A_{tQ} T \left( \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{\alpha_1! \alpha_2!} \right. \int_{\mathbb{R}^n} \frac{(x - \cdot)^{\alpha_1 + \alpha_2} D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2}{|x - \cdot|^m} f_1(\cdot) \]
\[ + A_{tQ} T \left( \prod_{j=1}^{2} R_{m_j+1}(\tilde{b}_j; x, \cdot) \right), \]
thus

\[
\frac{1}{|Q|^{1+2\beta/n}} \int_{Q} |T^b(f)(x) - A_{tQ} T^b(f)(x)| \, dx \\
\leq \frac{1}{|Q|^{1+2\beta/n}} \int_{Q} \left| T \left( \prod_{j=1}^{2} R_{m_j} \left( \tilde{b}_j; x, \cdot \right) \right) f_1 \right| \, dx \\
+ \frac{C}{|Q|^{1+2\beta/n}} \int_{Q} T \left( \sum_{|\alpha_1|=m_1} R_{m_2} \left( \tilde{b}_2; x, \cdot \right) (x - \cdot)^{\alpha_1} D^{\alpha_1} \tilde{b}_1 \right) \left| f_1 \right| \, dx \\
+ \frac{C}{|Q|^{1+2\beta/n}} \int_{Q} T \left( \sum_{|\alpha_2|=m_2} R_{m_1} \left( \tilde{b}_1; x, \cdot \right) (x - \cdot)^{\alpha_2} D^{\alpha_2} b_2 \right) \left| f_1 \right| \, dx \\
+ \frac{C}{|Q|^{1+2\beta/n}} \int_{Q} T \left( \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} (x - \cdot)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} b_2 \right) \left| f_1 \right| \, dx \\
+ \frac{1}{|Q|^{1+2\beta/n}} \int_{Q} A_{tQ} T \left( \prod_{j=1}^{2} R_{m_j} \left( \tilde{b}_j; x, \cdot \right) \right) \left| f_1 \right| \, dx \\
+ \frac{C}{|Q|^{1+2\beta/n}} \int_{Q} A_{tQ} T \left( \sum_{|\alpha_1|=m_1} R_{m_2} \left( \tilde{b}_2; x, \cdot \right) (x - \cdot)^{\alpha_1} D^{\alpha_1} \tilde{b}_1 \right) \left| f_1 \right| \, dx \\
+ \frac{C}{|Q|^{1+2\beta/n}} \int_{Q} A_{tQ} T \left( \sum_{|\alpha_2|=m_2} R_{m_1} \left( \tilde{b}_1; x, \cdot \right) (x - \cdot)^{\alpha_2} D^{\alpha_2} b_2 \right) \left| f_1 \right| \, dx \\
+ \frac{C}{|Q|^{1+2\beta/n}} \int_{Q} A_{tQ} T \left( \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} (x - \cdot)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} b_2 \right) \left| f_1 \right| \, dx \\
+ \frac{1}{|Q|^{1+2\beta/n}} \int_{Q} \left( T - A_{tQ} \right) \left( \prod_{j=1}^{2} R_{m_j+1} \left( \tilde{b}_j; x, \cdot \right) \right) \left| f_2 \right| \, dx \\
:= I_0 + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\]

Now, let us estimate $I_0, I_1, I_2, I_3$ and $I_4$, respectively. First, by Lemma 2, we get, for $x \in Q$ and $y \in \tilde{Q}$,

\[
|R_m(\tilde{b}_j; x, y)| \leq C |x - y|^m \sum_{|\alpha|=m} \sup_{x \in \tilde{Q}} |D^{\alpha} b_j(x) - (D^{\alpha} b_j)_{\tilde{Q}}| \leq C |x - y|^m |Q|^\beta/n \sum_{|\alpha|=m} ||D^{\alpha} b_j||_{Lip_\beta},
\]
thus, by the Hölder’s inequality and $L^r$-boundedness of $T$, we obtain

