FUZZY RELATIONAL CALCULUS

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*Abstract*___We provide a self contained survey of the state of art of the fuzzy binary relations and some of their applications.

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1 Introduction

For centuries probability theory and error calculus have been the only models to treat imprecision and uncertainty. However recently a lot of new models have been introduced for handling incomplete information. The fact that crisp relations fail in interpreting real life phenomenon was first expressed by Poincare [73] in 1902 . Half a century later, Menger [61] addressed the issue raised by Poincare and proposed his "Probabilistic relations". According to Menger in order to be in harmony with real life continuum, we should sacrifice transitivity and classical definition of relations should be changed and a probability of being related should be allocated to every pair of points belonging to the universe under consideration. Even after this development there remained a silence regarding re-building a rigorous theory of relations with different probabilities associated with them. Undoubtedly the notion of fuzzy set theory initiated by Zadeh [92] in 1965 in a seminal paper, plays the central role for further development. This notion tries to show that an object corresponds more or less to the particular category we want to assimilate it to; that was how the idea of defining the membership of an element to a set not on the Aristotelian pair {0,1} any more but on the continuous interval $[0, 1]$ was born. The notion of a fuzzy set is completely non-statistical in nature and the concept of fuzzy set provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables. In fact the idea of describing all shades of reality was for long the obsession of some logicians [60] [78]. During last four decades the fuzzy set theory has rapidly developed into an area which scientifically as well as from the application point of view, is recognized as a very valuable contribution to the existing knowledge (see [13],[14], [16], [17], [18], [19], [20], [29], [44], [48], [49], [53], [54], [55], [58], [63], [72], [94], [95] and [96]). After the emergence of fuzzy set theory in

1965 [92], the simple task of looking at relations as fuzzy sets on the universe $X \times X$ was accomplished in a celebrated paper by Zadeh [93], he introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy ordering. Fuzzy relations have broad utility. Compared with crisp relations, they have greater expressive power. They are considered as softer models for expressing the strength of links between elements. Starting in early seventies, fuzzy relations have been defined, investigated, and applied in many different ways e.g., in fuzzy modeling, fuzzy diagnosis, and fuzzy control. They also have applications in fields such as Artificial Intelligence, Psychology, Medicine, Economics, and Sociology. In this survey article our aim is to assemble a summary for the theory of fuzzy relations developed so far.

To provide a self contained survey of the state of art of the fuzzy binary relations and their applications, section 2 is devoted to a comprehensive introduction to the basic definitions of fuzzy set theory. Major source for definitions included in this section are [96], [46], [63], [53] and [36]. It is important to mention that the fuzzy relational compositions and images were initially defined and studied by Bandler and Kohout [3] [4] [5]. Afterwards De Baets and Kerre further developed the theory [31], [32], [34] and [36]. In Section 3, we state the properties of fuzzy relations fuzzified from those of crisp relations by Zadeh [93]. Next we focus on different forms of fuzzy equivalence relations found in literature as Similarity relations, Likeness relations, Probabilistic relations and T- equivalence relations. In this context, first we summarized Menger's [61] work on Probabilistic theory of relations. After that we study Similarity relations. Then the Likeness relations developed by Bezdek [22] are given. All these relations claim to model approximate equality and are supposed to solve the Poincare paradox: that in physical continuum Equality is nontransitive. All these relations were also generalized by Trillas [83] under the name T-

indistinguishability operators. Now we review the definition of T- transitivity and that of T- transitive equivalence relations due to Boixader [27], Valverde [87], Jacas [50], Ovchinnikov and Riera [70] and [88]. A thorough discussion on construction of fuzzy equivalence relations from a single criteria and from multi criteria is also included.

Section 5 deals with the task of studying fuzzy orderings. We start from the fuzzy orderings defined on the basis of max-min transitivity. Different types of fuzzy orderings, selected from the papers of Ovchinnikov [69], [68] and concepts related with them are presented . A number of examples are also included in order to elaborate the notion of different types of fuzzy orderings. Theorems about construction and numerical representation of quasi orderings are also included which are the significant achievement of Ovchinnikov. Beg [11], [7], [12] has successfully used definitions for fuzzy orderings with some new forms of reflexivity, antisymmetry and max-min transitivity. Beg's results, on fuzzy ordered sets, fixed point theory in fuzzy ordered sets, fuzzy chains, fuzzy maximal and minimal elements, fuzzy Zorn lemma and the extension of fuzzy Zorn lemma are included. Bodenhofer et al [24], [23], [26] and [25] results on T- E- orderings, their properties, linearity of fuzzy orderings are presented.

Fuzzy subsethood or fuzzy inclusion is an important concept in the field of fuzzy set theory and it provides a basis for fuzzy similarity and measures of similarity. First attempt to define fuzzy subsethood was made by Zadeh [92]. Later on it was realized that defining fuzzy subsethood in this way is though highly appreciable and useful but is still against the spirit of fuzzy set theory, in the sense that it represents a crisp decision about being a subset or not (see [4] and [46]). Researchers working in the area of fuzzy inclusion remained interested in assigning a degree of inclusion of one fuzzy set into another (see [28], [46], [52] and [81]). So fuzzy inclusion is defined as a fuzzy relation on set $F(X)$ of all fuzzy subsets of X. We hold that the fuzzy inclusion relation is a very important type of fuzzy relation and we have therefore opted to include an introduction to fuzzy inclusion relation in Section 7 from the work of [81], [28] and [52]. Fuzzy preference structures as a triplet of fuzzy relation satisfying certain properties played an important role while working on fuzzy relations. In this context we have made selections from Roubens and Vincke [75], [76], De Baets and Fodor [30] and De Baets, Van de Walle, and Kerre [39]. Some of the definitions for the constructions of fuzzy preference structures are from. Orlovski [64] and [65]. In the end, few definitions and results from [8] and [86] are included on continuity and linearity property of fuzzy multivalued mappings or fuzzy relations.

2 Fuzzy Sets

2.1 Basic Definitions

Throughout this paper we study only the fuzzy binary relations and we do not make any reference to n-ary relation so we use the term fuzzy relations instead of fuzzy binary relations.

Definition 2.1.1 [92] A *fuzzy set* A in a universe X is a mapping from X to $[0, 1]$. For any $x \in X$ the value $A(x)$ (or $\mu_A(x)$) is called the *degree of membership* of x in A. X is called the *carrier* of the fuzzy set A. The degree of membership can also be represented by x instead of $A(x)$. The class of all fuzzy sets in X is denoted by $F(X)$.

The fuzzy sets taking their values only on the boundary of $[0, 1]$ are called crisp sets. For example the whole universe X and the empty set $\emptyset$ are fuzzy sets defined as: $X(x) = 1$ for all $x \in X$ and $\emptyset(x) = 0$ for all $x \in X$.

Example 2.1.2 [36] As an example of fuzzy set consider the *fuzzy set of large numbers* L in the set of natural numbers whose membership function is a mapping from N to $[0,1]$ defined by:

$$L(n) = (1 + (\frac{100}{n})^3)^{-1}.$$

The above function associates with every natural number n, a degree to which it satisfies the description large number. For example, the number 100 is a large number to the degree 0.5 and the number 10 is a large number to the degree $\frac{1{,}000{,}000}{1{,}000{,}001}$.

Remark 2.1.3 [49] One can easily connect the above definition to $F(X) = I^X$ = the set of all $[0,1]-$valued mappings on X.

Definition 2.1.4 [92] Let $A, B \in F(X)$, The *inclusion* of A into B and the *equality* of A and B are defined as:

(i) $A \subseteq B$ if and only if $A(x) \leq B(x)$, for all $x \in X$.

(ii) $A = B$ if and only if $A(x) = B(x)$ for all $x \in X$.

Remark 2.1.5 [36] The choice of a membership function is both context and observer dependent. Initially, $[0,1]$ was taken to evaluate the degrees of membership. However, in some applications, this choice is too restrictive. The unit interval is totally ordered and does not allow incomparable degrees of membership. Goguen [47] has extended the concept of a fuzzy set by using a complete lattice to evaluate the degrees of membership. In such cases fuzzy set theory can be described as the Mathematics of lattice-valued mappings (see [49]).

Definition 2.1.6 [96] Let A be a fuzzy set in X and $A(x)$ be its membership function, then two important operators: the *height operator* $hgt(A)$ and the *plinth operator* $plinth(A)$ operators are defined as follows:

$$\begin{aligned} hgt(A) &= \sup_{x \in X} A(x), \\ plinth(A) &= \inf_{x \in X} A(x). \end{aligned}$$

A fuzzy set A is called *normal* if $hgt(A) = 1$. A nonempty fuzzy set A can always be normalized by dividing $A(x)$ by $hgt(A)$.

Definition 2.1.7 [98] The *scalar cardinality* of a fuzzy subset A of X is defined as:

$$|A| = \sum_{x \in X} A(x).$$

Definition 2.1.8 [96] A fuzzy set A in $\mathbb{R}$ is *convex* if

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2)),$$

for all $x_1, x_2 \in A$ and $\lambda \in [0,1]$.

Definition 2.1.9 [74] A fuzzy set in X is called a *fuzzy point* x_λ (or (x, λ)) if and only if it is of the following form:

$$(x,\lambda)(y) = \begin{cases} 1, & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

where, point x is called its *support*. The fuzzy point x_λ is said to be contained in a fuzzy set A if $\lambda \leq A(x)$.

Definition 2.1.10 [96] A *fuzzy number* M is a convex normalized fuzzy set M of real line $\mathbb{R}$ such that:

1. There exists exactly one $x_0 \in \mathbb{R}$, such that:

$$M(x_0) = 1,$$

   x_0 is then called the *mean value* of M.

2. $M(x)$ is piecewise continuous.

A fuzzy number M is called *positive (negative)* if its membership function is such that $\mu_M(x) = 0$, for all $x < 0$

(for all $x > 0$).

Definition 2.1.11 [96] The *weak α-cuts* A_α and the *strong α-cuts* $A_{\overline{\alpha}}$, are defined as follows:

$$A_\alpha = \{x \mid x \in X \text{ and } A(x) \geq \alpha\},$$

for all $\alpha \in]0,1]$;and

$$A_{\overline{\alpha}} = \{x \mid x \in X \text{ and } A(x) > \alpha\},$$

for all $\alpha \in [0, 1[$.

The value $\alpha = 0$, is excluded for weak α-cuts and $\alpha = 1$, for the strong α-cuts since A_0 and $A_{\overline{1}}$ are both independent of A, and hence yield no new information. $A_{\overline{0}}$ and A_1 are called *support* and *core* of A respectively.

Remark 2.1.12 [46] The α-cuts of A are nested in the sense that $\alpha > \beta$ implies that $A_\alpha \subseteq A_\beta$. In particular,

$$A_\alpha = \bigcap_{\beta<\alpha} A_\beta.$$

Going from the level cut representation to the membership function and back is easy. The membership function can be recovered from the level cuts as follows:

$$A(u) = sup\{\alpha : u \in A_\alpha\} = \sup_{\alpha\in[0,1]} \min(\alpha, A(\alpha)),$$

with $A_\alpha(u) = 1$ if $u \in A_\alpha$ and 0 otherwise.

Negoita and Ralescu [62] obtained a representation theorem according to which, a given fuzzy set can be represented by a combination of its α- level cuts and conversely given a family of crisp sets one can construct a fuzzy set from them under certain conditions.

2.2 Fuzzy Conjunction and Disjunction

Definition 2.2.1 [92] If A,$B \in F(X)$, then for all $x, y \in X$, following fuzzy sets were defined by Zadeh:

(i) $A \cup B(x) = max(A(x), B(x))$;

(ii) $A \cap B(x) = min(A(x), B(x))$;

(iii) $A^c(x) = 1 - A(x)$;

Remark 2.2.2 [36] These operations follow *De Morgan's laws, idempotency laws, commutativity laws, associativity laws, absorption laws and distributivity laws,* but the *law of contradiction* and *the law of excluded middle* no more hold. Anyhow, the overlap between a set and its complement is bounded by the following way:

$$\forall x \in X,\ A \cap A^c(x) \leq 0.5,$$

and the corresponding weakened law of excluded middle is:

$$\forall x \in X,\ A \cup A^c(x) \geq 0.5.$$

The *max* and *min* operations play a key role in the literature of fuzzy sets but these are not the only candidates as fuzzy extensions of the crisp disjunction and conjunction. Zimmermann and Zysno [97] proved that any t-norm T and any t-conorm δ can be used to model fuzzy intersection and union respectively.

Definition 2.2.3 [79] The *triangular norm (t-norm)* T and *triangular conorm (t-conorm)* δ are increasing, associative, commutative and $[0,1]^2 \to [0,1]$ mappings satisfying: $T(1,x) = x$ and $\delta(x,0) = x$, for all $x \in [0,1]$.

To every t-norm T there corresponds a t-conorm T^* called the *dual t-conorm*, defined by:

$$T^*(x,y) = 1 - T(1-x, 1-y).$$

Remark 2.2.4 [79] Following are some popular choices for t-norms:

(i) The minimum operator $M:\ M(x,y) = min(x,y)$.

(ii) The Lukasiewicz's t-norm $W:\ W(x,y) = max(x+y-1, 0)$.

(iii) The product operator $P:\ P(x,y) = xy$.

The corresponding dual t-conorms are**:**

(i) The maximum operator $M^*:\ M^*(x,y) = max(x,y)$.

(ii) The bounded sum $W^*:\ W^*(x,y) = min(x+y, 1)$.

(iii) The probabilistic sum $P^*\ :\ P^*(x,y) = x + y - xy$.

Remark 2.2.5 [53] Here is a table that gives us information about the relationship of these fuzzy conjunctions and disjunctions with laws satisfied by their crisp counter parts:

	(M,M^*)	(W,W^*)	(P,P^*)
De Morgan laws	Y	Y	Y
Commutativity	Y	Y	Y
Associativity	Y	Y	Y
Idempotency	Y	N	N
Absorption law	Y	N	N
Distributivity laws	Y	N	N

It is important to note that if a system disobeys some law, most of the times it satisfies some weaker version of the law. The behavior of Lukasiewicz system is different in the sense that it does not satisfy even the weakened distributive laws but the law of contradiction and the law of excluded middle hold for it.

An important concept associated with t-norms and t-conorms is that of compatibility. Two crisp sets are compatible if they have at least one element in common. Similarly, in fuzzy context it is defined as the maximum degree of overlap i.e. as the height of the intersection of these fuzzy sets.

