On the Distance Paired-Domination of Circulant Graphs *

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Abstract

Let \( G = (V, E) \) be a graph without isolated vertices. A set \( D \subseteq V \) is a \( d \)-distance paired-dominating set of \( G \) if \( D \) is a \( d \)-distance dominating set of \( G \) and the induced subgraph \( \langle D \rangle \) has a perfect matching. The minimum cardinality of a \( d \)-distance paired-dominating set for graph \( G \) is the \( d \)-distance paired-domination number, denoted by \( \gamma_p^d(G) \). In this paper, we study the \( d \)-distance paired-domination number of circulant graphs \( C(n; \{1, k\}) \) for \( 2 \leq k \leq 4 \). We prove that for \( k = 2 \), \( n \geq 5 \) and \( d \geq 1 \),

\[
\gamma_p^d(C(n; \{1, k\})) = 2\left\lceil \frac{n}{2kd + 3} \right\rceil,
\]

for \( k = 3 \), \( n \geq 7 \) and \( d \geq 1 \),

\[
\gamma_p^d(C(n; \{1, k\})) = 2\left\lceil \frac{n}{2kd + 2} \right\rceil,
\]

and for \( k = 4 \) and \( n \geq 9 \),

(i) if \( d = 1 \), then

\[
\gamma_p(C(n; \{1, k\})) = \begin{cases} 
2\left\lceil \frac{2n}{4kd + 1} \right\rceil + 2, & \text{if } n \equiv 15, 22 \pmod{23}; \\
2\left\lceil \frac{2n}{23} \right\rceil, & \text{otherwise}.
\end{cases}
\]

(ii) if \( d \geq 2 \), then

\[
\gamma_p^d(C(n; \{1, k\})) = \begin{cases} 
2\left\lceil \frac{2n}{4kd + 1} \right\rceil + 2, & \text{if } n \equiv 2kd, 4kd - 1, 4kd \pmod{4kd + 1} \\
2\left\lceil \frac{2n}{4kd + 1} \right\rceil, & \text{otherwise}.
\end{cases}
\]

Keywords: Paired-domination number; \( d \)-distance paired-domination number; Circulant graph.

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1 Introduction

All graphs considered in this paper are finite and simple. Let \( G = (V(G), E(G)) \) be a graph without isolated vertices. The open neighborhood and the closed neighborhood of a vertex \( v \in V(G) \) are denoted by \( N(v) = \{ u \in V(G) : vu \in E(G) \} \) and \( N[v] = N(v) \cup \{ v \} \), respectively. For a vertex set \( D \subseteq V(G) \), \( N(D) = \bigcup_{v \in D} N(v) \) and \( N[D] = \bigcup_{v \in D} N[v] \). For \( D \subseteq V(G) \), let \( \langle D \rangle \) be the subgraph induced by \( D \).

A set \( D \subseteq V(G) \) is a dominating set if every vertex in \( V(G) - D \) is adjacent to at least one vertex in \( D \). A set \( D \subseteq V(G) \) is a paired-dominating set of \( G \) if it is dominating and the induced subgraph \( \langle D \rangle \) has a perfect matching. The paired-domination number \( \gamma_p(G) \) is the cardinality of a smallest paired-dominating set of \( G \). This type of domination was introduced by Haynes and Slater in [9, 10] and is well studied, for example, in [1–7, 11–13, 15].

For two vertices \( x \) and \( y \), let \( d(x, y) \) denote the distance between \( x \) and \( y \) in \( G \). A set \( D \subseteq V(G) \) is a \( d \)-distance dominating set of \( G \) if every vertex in \( V(G) - D \) is within distance \( d \) of at least one vertex in \( D \). The \( d \)-distance domination number \( \gamma^d(G) \) of \( G \) is the minimum cardinality among all \( d \)-distance dominating sets of \( G \). For a more detailed treatment of domination-related parameters and for terminology not defined here, the reader is referred to [8].