$$I_0 \leq \frac{C}{|Q|^{1+2\beta/n}} \int_Q \left| T \left( \frac{|Q|^{2\beta/n} |x-y|^{m_1+m_2} \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}b_j||_{Lip} \right) f_1 \right) (x) \right| dx$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}b_j||_{Lip} \right) \frac{1}{|Q|} \int_Q |T(f_1)(x)| \, dx$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}b_j||_{Lip} \right) \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |T(f_1)(x)|^r \, dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}b_j||_{Lip} \right) \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |f(x)|^r \, dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}b_j||_{Lip} \right) M_r(f)(\tilde{x});$$

$$I_1 \leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}b_2||_{Lip} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T(D^{\alpha_1}b_1 f_1)(x)| \, dx$$

$$\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}b_2||_{Lip} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |T(D^{\alpha_1}b_1 f_1)(x)|^r \, dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}b_2||_{Lip} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |(D^{\alpha_1}b_1 f_1(x) - (D^{\alpha_2}b_2 f_1(x)) f_1(x)|^r \, dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}b_j||_{Lip} \right) \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |f(x)|^r \, dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}b_j||_{Lip} \right) M_r(f)(\tilde{x}).$$

For $I_2$, similar to the proof of $I_2$, we get

$$I_2 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}b_j||_{Lip} \right) M_r(f)(\tilde{x}).$$
Similarly, for $I_3$, we obtain

$$I_3 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n} \left( \frac{1}{|Q|} \int \left| T(D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2f_1)(x) \right|^r dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n} \left( \frac{1}{|Q|} \int \left| D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)f_1(x) \right|^r dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j}b_j\|_{\text{Lip}_3} \right) \left( \frac{1}{|Q|} \int \int |f(x)|^r dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j}b_j\|_{\text{Lip}_3} \right) M_r(f)(\tilde{x}).$$

For $I_4$, $I_5$, $I_6$ and $I_7$, by Lemma 5 and similar to the proof of $I_0$, $I_1$, $I_2$ and $I_3$, we get

$$I_4 + I_5 + I_6 + I_7 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j}b_j\|_{\text{Lip}_3} \right) \frac{1}{|Q|} \int \int |A_{tQ}T(f_1)(x)| dx$$

$$+ C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2}b_2\|_{\text{Lip}_3} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|^{1+\beta/n}} \int \int |A_{tQ}T(D^{\alpha_1}\tilde{b}_1f_1)(x)| dx$$

$$+ C \sum_{|\alpha_1|=m_1} \|D^{\alpha_1}b_1\|_{\text{Lip}_3} \sum_{|\alpha_2|=m_2} \frac{1}{|Q|^{1+\beta/n}} \int \int |A_{tQ}T(D^{\alpha_3}\tilde{b}_2f_1)(x)| dx$$

$$+ C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|^{1+2\beta/n}} \int \int |A_{tQ}T(D^{\alpha_3}\tilde{b}_1D^{\alpha_2}\tilde{b}_2f_1)(x)| dx$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j}b_j\|_{\text{Lip}_3} \right) \sum_{|\alpha|=m} M_r(f)(\tilde{x}).$$

For $I_8$, we write

$$R_{m_j+1}\tilde{b}_j(x, \cdot) f_2(y) dy$$

$$= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} (K(x, y) - K_{tQ}(x, y)) f_2(y) dy$$

$$= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} (K(x, y) - K_{tQ}(x, y)) f_2(y) dy$$

$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{D^{\alpha_1}\tilde{b}_1(y)(x-y)^{\alpha_1} R_{m_2}(\tilde{b}_2; x, y)}{|x-y|^m} (K(x, y) - K_{tQ}(x, y)) f_2(y) dy$$

$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_2} R_{m_1}(\tilde{b}_1; x, y)}{|x-y|^m} (K(x, y) - K_{tQ}(x, y)) f_2(y) dy$$