Definition 2.2.6 [85] Consider two fuzzy sets A and B in X, and a t-norm T. The degree of compatibility $Com_T(A, B)$ of A and B is:

$$Com_T(A, B) = \sup_{x \in X} T(A(x), B(x)).$$

Remark 2.2.7 [46] A broad class of problems consists of representation of multi-place functions, in general, by composition of simpler functions of fewer variables [59] such as:

$$K(x, y) = g(f(x) + f(y)).$$

Abel [1] obtained the first such representation. He assumed in addition the commutativity, strict monotonicity and differentiability. Since Abel's result, a lot of contributions have been made in the field of representation theory of associative functions (and generally speaking of abstract semigroups). It was Ling [59] who proved that *min* cannot be represented in the above form assuming either f, g to be continuous or strictly decreasing functions. This negative result indicates that a suitable class of t-norms should be considered. The properties of these t-norms are given next. For any $x \in [0,1]$ and any associative binary operation K on $[0,1]$, $x_K^{(n)}$ denotes the *nth* power of x defined $x_K^{(n)} = K(x, ..., x)$ (see [46]).

Definition 2.2.8 [46] A t-norm T (resp. a t-conorm δ) is said to be:

(i) *continuous*, if T (resp.δ) is continuous as a function on the unit interval;

(ii) *Archimedean*, if $\lim_{n\to\infty} x_T^{(n)} = 0$ (resp $\lim_{n\to\infty} x_\delta^{(n)} = 1$), for all $x \in]0,1[$.

Note that the definition of the Archimedean property has been borrowed from the theory of semigroups.

Remark 2.2.9 [46] If T is Archimedean, then it satisfies:

$$T(x,x) < x \text{ for all } x \in]0,1[. \tag{1}$$

If a t-norm is continuous, then it is Archimedean, if and only if, it satisfies (1). Similarly, if δ is Archimedean, then it satisfies

$$\delta(x,x) > x \text{ for all } x \in]0,1[. \tag{2}$$

If a t-conorm is continuous, then it is Archimedean, if and only if it satisfies (2). Generally, $T(x,x) < x$ does not imply the Archimedean property for discontinuous t-norms (see [79] for counter example).

Definition 2.2.10 [45] A t-norm T has *zero divisors* if and only if there exist $(x,y) \in]0,1]^2$ such that $T(x,y) = 0$.

In this case, x and y are the zero divisors of T. For example, 0.25 and 0.75 are zero divisors of W. A t-norm without zero divisors is called *positive*.

Theorem 2.2.11 [46] A t-norm T is Archimedean if and only if it can be represented in the following form:

$$T(x, y) = g(f(x) + f(y)),$$

where,

(a) $f : [0, 1] \to \mathbb{R}^{+} = [0, \infty]$ is a continuous strictly decreasing function such that $f(1) = 0$;

(b) g is continuous function from R^{+} onto $[0, 1]$ such that $g(x) = f^{-1}(x)$ on $[0, f(0)]$ and $g(x) = 0$, for $x \in g(0)$.

In this case, f is said to be an *additive generator* of T.

An order preserving permutation of the unit interval is called a $[0, 1]$- *automorphism*. Any automorphism ϕ is continuous and satisfies the boundary conditions $\phi(0) = 0$ and $\phi(1) = 1$.

Definition 2.2.12 [63] Consider a t-norm T and a $[0, 1]-$automorphism ϕ.

(1). the $\phi-$transform of T is the t-norm defined by:

$$T_{\phi}(x, y) = \phi^{-1}(T(\phi(x), \phi(y))).$$

(2). the $\phi-$transform of a t-conorm δ is the t-conorm δ_{ϕ} defined by:

$$\delta_{\phi}(x, y) = \phi^{-1}(\delta(\phi(x), \phi(y))).$$

The *min* t-norm and the *max* t-conorm are not affected by ϕ transforms.

Remark 2.2.13 [30] Continuous Archimedean t-norms are further divided into two classes:

(1). the class of strict t-norms. A t-norm is called *strict* if it is continuous and all partial mappings $T(x, .)$ and $T(., y)$ are strictly increasing.

(2). the class of nilpotent t-norms. A t-norm is called *nilpotent* if it is continuous, Archimedean and non-strict.

Remark 2.2.14 [30] The strict and nilpotent t-norms can be characterized by means of $\phi-$transforms of basic t-norms:

(i) A t-norm T is strict if and only if there exists a [0,1]$-$automorphism $\phi-$ such that T is the $\phi-$transform of the algebraic product P.

(ii) A t-norm T is nilpotent if and only if there exists a [0,1]-automorphism $\phi-$ such that T is the $\phi-$transform of the Lukasiewicz t-norm W.

Another characterization of strict and nilpotent t-norms is based on the notion of zero divisors:

(i) A t-norm T is *strict* if and only if it is continuous, Archimedean and positive.

(ii) A t-norm T is *nilpotent* if and only if it is continuous, Archimedean and has zero divisors

Proposition 2.2.15 [63] Consider a $[0,1]-$ automorphism ϕ. The N_ϕ-dual t-conorm of the $\phi-$transform of a t-norm T is given by:

$$T_\phi^{N_\phi}(x,y) = \phi^{-1}(T^*(\phi(x), \phi(y))).$$

In particular, we have that the N_ϕ-dual t-conorm of the $\phi-$transform of the t-norm W is given by:

$$W_\phi^{N_\phi}(x,y) = \phi^{-1}(\min(\phi(x) + \phi(y), 1)).$$

2.3 The Negator Operator N

Definition 2.3.1 [63] A *negator* N is an order-reversing $[0,1] \to [0,1]$ mapping such that $N(0) = 1$ and $N(1) = 0$.

Negators are used to model complementation in the calculus of fuzzy set theory i.e., for a given fuzzy set A in the universe X, its complement A^c is defined by negator N as:

$$A^c(x) = N(A(x)) \text{ for all } x \in X.$$

Definition 2.3.2 [46] Some important types of negators are:

(i) A negator is *strict*, if it is strictly decreasing and continuous.

(ii) A negator is *involutive* if $N(N(x)) = x$, for all $x \in [0, 1]$.

(iii) A strict and involutive negator is called a *strong negator*.

A popular strong negator is the standard negator N_s (Zadeh's complement) defined as:

$$N_s(x) = 1 - x.$$

An example of a strict negator which is not strong can be given by $N(x) = 1 - x^2$.

Definition 2.3.3 [82] A family of strong negators including the standard one is defined by Sugeno in 1977 (under the name $\lambda-$*complement*) as follows:

$$N_\lambda(x) = \frac{1 - x}{1 + \lambda x}, \quad \lambda > -1, \; x \in [0, 1].$$

Because any strict negator is strictly decreasing and continuous function, one can define its inverse N^{-1} which is also a strict negator and in general differs from N. Clearly, we have $N^{-1} = N$ if and only if n is involutive too. Another very important property of the strict negators is that there exists a unique value $0 < S_N < 1$ such that $N(S_N) = S_N$. This value is called membership crossover point by Dubois and Prade [43].

Definition 2.3.4 [63] A triplet (T, S, N) of a t-norm T, a t-conorm δ and a strong negator N is called a De Morgan's triplet, if and only if

$$\begin{aligned} \delta(x, y) &= N(T(N(x), N(y))), \text{ for all } x, y \in [0, 1] \\ \text{or } \delta &= T^N. \end{aligned}$$

Proposition 2.3.5.[83] A $[0, 1] \to [0, 1]-$mapping N is an *involutive negator* if and only if, there exists a $[0, 1]-$automorphism ϕ such that for all $x \in [0, 1]$

$$N(x) = \phi^{-1}(1 - \phi(x)).$$

The involutive negator N defined by$N(x) = \phi^{-1}(1 - \phi(x))$ is called the ϕ-transform of the standard negator N_s, and will be denoted as N_ϕ.

2.4 Fuzzy Implication Operator

Definition 2.4.1 [4] A *fuzzy implicator* I is a binary operation on $[0,1]$ with order reversing first partial mappings and order preserving second partial mappings such that:

$$I(0,1) = I(0,0) = I(1,1) = 1, I(1,0) = 0.$$

Definition 2.4.2 [80] From an axiomatic point of view, the following properties are considered as the axioms for a fuzzy implicator by Smets and Magrez [80]. A fuzzy implication I is a $[0,1]^2 \longrightarrow [0,1]$ mapping satisfying the following axioms:

A1. Contraposition: $(\forall x, y \in [0,1]), I(x,y) = I(1-y, 1-x)$;

A2. Exchange Principle: $(\forall x, y, z \in [0,1]), I(x, I(y,z)) = I(y, I(x,z))$;

A3. Hybrid Monotonicity: stated in definition 2.4.1;

A4. Boundary Conditions; $x \leq y \Longleftrightarrow I(x,y) = 1$, for all $x, y \in [0,1]$;

A5. Neutrality Principle: $I(1,x) = x$, for all $x \in [0,1]$;

A6. Continuity: I is continuous.

To every implication operator, there corresponds an implication operator I^* defined by:

$$I^*(x,y) = I(1-y, 1-x).$$

An implicator satisfying neutrality principle is also called a *border implicator* or B-implicator.

Definition 2.4.3 [36] Consider a t-norm T, its corresponding t-conorm δ. Then the mappings: $[0,1]^2 \rightarrow [0,1]$ defined by:

(1). $I_1^T(x,y) = \delta\ (T(x,y), 1-x)$,

(2). $I_2^T(x,y) = \delta(T(1-x, 1-y), y)$,

(3). $I_3^T(x,y) = \delta(1-x, y)$,

(4). $I_4^T(x,y) = sup\{z \mid z \in [0,1] \text{ and } T(x,z) \leq y\}$,

(5). $I_5^T(x,y) = sup\{z \mid z \in [0,1] \text{ and } T(1-y,z) \leq 1-x\}$.

are implication operators [34].

For the list of properties of these operators, see [36]. The implicator listed in 4 is called *R-implication or residual implication.*

Lemma 2.4.4 [26] If T is left-continuous, and I is the R-implication associated with T, then following holds for all $x, y, z \in [0,1]$:

(1). $T(x,y) \leq z \Longleftrightarrow x \leq I(y,z)$;

(2). $x \leq y \Longleftrightarrow I(x,y) = 1$;

(3). $T(I(x,y), I(y,z)) \leq I(x,z)$;

(4). $I(1,y) = y$;

(5). $T(x, I(x,y)) \leq y$.

Remark 2.4.5 [53] The following is a list of some important implicators. It is noteworthy that the definitions of these implicators have been borrowed from Smets and Kerre's work (for details see [53]), so we adopt the same notations as adopted there. So, for all $x, y \in [0,1]$:

(i) $I_b(x,y) = max(1-x, y)$;

(ii) $I_a(x,y) = min(1-x+y, 1)$;

(iii) $I_*(x,y) = 1-x+xy$;

(iv) $I_\#(x,y) = min(max(1-x,y), max(x, 1-x), max(y, 1-y))$;

(v) $I_m(x,y) = max(1-x, min(x,y))$;

(vi) $I_g(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$

Remark 2.4.6 [53], [77] By the definitions of these implicators, it can be easily established that:

$$I_\# \leq I_m \leq I_b \leq I_* \leq I_a.$$

Remark 2.4.7 [77] All the implicators being used may not satisfy the axioms of fuzzy implication operators stated in Definition 2.4.5. Here is a list describing the properties possessed by each implicator:

Implicator	Axioms satisfied
I_a	*A1, A2, A3, A4, A5, A6.*
I_*	*A1, A2, A3, A5, A6.*
I_b	*A1, A2, A3, A4, A6.*
I_m	*A5, A6.*
$I_\#$	*A1, A2, A5, A6.*

Definition 2.4.8 [36] A mapping $\varepsilon : [0,1]^2 \longrightarrow [0,1]$ is called an *equivalence operator* if it satisfies the boundary conditions $\varepsilon(0,0) = \varepsilon(1,1) = 1$ and $\varepsilon(0,1) = \varepsilon(1,0) = 0$.

3 Crisp and Fuzzy Relational Calculus

In this section first the crisp relational calculus will be studied, and then fuzzy relational calculus will be studied as its step by step fuzzification.

3.1 Crisp Relational Calculus

Definition 3.1.1 [96] A crisp relation R from a universe X to a universe Y is a subset of $X \times Y$. The statement $(x,y) \in R$ is abbreviated as xRy and one says that x is in (binary) relation R with y. In this case, we define:
(i) the *after set* xR of x as :

$$xR = \{y \mid \; y \in Y \text{ and } xRy \};$$

(ii) the *foreset* Ry of y as:

$$Ry = \{x \mid \; x \in X \text{ and } xRy\};$$

(iii) the *domain* $dom(R)$ of R as:

$$dom(R) = \{x \mid x \in X \text{ and there exists } y \in Y \text{ such that } xRy\};$$

(iv) the *range* $rng(R)$ of R is the subset of Y defined as:

$$rng(R) = \{y \mid y \in Y \text{ and there exists } x \in X \text{ such that } xRy\};$$

(v) the *converse relation* (or *inverse relation*) R^t of R is the relation from Y to X defined as:

$$R^t = \{(y, x) \mid (y, x) \in Y \times X \text{ and } xRy\}.$$

(vi) the *complement relation* coR of R is the relation from Y to X defined as:

$$coR = \{(y, x) \mid (y, x) \in Y \times X \text{ and } (x, y) \notin R\}.$$

Definition 3.1.2 [36] Consider a relation R from X to Y and a subset A of X. The following four kinds of *images of a set under a relation* are defined as:
(i) the *direct image*:

$$\begin{aligned} R(A) &= \{y \mid y \in Y \text{ and there exists } x \in X \text{ such that } xRy\} \\ &= \{y \mid y \in Y \text{ and } A \cap Ry \neq \emptyset\}; \end{aligned}$$

(ii) the *lower image*:

$$R_{\triangleleft}(A) = \{y \mid y \in Y \text{ and } \emptyset \subset A \subseteq Ry\};$$

(iii) the *upper image*:

$$R_{\triangleright}(A) = \{y \mid y \in Y \text{ and } \emptyset \subset Ry \subseteq A\};$$

(iv) the *ultra direct image*:

$$R_{\diamond}(A) = \{y \mid y \in Y \text{ and } \emptyset \subset Ry = A\}.$$

Definition 3.1.3 [36] Given $B \subseteq Y$, the direct images of B under the converse relation R^t are called the *inverse images* of B under R.