The \( d \)-distance paired-domination was introduced by Joanna Raczek [14] as a generalization of paired-domination. For a positive integer \( d \), a set \( D \subseteq V(G) \) is a \( d \)-distance paired-dominating set if every vertex in \( V(G) - D \) is within distance \( d \) of a vertex in \( D \) and the induced subgraph \( \langle D \rangle \) has a perfect matching. The \( d \)-distance paired-domination number, denoted by \( \gamma_p^d(G) \), is the minimum cardinality of a \( d \)-distance paired-dominating set.

In the same paper, Joanna Raczek investigated properties of the \( d \)-distance paired-domination number of a graph. He also gave an upper bound and a lower bound on the \( d \)-distance paired-domination number of a non-trivial tree \( T \) in terms of the size of \( T \) and the number of leaves in \( T \) and characterized the extremal trees.

The circulant graph \( C(n; S) \) is the graph with the vertex set \( V(C(n; S)) = \{ v_i | 0 \leq i \leq n - 1 \} \) and the edge set \( E(C(n; S)) = \{ v_iv_j | 0 \leq i, j \leq n - 1, (i - j) \text{ mod } n \in S \} \), \( S \subseteq \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor \} \).

In this paper, we determine the exact \( d \)-distance paired-domination number of the circulant graphs \( C(n; \{1, k\}) \) for \( 2 \leq k \leq 4 \) and \( d \geq 1 \). We prove that for \( k = 2, n \geq 5 \) and \( d \geq 1 \),

\[
\gamma_p^d(C(n; \{1, k\})) = 2\left\lceil \frac{n}{2kd + 3} \right\rceil,
\]

for \( k = 3, n \geq 7 \) and \( d \geq 1 \),

\[
\gamma_p^d(C(n; \{1, k\})) = 2\left\lceil \frac{n}{2kd + 2} \right\rceil.
\]
and for \( k = 4 \) and \( n \geq 9 \),

(i) if \( d = 1 \), then

\[
\gamma_p(C(n; \{1, k\})) = \begin{cases} 
2\left[\frac{2n}{kd+1}\right] + 2, & \text{if } n \equiv 15, 22 \pmod{23}; \\
2\left[\frac{2n}{kd+1}\right], & \text{otherwise}.
\end{cases}
\]

(ii) if \( d \geq 2 \), then

\[
\gamma_p^d(C(n; \{1, k\})) = \begin{cases} 
2\left[\frac{2n}{kd+1}\right] + 2, & \text{if } n \equiv 2kd, 4kd - 1, 4kd \pmod{4kd + 1} \\
2\left[\frac{2n}{kd+1}\right], & \text{otherwise}.
\end{cases}
\]

In this paper, let \( D = \{x_i, y_i : i = 1, 2, \ldots, q\} \) be an arbitrary \( d \)-distance paired-dominating set of \( C(n; \{1, k\}) \), where \( \{x_i, y_i : i = 1, 2, \ldots, q\} \) is a perfect matching of \( \langle D \rangle \), and let

\( \mathcal{D}_p = \{(x_i, y_i) : i = 1, 2, \ldots, q\} \).

For each pair \((x_j, y_j) \in \mathcal{D}_p\), with \( j \in \{1, 2, \ldots, q\} \), for convenience, we denote \( x_j = v_{i_j} \), and \( y_j = v_{i_j+1} \) or \( y_j = v_{i_j+k} \), i.e., \((v_{i_j}, v_{i_j+1}) \in \mathcal{D}_p\) or \((v_{i_j}, v_{i_j+k}) \in \mathcal{D}_p\), where \( 0 = i_1 \leq i_2 \leq \cdots \leq i_q < n \).

We also denote

\[
\delta_j = (i_{j+1} - i_j) \mod n
\]

for \( j = 1, 2, \ldots, q \), where the subscripts are modulo \( q \).

For example, we consider the case for \( C(12; \{1, 4\}) \). Let \( d = 4 \), \( D = \{v_1, v_2, v_3, v_5, v_8, v_9\} \), and let \( \mathcal{D}_p = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\} \) where \((x_1, y_1) = (v_1, v_5), (x_2, y_2) = (v_2, v_3) \) and \((x_3, y_3) = (v_8, v_9) \). That is, \( i_1 = 1, i_2 = 2, i_3 = 8 \). We check that \( \delta_1 = (2 - 1) \mod 12 = 1 \), \( \delta_2 = (8 - 2) \mod 12 = 6 \) and \( \delta_3 = (1 - 8) \mod 12 = 5 \).