$$+ \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{\mathbb{R}^n} \frac{D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} (K(x, y) - K_{tQ}(x, y)) f_2(y) dy.$$
By Lemma 2 and the following inequality, for \( b \in \text{Lip}_p(R^n) \),
\[
|b(x) - b_Q| \leq \frac{1}{|Q|} \int_Q ||b||_{\text{Lip}_p} |x - y|^\beta \, dy \leq C ||b||_{\text{Lip}_p} (|x - x_0| + \sqrt{n}d/2)^\beta,
\]
we get
\[
|R_{m_j}(\tilde{b}_j; x, y)| \leq C \sum_{|\alpha| = m_j} ||D^\alpha b_j||_{\text{Lip}_p} (|x - x_0| + d)^{m_j + \beta}.
\]
Note that \(|x - y| \sim |x_0 - y|\) for \( x \in Q \) and \( y \in R^n \setminus \tilde{Q} \), so that, we obtain, by the conditions on \( K \) and \( K_{t_0} \),
\[
\left| (T - A_{t_0} T) \left( \prod_{j=1}^2 R_{m_{j+1}}(\tilde{b}_j; x, \cdot) \right) f_2 \right|
\leq C \int_{R^n \setminus \tilde{Q}} \frac{|x - x_0|^\delta}{|x_0 - y|^{m_2 + n + \delta}} \prod_{j=1}^2 |R_{m_j}(\tilde{b}_j; x, y)||f(y)|dy
\quad + C \sum_{|\alpha_1| = m_1} \int_{R^n \setminus \tilde{Q}} \frac{|D^{\alpha_1} \tilde{b}_1(y)||x - x_0|^\delta}{|x_0 - y|^{m_2 + n + \delta}} |R_{m_2}(\tilde{b}_2; x, y)||f(y)|dy
\quad + C \sum_{|\alpha_2| = m_2} \int_{R^n \setminus \tilde{Q}} \frac{|D^{\alpha_2} \tilde{b}_2(y)||x - x_0|^\delta}{|x_0 - y|^{m_1 + n + \delta}} |R_{m_1}(\tilde{b}_1; x, y)||f(y)|dy
\quad + C \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \int_{R^n \setminus \tilde{Q}} \frac{|D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)||x - x_0|^\delta}{|x_0 - y|^{n + \delta}} |f(y)|dy
\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_p} \right) \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} \frac{|x - x_0|^\delta}{|x_0 - y|^{n + \delta - 2k}} |f(y)|dy
\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_p} \right) |Q|^{2\beta/n} \sum_{k=0}^{\infty} 2^{k(2\beta - \delta)} \frac{1}{|2^{k+1} \tilde{Q}|} \int_{2^k \tilde{Q}} |f(y)|dy
\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_p} \right) |Q|^{2\beta/n} M_r(f)(\tilde{x}),
\]
thus
\[
I_8 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_p} \right) M_r(f)(\tilde{x}).
\]
We now put these estimates together and take the supremum over all \( Q \) such that \( \tilde{x} \in Q \), then, by Lemma 1,
\[
||T^b(f)||_{F^{2\beta, \infty}_{p, \delta}} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_p} \right) ||M_r(f)||_{L^p} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_p} \right) ||f||_{L^p}.
\]
This completes the proof of Theorem 1.
Proof of Theorem 2. By using the same argument as in proof of Theorem 1, we obtain
\[
\frac{1}{|Q|} \int_Q |T^b(f)(x) - A_{t_Q} T^b(f)(x)| dx \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_{\beta}} \right) M_{2\beta,r}(f),
\]
for any cube $Q$ and $1 < r < p < \infty$, thus, we get the sharp estimate of $T^b$ as following
\[
M_A^\# (T^b(f)) \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_{\beta}} \right) M_{2\beta,r}(f).
\]
By Lemma 3, $||M(g)||_{L^p} \leq C ||M_A^\# (g)||_{L^p}$ for every $g \in L^p(\mathbb{R}^n)$ and $1 < p < \infty$. Now, using Lemma 1 and 4, we get
\[
||T^b(f)||_{L^q} \leq ||M(T^b(f))||_{L^q} \leq C ||M_A^\# (T^b(f))||_{L^q} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{\text{Lip}_{\beta}} \right) ||f||_{L^p},
\]
This completes the proof of the theorem.