(i) the *direct inverse*:

$$\begin{aligned} R^{-1}(B) &= \{x \mid x \in X \text{ and there exists } y \in B \text{ such that } xRy)\} \\ &= \{x \mid x \in X \text{ and } B \cap xR \neq \emptyset\}; \end{aligned}$$

(ii) the *lower inverse:*

$$R_{\lhd}^{-1}(B) = \{x \mid x \in X \text{ and } \emptyset \subset B \subseteq xR\};$$

(iii) the *upper inverse:*

$$R_{\rhd}^{-1}(B) = \{x \mid x \in X \text{ and } \emptyset \subset xR \subseteq B\};$$

(iv) the *ultra direct inverse*:

$$R_{\diamond}^{-1}(B) = \{x \mid x \in X \text{ and } \emptyset \subset xR = B\}.$$

Example 3.1.4 [36] Consider a set of patients X and a set of symptoms Y. Let R be a relation from X to Y defined by:

$$pRs \iff \text{patient } p \text{ shows the symptom } s.$$

Further let $F \subseteq X$ be the set of all female patients in the population. Then the images are given by:

(i) $R(F)$ =the set of symptoms shown by at least one female patient;

(ii) $R_{\lhd}(F)$ =the set of symptoms shown by all female patients;

(iii) $R_{\rhd}(F)$ =the set of symptoms shown by at least one female patient and not by any male patient;

(iv) $R_{\diamond}(F)$ =the set of symptoms shown by all female patients and not by any male patient.

3.1.5 Properties of Images [36]

These images possess many interesting properties, some are listed below. For a detailed study we refer to [31].

(i) $R_\diamond(A) = R_\triangleleft(A) \cap R_\triangleright(A)$,

(ii) $R_\diamond(A) \subseteq R_\triangleleft(A) \subseteq R(A)$,

(iii) $R_\diamond(A) \subseteq R_\triangleright(A) \subseteq R(A)$,

(iv) $R_\triangleleft(A) = co(co(R)(A))$, if $A \neq \emptyset$,

(v) $R_\triangleright(A) = co(R(coA)) \cap rng(R)$, if $A \neq \emptyset$,

(vi) $A \subseteq B \Rightarrow R(A) \subseteq R(B)$,

(vii) $\emptyset \subset A \subseteq B \Rightarrow R_\triangleleft(A) \subseteq R_\triangleleft(B)$,

(viii) $A \subseteq B \Rightarrow R_\triangleright(A) \subseteq R_\triangleright(B)$.

Definition 3.1.6 [36] Consider a relation R from X to Y and a relation S from Y to Z. We define following four *types of compositions of relations* R and S as follows:

(i) the *direct or round composition*:

$$R \circ S = \{(x, z) \mid (x, z) \in X \times Z \text{ and } xR \cap Sz \neq \emptyset\};$$

(ii) the *lower composition*:

$$R \triangleleft S = \{(x, z) \mid (x, z) \in X \times Z \text{ and } \emptyset \subset xR \subseteq Sz\};$$

(iii) the *upper composition*:

$$R \triangleright S = \{(x, z) \mid (x, z) \in X \times Z \text{ and } \emptyset \subset Sz \subseteq xR\};$$

(iv) the *ultra composition*:

$$R \diamond S = \{(x, z) \mid (x, z) \in X \times Z \text{ and } \emptyset \subset Sz = xR\}.$$

Example 3.1.7 [36] Again let X =the set of all patients p, Y = the set of all symptoms s and Z = the set of all illnesses i.

R is a relation from X to Y and S is a relation from Y to Z defined by:
$pRs \Longleftrightarrow$patient p shows the symptom s
and $sSi \Longleftrightarrow s$ is a symptom of illness i

The compositions of R and S are given by:

(i) $p(R \circ S)i$ patient p shows at least one symptom of illness i;

(ii) $p(R \lhd S)i$ all symptoms shown by patient p are symptoms of illness i and patient p shows at least one symptom;

(iii) $p(R \rhd S)i$ patient p shows all symptoms of illness i; .

(iv) $p(R \diamond S)i$ all symptoms shown by patient p are exactly those of illness i.

3.1.8 Properties of Compositions

Next we list important properties of these compositions. For a close examination of relationship between these compositions and their properties see [31]:

(i) $R \diamond S = (R \lhd S) \cap (R \rhd S)$,

(ii) $R \diamond S \subseteq (R \lhd S) \subseteq (R \circ S)$,

(iii) $R \diamond S \subseteq (R \rhd S) \subseteq (R \circ S)$,

(iv) $R \circ (S \circ T) = (R \circ S) \circ T$,

(v) $R_1 \subseteq R_2 \Rightarrow R_1 \circ S \subseteq R_2 \circ S$,

(vi) $R_1 \subseteq R_2 \Rightarrow R_1 \lhd S \subseteq R_2 \lhd S$,

(vii) $R_1 \subseteq R_2 \Rightarrow R_1 \rhd S \subseteq R_2 \rhd S$,

(viii) $(R \circ S)^t = S^t \circ R^t$,

(ix) $(R \lhd S)^t = S^t \rhd R^t$,

(x) $(R \rhd S)^t = S^t \lhd R^t$,

(xi) $(R \diamond S)^t = S^t \diamond R^t$.

3.1.9 Properties of Binary Relations

All these definitions of potential properties of crisp relations are due to [3]. A binary relation R in a universe X is called:

(i) *covering* if and only if for all $x \in X$, there exists a $y \in X$ such that $(x, y) \in R$;

(ii) *locally reflexive* if and only if, for all $x \in X$, there exists a $y \in X$ such that $(x, y) \in R$ or $(y, x) \in R$ implies that $(x, x) \in R$;

(iii) *reflexive* if and only if for all $x \in X, (x, x) \in R$;

(iv) *symmetric* if and only if for all $x, y \in X$,

$$(x, y) \in R \implies (y, x) \in R;$$

(v) *antisymmetric* if and only if for all $x, y \in X$,

$$(x, y) \in R \text{ and } (y, x) \in R \implies x = y;$$

(vi) *strictly antisymmetric* if and only if for all $x, y \in X$,

$$(x, y) \in R \text{ implies that } (y, x) \notin R;$$

(vii) *transitive* if and only if for all $x, y, z \in X$,

$$(x, y) \in R \text{ and } (y, z) \in R \text{ imply that } (x, z) \in R.$$

These simple properties can be combined into more complex types of relations, as follows [3]:

A binary relation R in a universe X is called

(i) a *local tolerance relation* if and only if it is locally reflexive and symmetric;

(ii) a *tolerance relation* if and only if it is a local tolerance relation and it is covering;

(iii) a *local preorder relation* if and only if it is locally reflexive and transitive;

(iv) a *preorder relation* if and only if it is a local preorder relation and it is covering;

(v) a *local equivalence relation* if and only if it is locally reflexive, symmetric, and transitive;

(vi) an *equivalence relation* if and only if it is a local equivalence relation and it is covering;

(vii) a *local order relation* if and only if it is locally reflexive, antisymmetric, and transitive;

(viii) a *strict order relation* if and only if it is strictly antisymmetric and transitive.

3.1.10 Characteristic Functions of Relations [36]

A relation R from X to Y can be identified with its characteristic mapping $R : X \times Y \to \{0, 1\}$ defined as:

$$R(x,y) = \begin{cases} 1, & \text{if } (x,y) \in R \\ 0, & \text{otherwise.} \end{cases}$$

(1). The characteristic mapping of round composition is:

$$R \circ S(x,z) = \sup_{y \in X}(R(x,y) \wedge_B R(y,z)).$$

(2). The characteristic mapping of the subcomposition is:

$$R \lhd S(x,z) = (\inf_{y \in y} R(x,y) \Longrightarrow_B S(y,z) \wedge (\sup_{By \in Y} R(x,y)).$$

(3). The characteristic mapping of the supercomposition is:

$$R \rhd S(x,z) = (\inf_{y \in y} S(y,z) \Longrightarrow_B R(x,y) \wedge (\sup_{By \in Y} S(y,z)).$$

(4). The characteristic mapping of the ultra composition is:

$$\begin{aligned} R \Diamond S(x,z) &= (\inf_{y \in y} R(x,y) \Longleftrightarrow_B S(y,z) \wedge (\sup_{By \in Y} R(x,y)) \\ &= (\inf_{y \in y} S(y,z) \Longleftrightarrow_B R(x,y) \wedge (\sup_{By \in Y} S(y,z)). \end{aligned}$$

Similarly, the characteristic mappings of lower and upper images are as follows:

$$\begin{aligned} (i)\ R_{\lhd}(A)(y) &= \inf_{x \in X} A(x) \Longrightarrow_B R(x,y), \\ (ii)\ R_{\rhd}(A)(y) &= \inf_{x \in X} R(x,y) \Longrightarrow_B A(x) \end{aligned}$$

where, $\wedge_B$, $\Longrightarrow_B$ and $\Longleftrightarrow_B$ stand for Boolean conjunction, implication and equivalence respectively.

3.2 Fuzzy Relational Calculus

Definition 3.2.1 [69], [96] A *fuzzy relation* from a universe X to a universe Y is a fuzzy set in $X \times Y$. $R(x, y)$ is called the degree of relationship between x and y.

In this case:

(1). The *afterset* xR of x is the fuzzy set in Y defined by $xR(y) = R(x, y)$;

(2). The *foreset* Ry of y is the fuzzy set in X defined by $Ry(x) = R(x, y)$;

(3). The *domain* $dom(R)$ of R is the fuzzy set in X defined by:

$$dom(R)(x) = Hgt(xR);$$

(4). The *range* $rng(R)$ of R is the fuzzy set in Y defined by:

$$rng(R)(y) = Hgt(Ry);$$

(5). The *converse* fuzzy relation R^t of R is the fuzzy relation from Y to X is defined as:

$$R^t(y, x) = R(x, y);$$

(6). The α-cuts of a relation R_α are defined as follows:

$$R_\alpha = \{(x, y) \mid (x, y) \in X \times Y : R(x, y) \geq \alpha\};$$

(7). The *resolution form* of a relation $R = \bigvee_{\alpha \in [0,1]} R_\alpha$;

(8). The *complement relation*:

$$coR(x, y) = 1 - R(x, y).$$

It is interesting to remark that we can define a fuzzy relation on a universe X, with the help of two fuzzy sets on the given universe. In this procedure usually a conjunction operator such as *min* is used i.e., if A and B are two fuzzy subsets of a given universe X, then we can define a fuzzy relation R on X as:

$$R = \{(x, y), \min(A(x), B(y)\}.$$

This product can be extended to any finite number of fuzzy sets on X. Not only this but this concept can also be extended to fuzzy sets in different universes. The interesting aspect of these definitions is that they reduce to their crisp counterparts if the sets are crisp sets. These operations enjoy many beautiful properties for details see [96].

3.2.2 Crisp Composition and Images to Fuzzy Ones

As we have seen that the characteristic mappings of compositions use conjunction, disjunction and implicator operators. The fuzzy versions of all these operators have already been discussed in section on fuzzy set theory. The classical composition of relations was extended to its fuzzy counter part by Zadeh [93]. Next we shall review the fuzzy relational compositions which were initially defined by Bandler and Kohout in [5]. These definitions were then modified by De Baets and Kerre [31], [32], [34], [35] and [36]. We prefer to state only the modified forms. For the sake of clarity a subscript bk will be placed with the relations and compositions if they are being used in the forms originally defined by Bandler and Kohout.

Consider a fuzzy relation R from X to Y and a fuzzy relation S from Y to Z. The *sup min* composition $R \circ S$ of R and S is a fuzzy relation from X to Z defined by:

$$R \circ S(x, z) = \sup_{y \in Y} \min(R(x, y), S(y, z));$$

or in terms of after and fore sets:

$$R \circ S(x, z) = Hgt(xR \cap Sz).$$

where, $\cap$ is modelled by *min*. In general, if a t-norm T other than *min* is used, then:

$$R \circ^T S(x,z) = \sup_{y\in Y} T((R(x,y), S(y,z))).$$

The other compositions are extended as follows:

$$\begin{aligned}
R \lhd^I S(x,z) &= \min[\inf_{y\in Y} I(R(x,y), S(y,z)),\ \sup_{y\in Y} R(x,y), \sup_{y\in Y} S(y,z))]; \\
R \rhd^I S(x,z) &= \min[\inf_{y\in Y} I(S(y,z), R(x,y)),\ \sup_{y\in Y} R(x,y), \sup_{y\in Y} S(y,z))]; \\
\text{and } R \diamond^I S(x,z) &= \min(R \rhd^I S(x,z), R \lhd^I S(x,z)).
\end{aligned}$$

The second set of improved definitions is given by:

$$\begin{aligned}
R \lhd^{T,I} S(x,z) &= \min[\inf_{y\in Y} I(R(x,y), S(y,z)),\ \sup_{y\in Y}(I(R(x,y), S(y,z)))]; \\
R \rhd^{T,I} S(x,z) &= \min[\inf_{y\in Y} I(S(y,z), R(x,y)),\ \sup_{y\in Y}(I(R(x,y), S(y,z)))]; \\
R \diamond^{T,I} S(x,z) &= \min(R \lhd^{T,I} S(x,z),\ R \rhd^{T,I} S(x,z)).
\end{aligned}$$

Using the height and plinth operators the expressions for the triangular compositions can be written as:

$$\begin{aligned}
R \lhd_b^I S(x,z) &= \min[Plt(\ I(xR, Sz)), Hgt(xR), Hgt(Sz)]; \\
R \rhd_b^I S(x,z) &= \min[Plt(I(Sz, xR),\ Hgt(xR), Hgt(Sz))].
\end{aligned}$$

and the second set can be written in improved form as:

$$\begin{aligned}
R \lhd^{T,I} S(x,z) &= \min[Plt(\ I(xR, Sz)), Hgt(xR \cap_T Sz)]; \\
R \rhd^{T,I} S(x,z) &= \min[Plt(I(Sz, xR)),\ Hgt(xR \cap_T Sz)].
\end{aligned}$$

The second set of definitions is more restrictive than the first one, i.e., yields lower degrees of relationship, and that the first one in turn is more restrictive than the Bandler-Kohout compositions:

$$\begin{aligned}
R \lhd^{T,I} S &\subseteq R \lhd^I S \subseteq R \lhd_{bk}^I S. \\
R \rhd^{T,I} S &\subseteq R \rhd^I S \subseteq R \rhd_{bk}^I S.
\end{aligned}$$

As already indicated, an alternative way to define the ultra composition is by introducing an equivalence operator:

$$\begin{aligned} R \diamond^{I} S(x,z) &= \min[Plt(\ \varepsilon(xR,Sz)), Hgt(xR), Hgt(Sz)], \\ R \diamond^{T,I} S(x,z) &= \min[Plt(\varepsilon(Sz,xR)),\ Hgt(xR \cap_T Sz)]. \end{aligned}$$

3.2.3 [36] **Images of a Fuzzy Set under a Fuzzy Relation**

The same line of reasoning as for the compositions can be followed in this case as well. Consider a fuzzy relation R from X to Y and a fuzzy set A in X, then the direct image can be written as:

$$R^{T}(A)(y) = Hgt(A \cap_T Ry));$$

where, T represents the t-norm being used. De Baets and Kerre [36] suggest following two sets of improved definitions for triangular images:

$$\begin{aligned} R^{I}_{\triangleleft}(A)(y) &= \min(Plt(I(A,Ry), Hgt(A), Hgt(Ry)); \\ R^{I}_{\triangleright}(A)(y) &= \min(Plt(I(Ry,A), Hgt(A), Hgt(Ry)). \end{aligned}$$

The second set of improved definitions is:

$$\begin{aligned} R^{T,I}_{\triangleleft}(A)(y) &= \min(Plt(I(A,Ry), Hgt(A \cap_T Ry)); \\ R^{T,I}_{\triangleright}(A)(y) &= \min(Plt(I(Ry,A), Hgt(A \cap_T Ry)). \end{aligned}$$

Example 3.2.4 [36] Consider a set of patients X and set of symptoms Y. Let R be the fuzzy relation from X to Y defined by:

$R(p,s)$= the degree to which patient p shows symptom s.