Clearly,

\[
n = \delta_1 + \cdots + \delta_q.
\]

Throughout the paper, the subscripts are taken modulo \( n \) when it is unambiguous.

## 2 \( d \)-distance paired-domination number of \( C(n; \{1, 2\}) \)

In this section, we shall determine the exact \( d \)-distance paired-domination number of \( C(n; \{1, k\}) \) for \( k = 2 \) and \( d \geq 1 \).

For the circulant graphs \( C(n; \{1, k\}) \), if there exists \( \ell \in \{1, 2, \ldots, q\} \) such that \( \delta_\ell \geq (2d+1)k+2 \) for \( k \geq 2 \) and \( d \geq 1 \), then \( v_{i_\ell + (d+1)k+1} \) would not be dominated by \( D \). Hence, we have

**Observation 2.1.** Suppose \( k \geq 2 \) and \( d \geq 1 \). Then \( 1 \leq \delta_j \leq (2d+1)k+1 \) for every \( j \in \{1, 2, \ldots, q\} \).

**Theorem 2.2.** For \( k \geq 2, n \geq 2k+1 \) and \( d \geq 1 \),

\[
\gamma_p^d(C(n; \{1, k\})) \geq 2\left[\frac{n}{(2d+1)k+1}\right].
\]
Proof. By Observation 2.1, we have \( n = \delta_1 + \cdots + \delta_q \leq q \times ((2d+1)k + 1) \), and thus, \( q \geq \left\lceil \frac{n}{(2d+1)k+1} \right\rceil \), which implies \( \gamma_p^d(C(n; \{1, k\})) \geq 2\left\lceil \frac{n}{(2d+1)k+1} \right\rceil \). \( \Box \)

**Theorem 2.3.** For \( k = 2, n \geq 2k + 1 \) and \( d \geq 1 \), \( \gamma_p^d(C(n; \{1, k\})) = 2\left\lceil \frac{n}{2kd+3} \right\rceil \).

**Proof.** Let \( D \) be a \( d \)-distance paired-dominating set of \( C(n; \{1, k\}) \) for \( k = 2 \). Let \( m = \lfloor \frac{n}{2kd+3} \rfloor \), \( t = n \mod (2kd + 3) \) and

\[
D = \begin{cases} 
\{v(2kd+3)i, v(2kd+3)i+2 : 0 \leq i \leq m-1\}, & \text{if } t = 0; \\
\{v(2kd+3)i, v(2kd+3)i+2 : 0 \leq i \leq m-1\} \cup \{v(2kd+3)m-1, v(2kd+3)m\}, & \text{if } t = 1; \\
\{v(2kd+3)i, v(2kd+3)i+2 : 0 \leq i \leq m-1\} \cup \{v(2kd+3)m, v(2kd+3)m+1\}, & \text{if } t = 2; \\
\{v(2kd+3)i, v(2kd+3)i+2 : 0 \leq i \leq m\}, & \text{otherwise.}
\end{cases}
\]

It is not hard to verify that \( D \) is a \( d \)-distance paired dominating set of \( C(n; \{1, k\}) \) for \( k = 2 \) with \( |D| = 2\left\lceil \frac{n}{2kd+3} \right\rceil \). Hence, \( \gamma_p^d(C(n; \{1, k\})) \leq 2\left\lceil \frac{n}{2kd+3} \right\rceil \) for \( k = 2 \) and \( d \geq 1 \). On the other hand, by Theorem 2.2, we have that \( \gamma_p^d(C(n; \{1, k\})) \geq 2\left\lceil \frac{n}{2kd+3} \right\rceil \) for \( k = 2 \) and \( d \geq 1 \). The result immediately holds. \( \Box \)

In Figure 2.1, we show the \( d \)-distance paired-dominating sets of \( C(n; \{1, 2\}) \) for \( d = 1 \) and \( 7 \leq n \leq 14 \), and for \( d = 2 \) and \( 11 \leq n \leq 22 \), where the vertices of \( d \)-distance paired dominating sets are in dark.