4. Applications

In this section we shall apply Theorems of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [6][9]). Given $0 \leq \theta < \pi$, define
\[
S_\theta = \{ z \in \mathbb{C} : |\text{arg}(z)| \leq \theta \} \cup \{0\}
\]
and its interior by $S_\theta^0$. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. An closed operator $L$ on some Banach space $E$ is said to be of type $\theta$ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta, \pi]$, there exists a constant $C_\nu$ such that
\[
|\eta|||(\eta I - L)^{-1}|| \leq C_\nu, \quad \eta \notin \tilde{S}_\theta.
\]
For $\nu \in (0, \pi]$, let
\[
H_\infty(S_\mu^0) = \{ f : S_\theta^0 \to \mathbb{C} : f \text{ is holomorphic and } ||f||_{L^\infty} < \infty \},
\]
where $||f||_{L^\infty} = \sup\{|f(z)| : z \in S_\mu^0\}$. Set
\[
\Psi(S_\mu^0) = \left\{ g \in H_\infty(S_\mu^0) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \leq c \frac{|z|^s}{1 + |z|^{2s}} \right\}.
\]
If \( L \) is of type \( \theta \) and \( g \in H_\infty(S^0_\mu) \), we define \( g(L) \in L(E) \) by
\[
g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta) d\eta,
\]
where \( \Gamma \) is the contour \( \{ \xi = re^{\pm i\phi} : r \geq 0 \} \) parameterized clockwise around \( S_\theta \) with \( \theta < \phi < \mu \). If, in addition, \( L \) is one-one and has dense range, then, for \( f \in H_\infty(S^0_\mu) \),
\[
f(L) = [h(L)]^{-1}(fh)(L),
\]
where \( h(z) = z(1 + z)^{-2} \). \( L \) is said to have a bounded holomorphic functional calculus on the sector \( S_\mu \), if
\[
\|g(L)\| \leq N\|g\|_{L_\infty}
\]
for some \( N > 0 \) and for all \( g \in H_\infty(S^0_\mu) \).

Now, let \( L \) be a linear operator on \( L^2(R^n) \) with \( \theta < \pi/2 \) so that \(( -L ) \) generates a holomorphic semigroup \( e^{-zL} \), \( 0 \leq |\arg(z)| < \pi/2 - \theta \). Applying Theorem 6 of [9], we get

**Theorem 3.** Let \( 0 < \beta < 1 \) and \( 1 \leq p < \infty \). The Triebel-Lizorkin space associated with the “approximations to the identity” \( \{ A_t, t > 0 \} \) is defined by
\[
\dot{F}_{p,A}^{\beta,\infty}(R^n) = \{ f \in L^1_{loc}(R^n) : ||f||_{\dot{F}_{p,A}^{\beta,\infty}} < \infty \},
\]
where
\[
||f||_{\dot{F}_{p,A}^{\beta,\infty}} = \left\| \sup_{Q \supset \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - A_{tQ}(f)(x)| dx \right\|_{L^p},
\]
Assume the following conditions are satisfied:

(i). The holomorphic semigroup \( e^{-zL} \), \( 0 \leq |\arg(z)| < \pi/2 - \theta \) is represented by the kernels \( a_z(x,y) \) which satisfy, for all \( \nu > \theta \), an upper bound
\[
|a_z(x,y)| \leq c_\nu h_{|z|}(x,y)
\]
for \( x,y \in R^n \), and \( 0 \leq |\arg(z)| < \pi/2 - \theta \), where \( h_t(x,y) = Ct^{-n/2} s(|x - y|^2/t) \) and \( s \) is a positive, bounded and decreasing function satisfying
\[
\lim_{r \to \infty} r^{n+\epsilon} s(r^2) = 0.
\]
(ii). The operator $L$ has a bounded holomorphic functional calculus in $L^2(\mathbb{R}^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S^0_L)$, the operator $g(L)$ satisfies

$$\|g(L)(f)\|_{L^2} \leq c_\nu \|g\|_{L^\infty} \|f\|_{L^2}.$$ 

Let $D^\alpha b_j \in Lip_\beta(\mathbb{R}^n)$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \ldots, l$.

(a). If $0 < \beta < \min(1, \epsilon, \delta/l)$, then the multilinear operator $g(L)^b$ associated to $g(L)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p,A}(\mathbb{R}^n)$ for any $1 < p < \infty$;

(b). If $0 < \beta < 1$, then $g(L)^b$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for any $1 < p < n/l\beta$ and $1/p - 1/q = l\beta/n$.

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**References**


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