Let O be the fuzzy set of old patients in the population X, then the direct images of O under R (for either one of the improved definitions) are given by:

- $R^{T}(O)$ is the fuzzy set of symptoms shown by at least one old patient;
- $R^{I}_{\triangleleft}(O)$ is the fuzzy set of symptoms shown by all old patients;
- $R^{I}_{\triangleright}(O)$ is the fuzzy set of symptoms shown by at least one old patient and not by any non-old patient.

Properties 3.2.5 [36] Consider three universes X, Y and Z and fuzzy relations R from X to Y, a fuzzy relation S from Y to Z, and a finite family of fuzzy relations R_i from X to Y, then following properties are satisfied by fuzzy relational compositions:

(1) *containment*:

$$
\begin{aligned}
R \lhd^{T,I} S &\subseteq R \circ^T S,\\
R \rhd^{T,I} S &\subseteq R \circ^T S,
\end{aligned}
$$

(2) *convertibility*:

$$
\begin{aligned}
(R \circ^T S)^t &= S^t \circ^T R^t,\\
(R \lhd S)^t &= S^t \lhd R^t,\\
(R \rhd S)^t &= S^t \rhd R^t,\\
(R \diamond S)^t &= S^t \diamond R^t,
\end{aligned}
$$

(3) *monotonicity*, for a hybrid monotonous implication operator:

$$
\begin{aligned}
R_1 &\subseteq R_2 \Longrightarrow R_1 \circ S \subseteq R_2 \circ S,\\
dom(R_1) &= dom(R_2) \text{ and } R_1 \subseteq R \Longrightarrow R_1 \lhd S \subseteq R_2 \lhd S,\\
R_1 &\subseteq R_2) \Longrightarrow R_1 \rhd S \subseteq R_2 \rhd S.
\end{aligned}
$$

(4) *interaction with union*, for a hybrid monotonous implication operator:

$$
\left[\bigcap_{i=1}^{n} R_i\right] \circ S \subseteq \bigcap_{i=1}^{n} (R_i \circ S),
$$

(5). *interaction with intersection*, for a hybrid monotonous implication operator:

$$
\left[\bigcap_{i=1}^{n} R_i\right] \rhd S = \bigcap_{i=1}^{n} (R_i \rhd S).
$$

Next we shall briefly review the results about the $\alpha-$cuts of these relations. For a detailed study see [35] . Consider a fuzzy relation R from X to Y and a fuzzy relation S from Y to Z. It is well known that the following equality holds, for all α in $[0,1]$, provided that Y is finite:

$$(R \circ S)_{\overline{\alpha}} = R_{\overline{\alpha}} \circ S_{\overline{\alpha}}.$$

The converse inclusion only holds when T possesses the following property:

$$x \geq \alpha \text{ and } y \geq \alpha \text{ implies } T(x,y) \leq \alpha, \text{ for all } x, y \in [0,1].$$

The only triangular norm satisfying this property for all α in $[0,1]$ is the *min* operator M. The cuttability of the Bandler-Kohout compositions and their improved versions is far more complex and can be found in [5].

3.2.6 Matrix Representation [36]

Fuzzy relations and their compositions have a special feature that when dealing with finite universes, as is often the case, relations and fuzzy relations can be represented by means of a matrix. A fuzzy relation R from $X = \{x_1, x_2, \ldots, x_l\}$ to $Y = \{y_1, y_2 \ldots, y_m\}$ can be represented by means of an $l \times m$ matrix, as follows:

$$R = \begin{bmatrix} R_{11} & . & . & R_{1m} \\ . & & & . \\ . & & & . \\ R_{l1} & . & . & R_{lm} \end{bmatrix},$$

where, R_{ij} stands for $R(x_i, y_j)$. A fuzzy set A in X can be represented by means of a row vector l with entries:

$$A = (A_1, A_{2,\ldots,} A_l),$$

where, A_j stands for $A(x_j)$. The direct image of A under R can be written as follows:

$$R^T(A) = (A_1, A_{2,}..., A_l) \begin{bmatrix} R_{11} & . & . & R_{1m} \\ . & & & . \\ . & & & . \\ R_{l1} & . & . & R_{lm} \end{bmatrix},$$

where, the matrix product is calculated using the triangular norm as multiplication and the maximum operator as addition. Now consider a fuzzy relation S from Y to $Z = \{z_1, z_2, ..., z_n\}$. The $max - T$ composition of R and S can be written as follows:

$$R \circ^T S = \begin{bmatrix} R_{11} & . & . & R_{1m} \\ . & & & . \\ . & & & . \\ R_{l1} & . & . & R_{lm} \end{bmatrix} \begin{bmatrix} S_{11} & . & . & S_{1n} \\ . & & & . \\ . & & & . \\ S_{m1} & . & . & S_{mn} \end{bmatrix}$$

The $max - T$ composition is similar to the well-known matrix product, again by using the triangular norm as multiplication and the maximum operator as addition.

Example 3.2.7 [36] Consider the triangular norm W and the Lukasiewicz implication operator $I(x, y) = min(1, 1 - x + y)$. The different fuzzy relational compositions are illustrated on the following fuzzy relations:

$$R = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0 \\ 0.9 & 0.6 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0.5 & 1 & 0 \\ 0.6 & 0.8 & 0.2 \\ 0.7 & 0.6 & 0.3 \end{bmatrix}$$

The compositions of R and S are given by [36], it is to be noted that the subscript bk represents Bandler and Kohout's defined compositions while b and k stand for Baets and Kerre's improved definitions of these compositions respectively.

$$R \circ^{W} S = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0 & 0.5 & 0 \\ 0.7 & 0.9 & 0.3 \end{bmatrix}$$

$$R \lhd_b^I S = \begin{bmatrix} 0.4 & 0.4 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.1 \end{bmatrix}, \quad R \rhd_b^I S = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.7 & 0.8 & 0.3 \end{bmatrix}$$

$$R \lhd_k^{W,I} S = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0 & 0.5 & 0 \\ 0.6 & 0.6 & 0.1 \end{bmatrix}, \quad R \rhd_k^{W,I} S = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0 & 0.4 & 0 \\ 0.7 & 0.8 & 0.3 \end{bmatrix}$$

$$R \lhd_{bk}^I S = \begin{bmatrix} 1 & 1 & 0.8 \\ 1 & 1 & 0.5 \\ 0.6 & 0.6 & 0.1 \end{bmatrix}, \quad R \rhd_{bk}^I S = \begin{bmatrix} 0.7 & 0.2 & 1 \\ 0.3 & 0.4 & 0.7 \\ 1 & 0.8 & 1 \end{bmatrix}$$

$$R \Diamond_b^I S = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.6 & 0.6 & 0.1 \end{bmatrix}, \quad R \Diamond_k^{T,I} S = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0 & 0.4 & 0 \\ 0.6 & 0.6 & 0.1 \end{bmatrix}$$

Remark 3.2.8 [93] Most of the following properties of binary fuzzy relations were defined by Zadeh . In all these definitions $\vee$ and $\wedge$ represent *max* and *min* operators respectively. A binary fuzzy relation R in a universe X is called:

(1) *covering* if and only if for all $x \in X$ $\exists$ $y \in X$ such that $R(x, y) = 1$;

(2) *locally reflexive* if and only if for all $x \in X$,

$$R(x, x) = \sup_{y \in X} \max(R(x, y).R(y, x));$$

(3) *reflexive* if and only if for all $x \in X$,

$$R(x, x) = 1;$$

(4) *symmetric* if and only if for all $x, y \in X$,

$$R(x, y) = R(y, x);$$

(5) *antisymmetric* if and only if for all $x, y \in X$, and $x \neq y$ implies

$$\min(R(x, y), R(y, x)) = 0;$$

(6) *strictly antisymmetric* if and only if for all $x, y \in X$,

$$\min(R(x, y), R(y, x)) = 0;$$

(7) *min-transitive* if and only if for all $x, y, z \in X$,

$$\min(R(x, y), R(y, z)) \leq R(x, z);$$

(8) *irreflexive* if and only if for all $x \in X$,

$$R(x, x) = 0;$$

(9) *asymmetric* if and only if for all $x, y \in X$,

$$R(x, y) \wedge R(y, x) = 0;$$

(10) *weakly asymmetric* if and only if for all $x, y \in X$,

$$R(x, y) \wedge R(y, x) < 1;$$

(11) *negatively transitive* if and only if for all $x, y, z \in X$,

$$R(x, y) \vee R(y, z) \geq R(x, z);$$

(12) *complete* if and only if for all $x, y \in X$,

$$R(x, y) \vee R(x, y) > 0;$$

(13) *strongly complete* if and only if for all $x, y \in X$

$$R(x, y) \vee R(x, y) = 1.$$

Next we shall study the different types of important relations originating from these properties.

4 The Indistinguishability Relations

In this section we shall study the relations formulated to represent fuzzy counterparts of crisp equivalence relations. In the scenario of fuzzy relations the similarity relations defined and studied by Zadeh [93] appear as first candidate in this sequence. He studied the properties of similarity relations and successfully constructed the fuzzy counterparts of equivalence classes. Though the similarity relations were the first to appear after development of fuzzy relations, but we will start with summary of Menger's work. Menger's paper [61] "Probabilistic Theory of Relations" appeared in 1951, much earlier than even the birth of fuzzy sets. Later on when it was proved by Zimmermann and Zysno [97], that the t-norms are the best fit candidates for modelling fuzzy conjuntion, the Probabilistic equivalence relations became a part of the T-transitive relations with Product t-norm taking place of T in the definition of T-transitivity. After a brief review of Likeness relations we shall study T-transitive relations and their generators with any general t-norm T.

4.1 Probabilistic Relations [61]

Poincare [73] repeatedly emphasized that only in the mathematical continuum the equalities $A = B$ and $B = C$ imply that $A = C$. In the observable physical continuum, "equal" means "indistinguishable" and $A = B$ and $B = C$ by no means imply that $A = C$. The raw result of experience may be expressed by the relation

$$A = B, B = C \text{ but } A < C$$

which may be regarded as the formula for the physical continuum. According to Poincare, physical equality is a non-transitive relation.

According to Menger, a closer examination to the physical continuum suggests that in describing our observation we should sacrifice more than the transitivity of equality. He suggested to associate a number to the sets A and B, called the probability of finding A and B indistinguishable. Menger designed following definition for his equality relation.

Definition 4.1.1 [61] If $E(A, B)$ denotes the probability that A and B be equal, the following postulates seem to be rather natural:

(1). $E(A, A) = 1$ for every A;

(2). $E(A, B) = E(B, A)$ for every A and B;

(3). $E(A, B).E(B, C) \leq E(A, C)$ for every A, B and C.

The first two correspond to reflexivity and symmetry of the equality relation, while (3) expresses a minimum of transitivity.

Proposition 4.1.2 [61] If we set

$$-\log\ E(A, B) = d(a, b),$$

then

$(1'_a)$ $d(A, A) = 0$; $(1'_b)$ $d(A, B) \geq 0$; $(1'_c)$ $d(A, B) \neq 0$ if $A \neq B$;

$(2')$ $d(A, B) = d(B, A)$ for every A and B;

$(3')$ $d(A, B) + d(B, C) \geq d(A, C)$ for every A, B and C.

These are Fŕechet's postulates for the distance in a metric space. In particular (3) is triangle inequality. Conversely if disjoint sets A, B,... form a metric space with the distance $d(A, B)$ and we set $E(A, B) = e^{-d(a,b)}$ for each element a of A and b of B, then $E(A, B)$ satisfies the postulates (1), (2), (3) of a probability of equality. The system of probabilities of equality in a set, S, are thus identical with the systems of negative antilogarithms of the distance for various possible metrizations of S.

Since $d(a, b) < \infty$, for every two points of a metric space, it follows that $E(a, b) > 0$. We may find it more desirable to assume that every element b differing from a by more than a certain exterior threshold be certainly distinguishable from a. But *then we have to give up even the minimum of transitivity expressed in Postulate* (3) in Definition 4.1.1.

4.2 Proximity and Similarity Relations

Definition 4.2.1 [68] Let U be a fuzzy set in a universe X. A symmetric relation R is said to be a *proximity relation* on U if for all $x, y \in X$:

(1). $R(x,x) = U(x)$,

(2). $R(x,y) \leq R(x,x) \wedge R(y,y)$.

Definition 4.2.2 [68] Let U be a fuzzy set in X. A family of fuzzy sets $\Sigma = \{P_i\}_{i\in J}$ is said to be a fuzzy covering of U if $U = \underset{i\in J}{\cup} P_i$.

Definition 4.2.3 [93] A fuzzy binary relation S on X is called a *similarity relation* if and only if it is reflexive, symmetric and $max - min$ transitive.

Examples 4.2.4

a) [68] If $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, then the fuzzy relation R defined by following relational matrix is a similarity relation on X:

$$\begin{bmatrix} 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ 0.6 & 0.6 & 0.6 & 0.8 & 0.2 & 1 \end{bmatrix}.$$

b) [36] If $X = [0, +\infty[$, then the fuzzy relation $R : [0, +\infty[^2 \rightarrow [0, 1]$ defined as:

$$R(x,y) = \begin{cases} 1, & \text{if } x = y \\ e^{-max(x,y)}, & \text{else.} \end{cases}$$

is a similarity relation on X.