\( G_{n,k} \) stands for \( C(n; \{1, k\}) \) in all figures of this paper.

**Figure 2.1:** The \( d \)-distance paired dominating sets of \( C(n; \{1, 2\}) \)

for \( d = 1 \) and \( 7 \leq n \leq 14 \), and for \( d = 2 \) and \( 11 \leq n \leq 22 \)
3  $d$-distance paired-domination number of $C(n; \{1, 3\})$

In this section, we shall determine the exact $d$-distance paired-domination number of $C(n; \{1, k\})$ for $k = 3$ and $d \geq 1$.

**Lemma 3.1.** For $k = 3$, $n \geq 2k+1$ and $d \geq 1$, $\gamma_p^d(C(n; \{1, k\})) \leq 2\lceil \frac{n}{2kd+2} \rceil$.

**Proof.** Let $D$ be a $d$-distance paired-dominating set of $C(n; \{1, k\})$ for $k = 3$. Let $m = \lfloor \frac{n}{2kd+2} \rfloor$, $t = n \mod (2kd + 2)$ and

$$D = \begin{cases} 
\{v(2kd+2)i, v(2kd+2)i+1 : 0 \leq i \leq m-1\}, & \text{if } t = 0; \\
\{v(2kd+2)i, v(2kd+2)i+1 : 0 \leq i \leq m-1\} \cup \{v(2kd+2)m-1, v(2kd+2)m\}, & \text{if } t = 1; \\
\{v(2kd+2)i, v(2kd+2)i+1 : 0 \leq i \leq m\}, & \text{otherwise.}
\end{cases}$$

It is not hard to verify that $D$ is a $d$-distance paired dominating set of $C(n; \{1, k\})$ for $k = 3$ with $|D| = 2\lceil \frac{n}{2kd+2} \rceil$. Hence, $\gamma_p^d(C(n; \{1, k\})) \leq 2\lceil \frac{n}{2kd+2} \rceil$ for $k = 3$ and $d \geq 1$. \qed

In Figure 3.1, we show the $d$-distance paired-dominating sets of $C(n; \{1, 3\})$ for $d = 1$ and $8 \leq n \leq 16$, and for $d = 2$ and $14 \leq n \leq 28$, where the vertices of $d$-distance paired dominating sets are in dark.

![Figure 3.1: The $d$-distance paired dominating sets of $C(n; \{1, 3\})$](image)

for $d = 1$ and $8 \leq n \leq 16$, and for $d = 2$ and $14 \leq n \leq 28$

**Lemma 3.2.** For $k = 3$, $n \geq 2k+1$ and $d \geq 1$, $\gamma_p^d(C(n; \{1, k\})) \geq 2\lceil \frac{n}{2kd+2} \rceil$. 

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Lemma 4.1. For $C(n; \{1, k\})$ for $k = 3$ with the minimum cardinality. By Observation 2.1, we have that

$$1 \leq \delta_j \leq 2kd + 4$$ (1)

for every $j \in \{1, 2, \ldots, q\}$.

Suppose that there exists $\ell \in \{1, 2, \ldots, q\}$ such that $\delta_\ell \geq 2kd+3$. Then $v_{i\ell+kd+2}$ would not be dominated by $(x_\ell, y_\ell)$ and $(x_{\ell+1}, y_{\ell+1})$. To dominate $v_{i\ell+kd+2}$, we have $v_{i\ell+2} \in D$. It follows that $v_{i\ell-1} \in D$, which implies $(x_{\ell-1}, y_{\ell-1}) = (v_{i\ell-1}, v_{i\ell+2})$, and thus

$$\delta_{\ell-1} = 1.$$ (2)