Theorem 4.2.5 [93] Following hold for a fuzzy relation R on X :

(1). If R is a similarity relation, then for all $x, y, z \in X$ at least two of the degrees $R(x,y), R(y,z)$ and $R(x,z)$ are equal.

(2). R is a similarity relation, if and only if

R_α is an equivalence relation for all $\alpha \in]0, 1]$.

(3). R is a similarity relation if and only if coR is a $[0, 1]-$*valued pseudo ultrametric* on X i.e., it satisfies for all $x, y, z \in X^3$:

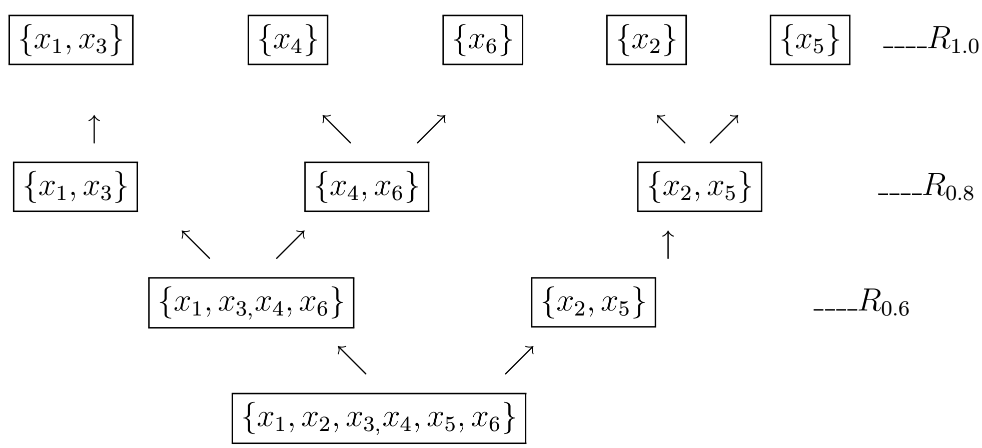
(M1) Non-negativity: $coR(x, y) = 0$;

(M2) Pseudo-separation: $x = y \Longrightarrow coR(x, y) = 0$;

(M3) Symmetry : $coR(x, y) = coR(y, x)$;

(M4) Strong triangle inequality: $coR(x, z) \leq max(coR(x, y), coR(y, z))$.

Remark 4.2.6 [68] An important concept associated with a similarity relation is its partition tree, the fuzzy analogue of the quotient set of crisp equivalence relation. As mentioned before, the α-cuts, R_α of a similarity relation R are equivalence relations. To each of these equivalence relations corresponds a partition Π_α of the universe X. These partitions become finer with increasing α. The fact that these partitions are nested can be visualized by means of a partition tree. We illustrate this construction procedure on the Example 3.2.4 (a). It suffices to consider those α-cuts for which α is effectively used as a degree of relationship. The partition tree of these $\alpha-$cuts is given by:



Definition 4.2.7 [69] Let S be a similarity relation on X. For any given $a \in X$, a *similarity class* of a is a fuzzy set $S[a]$ on X defined by the membership

function

$$S[a](x) = S(a, x)$$

for all $x \in X$.

Theorem 4.2.8 [69] Let S be a similarity relation on X. The following hold for all $a, b \in X$.

(1). $S[a] = S[b]$ if and only if $S(a, b) = 1$.

(2). If $S[a] \neq S[b]$, then $hgt(S[a] \cup S[b]) \leq 1$.

Definition 4.2.9 [69] A fuzzy covering Σ of X is called a *fuzzy partition* of X, if there exists a similarity relation S on X such that Σ is the set of all distinct similarity classes of S.

4.3 Likeness Relations

Definition 4.3.1 [22] A *likeness relation* is a fuzzy relation that is reflexive, symmetric and $sup{-}W$ transitive. Since $sup{-}min$ transitivity implies $sup{-}W$ transitivity so, it follows that every similarity relation is a likeness relation, but the converse may not be true as is shown in the following example.

Example.4.3.2 [22]. Consider the single-valued attribute BUILD of a person, with the following possible linguistic descriptions: *thin*, *slim*, *middling*, *sturdy* and *corpulent*. It is clear that these terms overlap to a certain extent. In order to take this into account we introduce a likeness relation on the domain of this attribute:

$$\begin{bmatrix} 1 & 0.9 & 0.5 & 0.3 & 0 \\ 0.9 & 1 & 0.6 & 0.4 & 0.1 \\ 0.5 & 0.6 & 1 & 0.8 & 0.4 \\ 0.3 & 0.4 & 0.8 & 1 & 0.6 \\ 0 & 0.1 & 0.4 & 0.6 & 1 \end{bmatrix}$$

One easily verifies that this fuzzy relation is a likeness relation and not a similarity relation. Some $\alpha-$cuts of this fuzzy relation are given next (with

t = thin, s =slim, m = middling, st = sturdy and c = corpulent):

$R_{1.0} = \{(t,t),(s,s),(m,m),(st,st),(c,c)\}$,

$R_{0.9} = R_{1.0} \cup \{(t,s),(s,t)\}$,

$R_{0.8} = R_{0.9} \cup \{(m,st),(st,m)\}$,

$R_{0.6} = R_{0.8} \cup \{(s,m),(m,s),(st,c),(c,st)\}$.

Remark 4.3.3 [27] Bezdek and Harris and later Ruspini indicate as one of the main reasons for introducing likeness relations is that the associated metric is the restriction to unit interval of the Euclidean metric.

4.4 T-transitive Equivalence Relations

In 1982 E. Trillas [83] summarized all the above mentioned results under the name $T-$indistinguishability operators since they are the key idea in order to solve the Poincare paradox. The T-transitive equivalence relations are defined as the fuzzy counter parts of the crisp equivalence relation. As it is well known, within a classical context, an equivalence relation in a set defines a partition or a classification in it, and vice versa. There have been several attempts to extend these concepts to the fuzzy framework. In the existing literature on this subject, two different trends have been followed. The first one puts its emphasis on the definition of fuzzy partition and then, studies the properties of the associated relation, if it exists. The papers by Bezdek & Harris [22] and Ovchinnikov & Riera [70] are representative of this research line. The following concepts are being recalled here for the sake of completeness; nevertheless, further details can be found, for instance, in Trillas and Valverde [84], Valverde [87] and Jacas [50].

Definition 4.4.1 [27] Let R be a fuzzy relation on a universe X, R is called T-*transitive* if for all $x, y, z \in X$

$$T(E(x,y), E(y,z)) \leq E(x,z).$$

Definition 4.4.2 [27] Let R be a fuzzy relation, its T-*transitive closure* $\overline{R}$ is the smallest transitive superset of a fuzzy relation and is obtained as follows:

$$\overline{R} = \sup_{n \in N} R^n.$$

where, $R^1 = R$ and $R^{n+1} = R^n \circ R$, moreover the t-norm T is used in the direct product.

Proposition 4.4.3 [27] If R is a reflexive and symmetric fuzzy relation on a finite universe X of cardinal n, then $\overline{R} = R^{n-1}$.

Example 4.4.4 [27] Let R be the relation given by the matrix

$$R = \begin{bmatrix} 1 & 0.3 & 0.5 & 0.7 \\ 0.3 & 1 & 0.2 & 0.8 \\ 0.5 & 0.2 & 1 & 0.3 \\ 0.7 & 0.8 & 0.3 & 1 \end{bmatrix}.$$

Then the transitive closures $\overline{R}_W$,$\overline{R}_P$, and $\overline{R}_M$ of R with respect to the t-norms of W, P and M respectively are:

$$\overline{R}_W = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.7 \\ 0.5 & 1 & 0.2 & 0.8 \\ 0.5 & 0.2 & 1 & 0.3 \\ 0.7 & 0.8 & 0.3 & 1 \end{bmatrix}$$

$$\overline{R}_P = \begin{bmatrix} 1 & 0.56 & 0.5 & 0.7 \\ 0.56 & 1 & 0.28 & 0.8 \\ 0.5 & 0.28 & 1 & 0.35 \\ 0.7 & 0.8 & 0.35 & 1 \end{bmatrix}$$

$$\overline{R}_M = \begin{bmatrix} 1 & 0.7 & 0.5 & 0.7 \\ 0.7 & 1 & 0.5 & 0.8 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.7 & 0.8 & 0.5 & 1 \end{bmatrix}$$

Proposition 4.4.5 [27] Given a reflexive and symmetric fuzzy relation on a set X, let A be the set of fuzzy T-transitive equivalence relations on X greater than or equal R and $\overline{R}$ its transitive closure. Then, for any $x, y \in X$,

$$\overline{R}(x, y) = \inf_{E \in A} \{E(x, y)\}.$$

Definition 4.4.6 [27] A $T-$*indistinguishability* relation E is a reflexive, symmetric and T-transitive fuzzy relation.

Definition 4.4.7 [27] An *S-pseudometric* m is a map from $X \times X$ into $[0, 1]$ such that for all $x, y, z \in X$,

(i) $m(x, x) = 0$

(ii) $m(x, y) = m(y, x)$

(iii) $S(m(x, y), m(y, z)) \geq m(x, z)$ ($S-$triangle inequality)

An S-metric is defined in the usual way i.e., replacing(i) by $m(x, y) = 0$ if and only if $x = y$.

The very first property of T-indistinguishability relations is their close relation with S- pseudometrics as is shown in the following theorem:

Theorem 4.4.8 [27] Let E be a T-indistinguishability relation and let ϕ be a continuous and order-reversing bijection from $[0, 1]$ into itself, then

$$m_E(x, y) = \phi(E(x, y))$$

is a S-pseudometric and vice-versa, where $S(x, y) = \phi^{-1}(T(\phi(x), \phi(y))$.

4.4.9 [88] **Construction of T-equivalence relations**

For a long time, the only available methods to build up fuzzy transitive relations (FER) have been the transitive closure and related methods. To be more concrete in order to apply the transitive closure method (Proposition 4.4.3) to construct in general a fuzzy T-transitive relation, a reflexive and symmetric fuzzy relation has to be used as a starting point. In other words an index of similarity relating each couple of elements in the sample space has to be given: each two elements should be matched, in some way, and then a method is applied to obtain either a similarity or dissimilarity relations.

At this point, the first arising question is: Does it mean that, from a single criterion, or from the matching of all elements to one given, no similarity measure can be given?

The obvious negative answer can be stated by assuming that as a result of the single criterion evaluation or the matching-to-one process, a function

$$h : X \to [0, 1],$$

is given, $h(x)$ representing the degree to which x fits the given conditions; with such assumption it is easy to check that

$$m(x, y) = |h(x) - h(y)|\,,$$

is a pseudo-distance (i.e., a dissimilarity measure) such that

$$m(x, x_0) = h(x),$$

for any $x_0 \in h^{-1}(\{0\})$ so that such pseudo-distance may be considered as truely induced by the data. It is also quite obvious that

$$E(x, y) = 1 - m(x, y) = 1 - |h(x) - h(y)|\,,$$

is a likeness relation on X for which $E(y, y_0) = h(y)$ for any $y_0 \in h^{-1}(\{1\})$ i.e., $h(y)$ itself is the measure of similarity between the element y, and any perfect prototype.

Lemma 4.4.10 [88] The construction of fuzzy equivalence relations can be extended in order to get T-transitive fuzzy relations for any t-norm T. If I stands for the R-implication associated with the t-norm T then it is easy to check that

$$E_h(x, y) = I(\max(h(x), h(y)), \min(h(x), h(y))),$$

is a T-fuzzy transitive relation, such that:

$$E_h(x, x_0) = h(x),$$

for any $x_0 \in h^{-1}(\{1\})$.

Thus, for instance,

$$E_h(x, y) = \begin{cases} \min(h(x), h(y)), & \text{if } h(x) \neq h(y) \\ 1, & \text{otherwise,} \end{cases}$$

is a similarity relation induced by h , i.e. E_h is *min*-transitive. Consequently,

$$m_h(x, y) = 1 - E_h(x, y),$$

is an ultrametric.

If the product t-norm P is used, then

$$E_h(x, y) = \begin{cases} \frac{\min(h(x), h(y))}{\max(h(x), h(y))} & \text{if } h(x) \neq h(y) \\ 1, & \text{otherwise,} \end{cases}$$

is a probabilistic relation, i.e. transitive with respect to the t-norm and P, and

$$m_h(x, y) = 1 - E_h(x, y),$$

is generalized pseudo-metric with respect to the t-conorm P^*.

The most important of all if W, the Lukasiewicz t-norm is used then

$$E_h(x, y) = 1 - |h(x) - h(y)|.$$

This fact was mentioned in the Remark 4.3.3 about Likeness relations.

Let it be noticed that, if the t-norm is strict, like the *min* and P t-norms and

$$h(x) = \{0, a\} \text{ with a } \in]0, 1],$$

then the induced relation is nothing but the classical equivalence relation associated with h, that is

$$E_h(x, y) = \begin{cases} 1, & \text{if } h(x) = h(y) \\ 0, & \text{otherwise.} \end{cases}$$

In general, such relation can be found as the 1-level set of the induced fuzzy relation.

Summing up, the above considerations show what to do in order to obtain a similarity (or dissimilarity) measure which matches to the data from a single subjective evaluation of the degrees of similarity in the sample set.

Next, suppose that several criteria or prototypes are given in the form of a family of functions:

$$h_j : X \to [0,1]; \quad j = 1, ..., n.$$

In this case the most natural procedure seems, first, to get the similarity measure in the form of a fuzzy transitive relation for a fixed t-norm T- associated with each h_j, E_j, and then to take as the degree of the similarity of two elements, $E(x, y)$, the minimum of all the degrees i.e.

$$E(x, y) = \inf_{j=1,...,n} E_j(x, y),$$

which, is also a T-transitive relation. Obviously, there are other ways to combine fuzzy transitive relations which also preserve the transitive character of the relation, but the one chosen here is canonical, in the sense expressed by the following representation theorem:

Theorem 4.4.11 [27] Let E be a fuzzy relation on a set X, i.e., a map from $X \times X$ into $[0, 1]$ and let T be a continuous t-norm. Then E is a reflexive, symmetric and T-transitive fuzzy relation (T-indistinguishability relation) if and only if, there exists a family $\{h_j\}_{j \in J}$ of fuzzy subsets of X, such that

$$E(x, y) = \inf_{j \in J}(I(h_j(x) \vee h_j(y), h_j(x) \wedge h_j(y))),$$

for all $x, y \in X$.

In other words, any reflexive, symmetric and T-transitive fuzzy relation on a set X is generated by a family of fuzzy subsets of the given set through the procedure described in this section. Valverde and Ovchinnikov [89] have shown that the above representation also holds for left-continuous T -norms. This fact is specially interesting when the minimal t-norm Z is considered. As it is known, this t-norm is defined by:

$$Z(a,b) = \begin{cases} \min(a,b), & \text{if } max(a,b) = 1 \\ 0, & \text{otherwise} \end{cases}$$

It is worth noting that reflexive, symmetric and Z-transitive fuzzy relations are simply those reflexive and symmetric relations for which the 1-level set is a classical equivalence relation. From this viewpoint, the Z-transitive relation S , obtained by applying the procedure implied by the representation theorem starting from a strict reflexive fuzzy relation, R , is simply the greatest symmetric relation contained in R , i.e.

$$S(x,y) = min(R(x,y), R(y,x)).$$

On the other hand, when the representation theorem is applied to build the $T-$indistinguishability relation E_R generated by a reflexive and symmetric fuzzy relation R, the representation theorem assures for both the existence of such a fuzzy relation and the method to compute it. Moreover, the use of the representation theorem no longer requires a complete fuzzy binary relation; neither reflexivity nor symmetry are required. As it has been shown the initial data may be just one function from the set X into $[0,1]$.

It is quite clear that, due to duality between indistinguishability and S-pseudometrics the Theorem has an immediate counter part for $S-$pseudometrics. The details for such results can be found in [27]. In other words, any S-pseudometric on a given set X comes from a family of fuzzy subsets of the given set and a metric on the unit interval.

Next the structure and characterization of the generators of a given T-indistinguishability relation E will be studied.

Definition 4.4.12 [87] A function from X into $[0,1]$ is termed a *generator* of a given T-indistinguishability relation E, if $E_h \geq E$.

H_E will denote the set of all generators of E. It follows immediately from the representation theorem that, given a T-indistinguishability relation E on X, the set $\{E(x,y)\}_{y\in X}$ of fuzzy subsets of X is a generating family of E and will be denoted by $\{h_y(x)\}_{y\in X}$. The next definition will play an important role in order to give a more convenient characterization of the generators of a T-indistinguishability relation E.

Definition 4.4.13 [87] Let E be a T-indistinguishability relation, then ϕ_E is the map from $F(X)$ into $F(X)$ defined by

$$\phi_E(h)(x) = \sup_{y\in X}\{T(E(x,y), h(y))\} \text{ for any } x \in X$$

It is worth noting that if X is a finite set then E is represented by a matrix and may be understood as the $max - T$ product of E by the column vector representing the fuzzy set h.

Theorem 4.4.14 [87] A fuzzy subset $h \in F(X)$ is a generator of a T-indistinguishability relation E on X, if and only if

$$\phi_E(h) = h.$$

Theorem 4.4.15 [87] Let E be a fuzzy relation on a set X, i.e. a map from $X \times X \longrightarrow [0,1]$, and let T be a continuous t-norm. Then E is a reflexive, symmetric and T-transitive fuzzy relation (T- indistinguishability relation) if and only if there exist a family $\{hj\}, j \in J$ of fuzzy subsets of X, such that:

$$E(x,y) = \inf_{j\in J} E_j(x,y) = \inf_{j\in J} T\{\frac{\{h_j(x) \vee h_j(y)\}}{\{h_j(x) \wedge h_j(y)\}}\},$$

for all $x, y \in X$.

5 Fuzzy Orders

Initially the fuzzy orders were defined with the assumption of $max - min$ transitivity as in [93]. Later some researchers preferred W-transitivity and formulated different forms for transitivity of fuzzy orders and inclusions. In general a fuzzy ordering is a fuzzy binary relation satisfying the T-transitivity property (see [66]). In addition it is desirable that the fuzzy ordering should satisfy some kind of asymmetric property. The most recent approach towards defining similarity based fuzzy orderings with the assumption of T-transitivity was made by Bodenhofer [24].

In this section we intend to summarize all the above mentioned results. We shall first summarize the work of Ovchinnikov about different types of fuzzy orderings defined with the help of $max - min$ transitivity. Some of their representation theorems will also be mentioned. Next we shall discuss the forms of asymmetry and fuzzy transitivity of fuzzy orderings formulated by Willmott [91] and Beg [9],[10],[11] and[7]. In the end we shall summarize the work of Bodenhofer [24],[25] and [23] on similarity based fuzzy T-transitive orderings and their representation theorems.

5.1 Max-min Transitive Fuzzy Orderings

Definition 5.1.1 [68] In order to define different types of fuzzy orderings we shall need following additional definitions of the fuzzy relations associated with R.

Symmetric part:

$$R^s(x, y) = R(x, y) \wedge R(y, x),$$

Dual relation:

$$R^d(x, y) = 1 - R(y, x),$$

for all $x, y \in X$. It can be readily verified that $R^s = R \cap R^{-1}$, $R^d = \overline{R^{-1}}$ and $(R^d)^d = R$, i.e., relations R^d and R are dual of each other.

Definitions 5.1.2 [68] Let R be a fuzzy relation on X. R is a:

(1) *pre-order or quasi–order if* it is reflexive and *max-min* transitive;

(2) *partial order* if it is asymmetric and *max-min* transitive;

(3) *weak ordering* if it is asymmetric and negatively transitive;

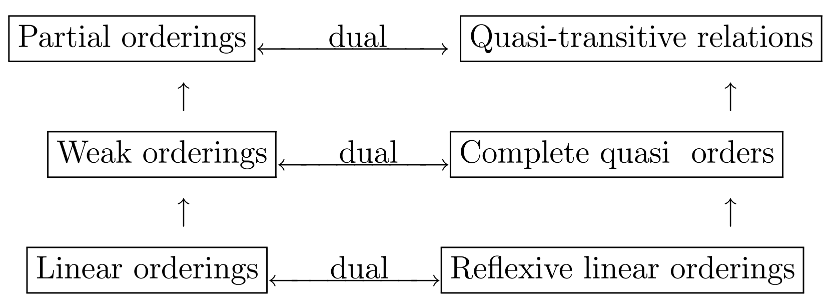
(4) *linear ordering* if it is a complete weak ordering;

(5) *quasi transitive relation* if R is dual to a partial ordering;

(6) *reflexive linear ordering* if it is dual to a linear ordering.

The nonempty set X with any of the order is called a *(quasi, partially, weakly, or linearly) ordered set.*

The classes of fuzzy orderings are presented in the form of a diagram borrowed from [68].



where vertical arrows indicate proper inclusions.

Example 5.1.3 [68] Let $X = \{x_1, x_2, x_3\}$. We define a fuzzy binary relation R on X by:

$$R = \begin{bmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ \gamma & \beta & 1 \end{bmatrix}$$

where $\alpha, \beta, \gamma \in [0,1]$. Then R is a strongly complete fuzzy binary relation on X. The dual relation $P = R^d$ and the symmetric part $I = R \cap R^{-1}$ associated with R are represented by the following matrices:

$$P = \begin{bmatrix} 0 & 1-\alpha & 1-\gamma \\ 0 & 0 & 1-\beta \\ 0 & 0 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & \alpha & \gamma \\ \alpha & 1 & \beta \\ \gamma & \beta & 1 \end{bmatrix},$$

It is observed that if $\gamma = \min(\alpha, \beta)$, then R is a strongly complete transitive relation, P a transitive fuzzy binary relation and I is a symmetric and transitive fuzzy binary relation. This demonstrates the facts proved in following theorems (for proofs see [68]).

Theorem 5.1.4 Let R be strongly complete and transitive fuzzy binary relation. The dual relation R^d associated with R is a transitive fuzzy binary relation. Moreover the symmetric part of a transitive fuzzy relation is a transitive fuzzy relation.

Remark 5.1.5 [69] Quasi orders enjoy a special importance; firstly because they are at the root of both fuzzy equivalence relations and fuzzy orders, and secondly because of their pleasant properties. For example (for proofs see [69]):

(1). The intersection of two quasi orders is again a quasi order.

(2). If R is a quasi order then for any given $k \in [0,1]$ a relation P given by:

$$P(x,y) = R(x,y) \vee k,$$

is a quasi order for all $x, y \in X$.

(3). Let $f : X \to Y$ be a function and Q be a quasi order on X. Then the function

$$R(x,y) = Q(f(x), f(y)), \quad x, y \in X,$$

is a quasi order on X.

(4). Let X and Y be two topological spaces and let f be a continuous function. Also let P and Q be two quasi orders on X and Y respectively. For any given

$k \in [0,1]$ a relation given by:

$$R(x,y) = (P(x,y) \vee k) \wedge Q(f(x), f(y)),$$

is a continuous quasi order for all $x, y \in X$.

Definition 5.1.6 [69] A quasi order R on X is said to be *strict* if:

$$R(x,y) \wedge R(y,x) < 1,$$

for all $x \neq y$ in X.

Definition 5.1.7 [69] Let X be a space and let R be a quasi order on X. The pair (X, R) is called a fuzzy relational system.

Definition 5.1.8 [69] A fuzzy relational system ($\mathbb{R}$, R) is called numerical fuzzy relational system. Here, $\mathbb{R}$ stands for the set of real numbers.

Definition 5.1.9 [69] A homomorphism from (X, P) to (Y, Q) is a continuous function $f : X \to Y$ such that:

$$P(x,y) = Q(f(x), f(y)),$$

for all $x, y \in X$.

Theorem 5.1.10 [69] There exists a numerical fuzzy relational system ($\mathbb{R}$, R) such that any fuzzy relational system (X, R) with finite X admits a numerical representation in ($\mathbb{R}$, R), provided R is positive and T is a strict t- norm.

For proof see [69].

Theorem 5.1.11 [68] A fuzzy relation R on X is a fuzzy pre-ordering if and only if there is a family F of positive real functions on X such that:

$$R(x,y) = \inf_{f \in F} \min \left\{ \frac{f(x)}{f(y)}, 1 \right\} \text{ for all } x, y \in X.$$

5.2 Fuzzy Order Relations

Beg has used fuzzy orders in several forms. Initially the quasi order defined in his work is in the same form as defined by Ovchinnikov (see Definition

5.1.2), but later he reformulated some other definitions of order relations. A summary of his work will be presented . His focus is on total fuzzy orders, fuzzy maximal elements, fuzzy chains etc.

Definition 5.2.1 [7] A quasi-order (see Definition 5.1.2) R is called *order* if it satisfies:

$$R(x,y) + R(y,x) > 1 \ \Rightarrow x = y.$$

Moreover, a quasi order is called *total* if

$$\text{for all } x \neq y \Rightarrow R(x,y) \neq R(y,x).$$

Other than this traditional definition, he constructed some other forms of fuzzy order relations. We next give one such definition.

Definition 5.2.2 [10] Let X be a crisp set. A fuzzy ordering relation on X is a fuzzy subset of $X \times X$ with the following properties:

(1). For all $x \in X$, $R(x,x) \in (0,1]$.

(2). For all $x, y \in X$,

$$R(x,y) + R(y,x) > 1 \Rightarrow x = y.$$

(3). For all $x, y, z \in X$,

$$R(x,y) \geq R(y,x) \text{ and } R(y,z) \geq R(z,y) \Rightarrow R(x,z) \geq R(z,x).$$

A set with a fuzzy order relation defined on it, is called a *fuzzy ordered set.*

Definition 5.2.3 [9] Let X be a set with a fuzzy order relation R. Then:

(a) A fuzzy ordered subset of X on which the fuzzy order R is total, is called a *fuzzy chain,* alternatively a fuzzy subset on which fuzzy preorder is linear is called a fuzzy chain.

(b) For a fuzzy subset $A \subseteq X$, *the fuzzy set $U(A)$ of upper bounds* is defined by

$$U(A)(x) = sup\{inf(R(y,x) - R(x,y) : y \in A\}, 0\}.$$

Thus an *upper bound* or a *strict upper bound* of A is an element $x \in X$ satisfying:

$$\text{Either } R(y,x) \geq R(x,y) \text{ or } R(y,x) > R(x,y)) \text{ for all } y \in Y.$$

Definition 5.2.4 Let X be a set with a fuzzy order relation R. A fuzzy subset B of X is said to be *pointwisely dominated* in X if for each $x \in B$ there exists a $y \in X$ such that $y \neq x$ and $R(x,y) \geq R(y,x)$. The fuzzy set B is called *strictly dominated* in X if there exists some $y \in B^c$ such that $R(x,y) > R(y,x) = 0$ for all $x \in B$. A pointwisely dominant $R-$fuzzy chain C in X is said to have *the dominant property* on X, if it is strictly dominated in X. When every pointwisely dominated R-fuzzy chain $C \subset X$ is strictly dominated in X, we say that the fuzzy relation R has *fuzzy chain dominant property.*

Definition 5.2.5 [9] An element x is called a *maximal element* of A if there is no $y \neq x$ in A for which $r(x,y) \geq r(y,x)$.

Definition 5.2.6 [9] An element $x \in A$ satisfying $r(y,x) \geq r(x,y)$ for all $y \in A$ is called the *greatest element* of A.

Similarly, we can define a *lower bound*, a *minimal* and a *least element* of A. In a chain the least and the greatest element are unique lower bound in X.

Theorem 5.2.7 [10] Let X be a fuzzy ordered set such that every decreasing chain in X has greatest lower bound in X. If $f : X \longrightarrow X$ is a map such that for all $x \in X$, $\mu(f(x), x) \geq \mu(x, f(x))$, then there is a v in X with $f(v) = v$.

Theorem 5.2.8 [9] Let (X, R) be an ordered set. If every R-fuzzy chain in X has an upper bound then X has a maximal element.

Theorem 5.2.9 [11] Let R be a fuzzy relation on a nonempty set X having the chain dominant property then there exists a maximal element x^* in X.

5.3 T-transitive Fuzzy Orderings

Definition 5.3.1 [23] A mapping $L : X^2 \to [0,1]$ is a *fuzzy ordering* on the non-empty crisp domain X with respect to a T-norm T, for brevity T-order if and only if the following axioms are satisfied for all $x, y, z \in X$:

(1) $L(x,x) = 1$ (reflexivity),

(2) $x \neq y \Rightarrow T(L(x,y), L(y,x)) = 0$ (T-antisymmetry),

(3) $T(L(x,y), L(y,z)) \leq L(x,z)$ (T-transitivity).

If the condition (2) is dropped the resulting relation is called a fuzzy preorder with respect to T. As studied in the previous section the symmetric preorders are called the T-transitive fuzzy equivalence relations. The strongly complete T-preorders are called *fuzzy weak orders* with respect to T or in short T-weak orders.

Doubts that $T-$antisymmetry could be too strong a requirement have appeared and they have motivated several researchers to propose generalizations. According to Bodenhofer "In opposition to Zadeh's our point of view is that an axiom of antisymmetry without reference to a concept of equality is meaningless"

Consequently Bodenhofer [23] reformulated the definition of fuzzy ordering by making fuzzy similarity its basis.

Definition 5.3.2 A fuzzy relation L on a universe X is called a fuzzy ordering with respect to a t-norm T and a T-*equivalence relation* E (on the same domain X) for brevity $T-E$-*ordering*, if and only if it is T-transitive and additionally fulfils two axioms:

(1) E-reflexivity, i.e. for all $x, y \in X$, $E(x,y) \leq L(x,y)$.

(2) $T-E$-antisymmetry, i.e. for all $x, y \in X$,

$$T(L(x,y), L(y,x)) \leq E(x,y).$$

It is easily verified that this definition of antisymmetry coincides with the one in (2) of 5.3.1

Lemma 5.3.3 [24] Some basic properties of T-E-ordering are as follows:

(1) Every crisp equality is a fuzzy ordering with respect to any t-norm and the crisp equality.

(2) A T-equivalence E is itself a $T-E$-ordering. Moreover, for a given T-equivalence E, it is the smallest $T-E$-ordering.

(3) The inverse of a $T-E-$ordering is itself a $T-E-$ordering.

Examples 5.3.4 [25] The fuzzy relation

$$L(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1-x+y,0) & \text{otherwise,} \end{cases}$$

is a fuzzy ordering on the real numbers $\mathbb{R}$ with respect to the Lukasiewicz t-norm W and the following well known W-equivalence

$$E(x,y) = \max(1-|x-y|,0).$$

Many other interesting results about the intersections, Cartesian products, implicit factorizations, constructions and representations can be found in [24].

Lemma 5.3.5 [25] Every $T-$preorder $R : X^2 \longrightarrow [0,1]$ fulfills the following equality for all $x,y \in X$.

$$R(x,y) = \inf_{z\in X} I(R(z,x),R(z,y)),$$

where I is the $R-$implication associated with T.

Theorem 5.3.6 [25] A fuzzy relation on a universe X is a weak $T-$order if and only if there exist a non-empty domain Y, a $T-$equivalence relation on Y, a strongly complete $T-E-$order L on $Y$and a mapping $f : X \longrightarrow Y$ such that the following equality holds for all $x,y \in X$:

$$R(x,y) = L(f(x),f(y)).$$

6 $\epsilon-$Fuzzy Equivalence Relations

The *max-min* transitivity is the most commonly used transitivity in fuzzy equivalence relations. It was introduced by Zadeh and agreed by Orlovsky [64] and [65]. Moreover, it is the only type of transitivity proposed by Roubens and Vincke [76]. But it faces a lot of criticism [57].

Examples 6.1.1

a) Bazu [6] develops a criticism against *max-min* transitivity and prefers some weighted transitivities. According to his argument: Let us suppose that $X = \{x, y, z\}$ and $R(x, y) = R(y, z) = 0.5$. If we want R to be *max-min* transitive relation, then the smallest level for $R(x, z)$ is 0.5. Let us now alternatively consider $R(x, y) = 1$ and $R(y, z) = 0.5$; even in this case, the smallest level for $R(x, z)$ is 0.5, other wise the relation violates *max-min* transitivity. Intuitively speaking for the second case, $R(x, z)$ should be assumed greater than the $R(x, z)$ of the first case.

Here is another example that captures the failure of the existing definition in giving fuzzy extension of the crisp transitivity.

b) [21] In case of $max - min$ transitivity, a fuzzy relation R is called *non-transitive* if it fails to keep the inequality even at a single triplet of points. Consider a fuzzy relation R such that:

$$\begin{aligned} R(x, y) &= 0.9, \\ R(y, z) &= 0.9, \\ R(x, z) &= 0.899999999. \end{aligned}$$

The inequality (1) is so strict that it would call the relation nontransitive while $R(x, z)$ has a considerable value. A large number of such relations are discarded being nontransitive by not only the definition of *max-min* transitivity but all the other T-transitivities fail in this respect.

Remark 6.1.2 [42], [56], [57] As mentioned earlier Similarity relations, Likeness relations and Probabilistic relations were all summarized by E. Trillas [83] under the name indistinguishability operators in an attempt to solve the Poincare paradox. It has been established by De Cock [42] and Klawonn [57] that even with T-transitive equivalence relations the Poincare paradox appears. Following example is constructed by De Cock and Kerre [42] in this context.

Let X be the universe of possible heights of men, T a t-norm and E a T-transitive equivalence relation on X. If E is used to represent "approximate equality", we could expect intuitively

$$E(1.50m, 1.51m) = E(1.51m, 1.52m) = 1, \text{ and } E(1.52m, 1.53m) = 1. \tag{3}$$

Now the T- transitivity behaves as follows:

$$T(E(1.50m, 1.51m), E(1.51m, 1.52m)) \leq E(1.50m, 1.52m).$$

So,

$$E(1.50m, 1.52m) = 1. \tag{4}$$

Combining (3) with (4) and using T-transitivity, we can derive in a similar way that

$$E(1.50m, 1.53m) = 1. \tag{5}$$

Similarly we get

$$E(1.50m, 1.54m) = 1$$

etc. Finally, it yields that all heights are approximately equal to degree 1 that is, the Poincare paradox. De Cock [42] establishes that when resemblance relation is used as model of approximate equality the Poincare paradox does not appear, but for the construction of a resemblance relation a pseudometric space must be at hand as can be seen from the Definition 6.1.3.

Definition 6.1.3 [42] For a universe X, a pseudometric space (M, d) and a mapping $g : X \to M$, a fuzzy relation E on X is called a (g, d)-*resemblance relation* on X if and only if for all $x, y, z, u \in X$:

(R.1) $E(x, x) = 1$,

(R.2) $E(x, y) = E(y, x)$,

(R.3) $d(g(x), g(y)) \leq d(g(z), g(u))$ implies $E(x, y) \geq E(z, u)$.

If X is already equipped with a suitable pseudometric, then $g = I$, the identity mapping on X. In this case (R.3) reduces to

$$d(x, y) \leq d(z, u) \text{ implies } E(x, y) \geq E(z, u).$$

S. Mazhar and I. Beg propose $\epsilon-$fuzzy equivalence relations as better candidates for the concept of fuzzy transitivity. They hold the opinion that the concept of $\epsilon-$fuzzy equivalence relations is self dependent and resemblance relations lie within the class of strong fuzzy equivalence relations see [15].

Definition 6.1.4 [21] Let R be a fuzzy relation on X. The *fuzzy transitivity relation* $tr_R^{I,T}$ is a fuzzy relation on X defined as:

$$tr_R^{I,T}(x, z) = \inf_{y \in X} I(T(R(x, y), R(y, z)), R(x, z));$$

where, I is any implication and T is a t-norm. The transitivity function so defined, assigns a degree of transitivity to the given fuzzy relation at each point of $X \times X$. Therefore, the given relation may have different degrees of transitivity at different pairs of points. When we work with a fixed relation t-norm and implicator we shall drop the superscripts and the subscript.

Definition 6.1.5 [15] For a given fuzzy relation R, the *measure of transitivity* of R is given by:

$$mTr(R) = m(tr_R);$$

where, m is Sugeno's measure [82]. In this paper *plinth* of a fuzzy set will be taken as the measure, consequently;

$$mTr(R) = Plinth(tr_R) = \inf_{x,y,z \in X}(I(T(R(x, y), R(y, z)), R(x, z)))$$

will be taken as the value for the measure of transitivity of a given fuzzy relation. A fuzzy relation R is called $\epsilon-$*fuzzy transitive* if $\epsilon = mTr(R)$, R is *non-transitive* if $\epsilon = 0$. ϵ expresses different shades of transitivity, infact it orders the fuzzy relations with respect to the degree of transitivity.

Remark 6.1.6 [15] If an $R-$implicator is used in calculation of degree of transitivity, then the T-transitive fuzzy relations are 1-*fuzzy transitive*. This is due to the fact that $R-$implications possess the property that $a \leq b$ implies $I(a, b) = 1$. So the T-transitivity i.e., $T(R(x, y), R(y, z)) \leq R(x, z)$ for all $x, y, z \in X$ implies $I(T(R(x, y), R(y, z)), R(x, z)) = 1$ for all $x, y, z \in X$. Hence the result.

Definition 6.1.7 [21] A fuzzy relation E on X is called an $\epsilon-$*fuzzy equivalence relation* if for all $x, y, z \in X$

(i) $E(x, x) = 1$;

(ii) $E(x, y) = E(y, x)$;

(iii) $mTr(E) = \epsilon$.

In this case E will be called an $\epsilon-$*equivalence relation*. For T-transitive equivalence relations $\epsilon = 1$. In general, if $\epsilon > 0$, then R will be called a *fuzzy equivalence relation* and application of a strong negator to R will convert it to what we shall call a *fuzzy dissimilarity relation*.

The following theorem proved in [15] connects this form of degree or measure of transitivity with a traditional property of crisp transitive stated as: "A relation R on a universe X is transitive if and only if $R \circ R \subseteq R$ ".

Theorem 6.1.8 [15] Let R be a fuzzy relation on X and $\circ$ stands for $max-min$ product. If a continuous implicator is used in both sides then

$$Inc(R \circ R, R) = Tr(R).$$

where, the mapping Inc is defined in Definition 7.1.3.

Theorem 6.1.9 [15] Let R be a fuzzy relation on a given universe X, then

$$Tr(R) = mInc(R \circ R, R) \in]0, 1].$$

and

$$mInc(R \lhd R, R) \geq Tr(R).$$

7 Fuzzy Inclusion Relation

Fuzzy inclusion relations constitute a very important part of the fuzzy set theory. The first inclusion of fuzzy sets was first introduced by Zadeh in his seminal paper [92]. Inspite of its immense practical utility, it was soon realized that the definition was not in accordance with the true spirit of fuzzy logic, infact it was a crisp definition. This intuition has inspired many researchers to consider $F(X) \times F(X) \rightarrow [0, 1]$ mapping *Inc*, such that the value $Inc(A, B)$ predicts to what extent A is included into B (see [4]). Moreover, many researchers formulated axioms of inclusion i.e., they provide a list of properties which a reasonable inclusion measure must satisfy. A significant contribution in this direction is by Sinha and Dougherty [81]; they list nine properties that a reasonable inclusion measure should satisfy and then proceed to introduce inclusion measures which have these properties. Cornelis [28] later proved that the scalar introduced by Bandler and Kohout in [4] as: $\inf_{x \in X} \{I(A(x), B(x)\}$ is a fuzzy inclusion satisfying all the axioms of fuzzy inclusion constructed by Sinha and Dougherty. Kehagias [52] made the most recent achievement in this context by extending the range of fuzzy inclusion to a lattice. Willmott [91] has addressed the issue that the form of transitivity expected to be satisfied by set inclusion must be different from the one set equivalence must satisfy.

Definition 7.1.1 [28] Let Inc be a $F(X) \times F(X) \rightarrow [0, 1]$ mapping, and A, B and C fuzzy sets in a given universe X. The *Sinha-Dougherty axioms imposed on* Inc are as follows:

Axiom 1. $Inc(A, B) = 1$ if and only if $A \subseteq B$ (in Zadeh's sense)

Axiom 2. $Inc(A, B) = 0$ if and only if $Ker(A) \cap supp(B) \neq \emptyset$, where, $ker(A = \{x \in X \mid A(x) = 1\}$, $supp(B) = \{x \in X \mid B(x) > 0\}$

Axiom 3. $B \subseteq C$ implies $Inc(A, B) \leq Inc(A, C)$,i.e., Inc has increasing second partial mappings.

Axiom 4. $B \subseteq C$ implies $Inc(B, A) \geq Inc(C, A)$,i.e., Inc has decreasing first partial mappings.

Axiom 5. $Inc(A, B) = Inc(S(A), S(B))$ where S is a $F(X) \to F(X)$ mapping defined by, for every $x \in X$, $S(A)(x) = A(s(x))$, s denoting an $X \to X$ mapping.

Axiom 6. $Inc(A, B) = Inc(B^c, A^c)$.

Axiom 7. $Inc(B \cup C, A) = min(Inc(B, A), Inc(C, A))$.

Axiom 8. $Inc(A, B \cap C) = min(Inc(A, B), (A, C))$.

Definition 7.1.2. [4] A fuzzy relation Inc on $F(X)$ is a *fuzzy inclusion* if and only if there is a contrapositive fuzzy implication I on X such that:

$$Inc(A, B) = \inf_{x \in X} I(A(x), B(x))$$

for all $A, B \in F(X)$.

Example 7.1.3 [85] The fuzzy inclusion Inc corresponding the Gödel implication operator I_g is defined for any A and B in $F(x)$ as follows:

$$Inc(A, B) = \inf_{A(x) > B(x)} B(x).$$

Remark 7.1.4 A list of some specific examples of fuzzy inclusion measures cited in [52] are given below, for comparison with these previous attempts. $i(A, B)$ in all these cases represent the degree of inclusion of A into B.

(1). $i(A, B) = \frac{|A \wedge B|}{|A|}$.

(2). $i(A, B) = \frac{|B|}{|A \vee B|}$.

(3). $i(A, B) = \frac{|A^c \wedge B^c|}{|B^c|}$.

(4). $i(A, B) = \frac{|B^c|}{|A^c \vee B^c|}$.

(5). $i(A, B) = \frac{\sum_{u \in U} 1 \wedge (1 - A(u) + B(u))}{|U|}$.

(6). $i(A, B) = \frac{\sum_{u \in U} (1 - A(u)) \vee B(u))}{|U|}$.

8 The Preference Structures

Now we are in a position to give a formal discussion on fuzzy preference structure [90] that come out of combinations of three fuzzy relations at a time. Some preliminaries of crisp preference structures are also presented.

8.1 Classical Preference Structure [76]

The study of classical preference structures is a well-known field of research in preference modeling. These structures consist of three fundamental relations (the strict preference, indifference and incomparability relations) fulfilling a number of intuitive conditions. To be more explanative, in decision problems a decision maker is usually confronted with a set of alternatives A, among which for instance, the best alternative has to be selected. In following we demand the decision maker to compare two alternatives a and b in A. It is then acceptable to assume that the decision maker either

- prefers a to b ;
- prefers b to a;
- is indifferent to a and b;
- is not able to compare a and b.

Further let P, I and J be three binary relations on A then:

• A couple of alternatives (a, b) belongs to the *strict preference relation* P if and only if the user prefers a to b

• A couple of alternatives (a, b) belongs to the *indifference relation* I if and only if the user is indifferent between alternatives a and b.

• A couple of alternatives (a, b) belongs to the *incomparability relation* J if and only if the user is unable to compare a and b, for instance caused by conflicting or insufficient information. These considerations lead to the following formal definition of the crisp preference structure.

Definition 8.1.1 [76] A *preference structure* on a set A is a triplet (P, I, J) of binary relations in A that satisfy :

(1) P is irreflexive, I is reflexive and J is irreflexive;

(2) P is asymmetrical , I is symmetrical and J is symmetrical;

(3) $P \cap I = \emptyset, P \cap J = \emptyset$ and $I \cap J = \emptyset$;

(4) $P \cup P^t \cup I \cup J = A^2$.

This definition is exhaustive: it lists all properties of the components P,I and J of a preference structure. The asymmetry of P can also be written as $P \cap P^t = \emptyset$ and it implies the irreflexivity of P. Condition (4) is called the completeness condition and expresses the ability of the decision maker to judge all pairs of alternatives. It is important to realize that this condition can be written equivalently in at least eight different ways:

C_1 : $co(P \cup I) = P^t \cup J$;

C_2 : $co(P \cup J) = P^t \cup I$;

C_3 : $co(P \cup P^t) = I \cup J$;

C_4 : $coP^t \cap coJ \cap coI = P$;

C_5 : $coP \cap coJ \cap coI = P^t$;

C_6 : $coP \cap coP^t \cap coJ = I$;

C_7 : $coP \cap coP^t \cap coI = J$;

C_8 : $P \cup P^t \cup I \cup J = A^2$.

Definition 8.1.2 [76] The binary relation $R = P \cup I$ is called the *large (or weak)* preference relation of the preference structure (P, I, J). The relation R

is often interpreted as expressing the relationship " as good as" among two alternatives, i.e., $(a, b) \in R$ means that a is as good as b.

Remark 8.1.3 [76] It is a well known fact that from any reflexive binary relation R in a set of alternatives A, a classical preference structure (P, I, J) can be constructed as follows:

(i) $P = R \cap coR^t$;

(ii) $I = R \cap R^t$;

(iii) $J = coR \cap coR^t$.

A beautiful interpretation of this structure can be found in [45]. Moreover any classical preference structure (P, I, J) can be reconstructed from its large relation $R = P \cup I$ in the following manner:

$$(P, I, J) = (R \cap coR^t, R \cap R^t, coR \cap coR^t).$$

By their crisp nature, the preference structures can not express the degrees of preference, indifference or incomparability, and are therefore too stringent in practice. Therefore the fuzzy relations are used to construct fuzzy preference structures.

8.2 Fuzzy Preference Structures [30]

At first sight, the generalization of the concept of a classical preference structure to that of a fuzzy preference structure, expressing degrees of strict preference, indifference and incomparability among a set of alternatives, seems to be a rather easy, formal task. Apparently the process should require only the choice of a fuzzy union, intersection and complementation operation on fuzzy relations, but it is not so. If a De Morgan triplet (S, T, N) is selected, the following are the eight different forms of the completeness conditions:

C_1 : $co(P \cup_S I) = P^t \cup_S J$;

C_2 : $co(P \cup_S J) = P^t \cup_S I$;

C_3 : $co(P \cup_S P^t) = I \cup_S J$;

C_4 : $co_N P^t \cap_T co_N J \cap_T co_N I = P$;

C_5 : $co_N P \cap_T co_N J \cap_T co_N I = P^t$;

C_6 : $co_N P \cap_T co_N P^t \cap_T co_N J = I$;

C_7 : $co_N P \cap_T co_N P^t \cap_T co_N I = J$;

C_8 : $P \cup_S P^t \cup_S I \cup_S J = A^2$.

Where, the subscripts T, S, N denote the t-norm, t-conorm and the negator to be used as conjunction, disjunction and complementation operators.

The notable thing is that there is no interrelationship amongst these eight conditions. Hence in the construction of fuzzy preference structure one has to take care of which completeness conditions to be used.

Definition 8.2.1 [30] Consider a strong De Morgan triplet $M = (T, S, N)$ and $i \in \{1, 2, \ldots, 8\}$. An $M-$*fuzzy preference structure* on A w.r.t completeness condition (C_i) is a triplet (P, I, J) of fuzzy binary relations in A that satisfy:

(M1) P is irreflexive, I is reflexive and J is irreflexive;

(M2) P is $T-$asymmetrical, I is symmetrical and J is symmetrical;

(M3) $P \cap_T I = \emptyset, P \cap_T J = \emptyset$ and $I \cap_T J = \emptyset$;

(M4) (P, I, J) satisfies completeness condition C_i.

However it is not guaranteed that for any triplet M there exist binary fuzzy relations P, I, J satisfying the above definition. Also note that T-asymmetry means that $P \cap_T P^t = \emptyset$.

Theorem 8.2.2 [30] Consider a De Morgan triplet $M = (T, S, N)$ with a positive t-norm and $i \in \{1, 2...8\}$. Then any $M - FPS$ on A with respect to (C_i) is a classical preference structure.

Therefore, we can not succeed in generalizing preference to the fuzzy case if we use a positive t-norm in the underlying de Morgan triplet. In other words T must have zero divisors. Restricting ourself to continuous t-norms, there are two possibilities: T is Archimedean, or T is not Archimedean. In case T

is Archimedean, it must be nilpotent, i.e., a ϕ-transform of the Lukasiewicz t-norm.

Theorem 8.2.3 [30] Consider a continuous strong De Morgan triplet $M = (T, S, N)$ with T a non-Archimedean t-norm with zero divisors and $i \in \{1, 2, ...8\}$. Then there exists a $c \in]0, 1[$ such that for any $M - FPS$ (P, I, J) on A with respect to (C_i) it holds that P, I, J can not take values in $[c, 1[$.

An important class of preference structures consists of those structures for which there are no couples of incomparable alternatives. A preference structure of the form $(P, I, J = \emptyset)$ is called a *preference structure without incomparability*, and will be denoted as: (P, I). The following theorem provides an important characterization of a preference structure in terms of its large preference relation. Recall that a binary relation is called complete if and only if

$$R \cup R^t = A^2.$$

Theorem 8.2.4 [76] A preference structure (P, I, J) on A is a preference structure (P, I) on A if and only if its large preference relation is complete.

Generally the following two classes of fuzzy preference structures may be distinguished.

Theorem 8.2.5 [39] A fuzzy preference structure (P, I, J) on A with fuzzy large preference relation R in A is a fuzzy preference structure (P, I) on A of type 1 if and only if for all $(a, b) \in A^2$

$$\max(R(a, b), R(b, a)) = 1.$$

Theorem 8.2.6 [39] A fuzzy preference structure (P, I, J) on A with fuzzy large preference relation R in A is a fuzzy preference structure (P, I) on A of type 2 if and only if for all $(a, b) \in A^2$

$$R(a, b) + R(b, a) \geq 1.$$

In both the cases the following relationship between the fuzzy strict preference relation P and the fuzzy large preference relation R holds:

$$P(a,b) = 1 - R(b,a) \text{ for all } (a,b) \in A^2.$$

8.2.7 Construction of Fuzzy Preference Structures

Now we give an overview of different proposals for constructing a fuzzy strict preference relation from a reflexive binary fuzzy relation R in A.

(1). Orlovski [64] was the first who explicitly defined fuzzy preference and indifference relations as follows:

$$\begin{aligned} P(a,b) &= max(R(a,b) - R(b,a), 0), \\ I(a,b) &= min(R(a,b), R(b,a)). \end{aligned}$$

(2). Ovchinnikov [66] investigated fuzzy relations in a different framework: The evaluation set was a partially ordered set . The fuzzy strict preference relation P is defined by:

$$P(a,b) = \begin{cases} R(a,b), & \text{if} \quad R(a,b) > R(b,a), \\ 0, & \text{otherwise.} \end{cases}$$

(3). Roubens and Vincke [76] were the first to simultaneously define fuzzy strict preference, indifference and incomparability relations. There is no doubt about the definition of I and J:

$$\begin{aligned} I(a,b) &= min(R(a,b), R(b,a)), \\ J(a,b) &= min(1 - R(a,b), 1 - R(b,a)). \end{aligned}$$

In addition to the previously stated forms of fuzzy preference they give the following definition:

$$P(a,b) = min(R(a,b), 1 - R(b,a)).$$

(4). Roubens [75] cosiders a t-norm T such that for all $x, y \in [0,1]$

$$(x + y \leq 1 \text{ implies } T(x,y) = 0).$$

And then he gives the following definition:

$$P(a,b) = T(R(a,b), 1 - R(b,a)).$$

Orlovski's definition is a particular case with $T = W$.

(5). Ovchinnikov and Roubens [71] give the general functional form of a fuzzy strict preference relation P, under the extra condition of it's *min-asymmetry:*

$$P(a,b) = p(R(a,b), R(b,a)),$$

where p is a $[0,1]^2 \to [0,1]$ mapping with increasing first and decreasing second partial mappings. Such a mapping is called a *strict preference generator*. As an example they propose the formula:

$$P(a,b) = T(R(a,b), N(R(b,a))),$$

where, T is a t-norm and N is an involutive negator. For further details and for study of the axiomatic approach see [30].

9 Continuous and Linear fuzzy Relations

9.1 Continuous Relations

In the study of relations continuous relations or continuous multivalued mappings possess a very important place. Generally two types of continuity are defined for multivalued mappings: the upper semicontinuity and the lower semicontinuity. These continuities need two types of inverse images the direct inverse image and the lower inverse image of an open set, in the presence of a topological space. These inverses have already been defined and generalized to their fuzzy counterpart as $R^{-1}(A)$ and $R^{-1}_{\triangleleft}(A)$ respectively in 3.2.2. Let us revise Aubin's definitions of continuity.

Definition 9.1.1 [2] Let X and Y are metric (or topological spaces). A set valued map $R : X \rightsquigarrow Y$ is:

(1). *upper semicontinuous* if the lower inverse of any open subset in Y is open in Y;

(2). *lower semicontinuous* if the direct inverse image of any open subset in Y is open in X.

Corollary 9.1.2 [2] Let X and Y are metric (or topological spaces). A set valued map $R : X \rightsquigarrow Y$ is:

(1). *upper semicontinuous* if the direct inverse of any closed subset in Y is closed in Y;

(2). *lower semicontinuous* if the lower inverse Image of any closed subset in Y is closed in X.

Remark 9.1.3 [86] As defined earlier a fuzzy relation R on a crisp universe X is a fuzzy subset of $X \times X$ while, a fuzzy multivalued mapping F assigns to each point of the universe X a fuzzy subset of the universe Y. The connection between the two is defined as follows:

$$R(x, y) = F(x).y,$$

a fuzzy multivalued mapping is called:

(i) *non-void* if and only if for all $F(x) \neq \phi$ for all $x \in X$;

(ii) *surjective* if and only if $rng(F) = Y$;

(iii) *normalized* if and only if for all $x \in X$ there exists a $y \in Y$ such that $F(x).y = 1$.

Before proceeding towards the continuity of fuzzy multivalued mappings we need to recall the definition of fuzzy topology.

Definition 9.1.4 [51] Given a nonempty set X. A fuzzy topology on X is a subset Φ of I^X satisfying:

(1) Φ contains every constant fuzzy set.

(2) If μ_1 and μ_2 then $\mu_1 \wedge \mu_2 \in \Phi$.

(3) If μ_i for each $i \in A$, then $\sup\limits_{i \in A} \mu_i \in \Phi$.

A set with fuzzy topology present on it is called a *fuzzy topological space.*

Definition 9.1.5 [8] A fuzzy multivalued mapping $F : X \rightsquigarrow Y$ between two fuzzy topological spaces X and Y is:

(a) *upper hemicontinuous* at the point x, if for every open neighborhood U of $F(x)$ in topological space Y, $F_{\triangleright}^{-1}(U)$ is a neighborhood of x in topological space X. The fuzzy multivalued mapping F is upper hemicontinuous on X, if it is upper hemicontinuous at every point of X.

(b) *lower hemicontinuous* at the point x, if for every open fuzzy set U which intersects with $F(x)$, $F^{-1}(U)$ is a neighborhood of x in X. The fuzzy multivalued mapping F is lower hemicontinuous on X, if it is lower hemicontinuous at every point of X.

(c) continuous if it is both lower and upper hemicontinuous.

Lemma 9.1.6 [8] Let $F : X \rightsquigarrow Y$ be a fuzzy multivalued mapping between two fuzzy topological spaces X and Y , then

(i) if F is upper hemicontinuous then $F_{\triangleright}^{-1}(\phi)$ is open,

(ii) if F is lower hemicontinuous then $F^{-1}(Y)$ is open.

Theorem 9.1.7 [8] Let $F : X \rightsquigarrow Y$ be a fuzzy multivalued mapping between two fuzzy topological spaces X and Y, then the following statements are equivalent:

(i) $F_{\triangleright}^{-1}(V)$ is open for each open fuzzy subset V of Y.

(ii) $F^{-1}(W)$ is closed for each closed fuzzy subset W of Y.

The compositions, unions and intersections of upper and lower semicontinuous multivalued mappings are upper and lower semicontinuous respectively.

Theorem 9.1.8 [8]*[Fuzzy Maximum Theorem]*

Let X and Y be fuzzy topological space and $f : X \rightsquigarrow Y$ be a continuous fuzzy multivalued function with nonempty fuzzy compact values and suppose $g : X \times Y \to \mathbb{R}$ is a continuous fuzzy function. Define the value fuzzy function $h : X \to \mathbb{R}$ by $h(x) = max\{g(x, y) : y \in f(x)\}$ and the fuzzy multivalued function $m : X \to Y$ of maximizers by $m(x) = \{y \in f(x) : g(x, y) = h(x)\}$ Then the fuzzy function h is continuous and fuzzy multivalued function m is upper hemicontinuous with compact values.

10 Conclusion

In this survey article we have presented basics of fuzzy relational calculus and some of their applications. Fuzzy relational calculus has more impressive applications in expert systems and in artificial intelligence, approximate reasoning, inference system, psychology, medical diagnosis, economics, sociology, knowledge representation, knowledge acquisition and validation, learning, in information process, in pattern analysis and classification, in fuzzy system science for fuzzy control and modelling, in decision making, in engineering for fault detection and diagnosis, in management, etc. In future to implement fuzziness in more complicated situations we require to further develop the theory of fuzzy relational calculus.

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