Let

$$S_1 = \{i : 1 \leq i \leq q, 2kd + 3 \leq \delta_i \leq 2kd + 4\},$$

$$S_2 = \{i : 1 \leq i \leq q, 2 \leq \delta_i \leq 2kd + 2\},$$

$$S_3 = \{i : 1 \leq i \leq q, \delta_i = 1\}.$$

By (1) and (2), we have that $\{1, 2, \ldots, q\} = S_1 \cup S_2 \cup S_3$, and there exists an injection $\phi : S_1 \to S_3$ defined by $\phi(i) = i - 1$ for any $i \in S_1$, i.e., $|S_1| \leq |S_3|$. It follows that

$$n = \delta_1 + \cdots + \delta_q$$

$$= \sum_{i \in S_1} \delta_i + \sum_{i \in S_2} \delta_i + \sum_{i \in S_3} \delta_i$$

$$\leq (2kd + 4)|S_1| + (2kd + 2)|S_2| + |S_3|$$

$$= (2kd + 2)(|S_1| + |S_2| + |S_3|) + 2(|S_1| - |S_3|) - (2kd - 1)|S_3|$$

$$\leq (2kd + 2)q,$$

which implies $q \geq \lceil \frac{n}{2kd+2} \rceil$, and thus $\gamma_d^p(C(n; \{1, k\})) \geq 2\lceil \frac{n}{2kd+2} \rceil$ for $k = 3$ and $d \geq 1$.

As an immediate consequence of Lemmas 3.1 and 3.2, we have the following

Theorem 3.3. For $k = 3$, $n \geq 2k + 1$ and $d \geq 1$, $\gamma_d^p(C(n; \{1, k\})) = 2\lceil \frac{n}{2kd+2} \rceil$.

4 $d$-distance paired-domination number of $C(n; \{1, 4\})$

In this section, we shall determine the $d$-distance paired domination number of $C(n; \{1, k\})$ for $k = 4$ and $d \geq 1$.

We shall first consider the case for $d = 1$. At this time, the $d$-distance paired-domination number $\gamma_d^p$ is just the paired-domination number $\gamma_p$.

Lemma 4.1. For $n \geq 9$,

$$\gamma_p(C(n; \{1, 4\})) \leq \begin{cases} 2\lceil \frac{3n}{23} \rceil + 2, & \text{if } n \equiv 15, 22 \pmod{23}; \\ 2\lceil \frac{3n}{23} \rceil, & \text{otherwise.} \end{cases}$$
Proof. It suffices to give a paired-dominating set $D$ of $C(n; \{1, 4\})$ with the cardinality equaling to the exact values mentioned in this lemma.

Let $m_1 = \lfloor \frac{n}{23} \rfloor$ and $t = n \mod 23$. Then $n = 23m_1 + t$.

For $2k + 1 \leq n \leq 22$, let

$$D = \begin{cases} 
\{v_0, v_1, v_7, v_8\}, & \text{if } 9 \leq n \leq 14 \text{ and } n \neq 12; \\
\{v_0, v_1, v_2, v_3\}, & \text{if } n = 12; \\
\{v_0, v_1, v_7, v_8, v_{13}, v_{14}\}, & \text{if } n = 15; \\
\{v_0, v_1, v_7, v_8, v_{14}, v_{15}\}, & \text{if } 16 \leq n \leq 21 \text{ and } n \neq 19; \\
\{v_0, v_1, v_7, v_{11}, v_{13}, v_{17}\}, & \text{if } n = 19; \\
\{v_0, v_1, v_7, v_8, v_{14}, v_{15}, v_{20}, v_{21}\}, & \text{if } n = 22. 
\end{cases}$$

For $n \geq 23$ and $t \neq 5$, let $m_2 = \lceil \frac{n}{23} \rceil$,

$$D_{01} = \{v_{23i}, v_{23i+1}, v_{23i+7}, v_{23i+11}, v_{23i+13}, v_{23i+17} : 0 \leq i \leq m_1 - 1\},$$
$$D_{02} = \{v_{23m_1+i}, v_{23m_1+i+7} : 0 \leq i \leq m_2 - 1\}$$

and

$$D = \begin{cases} 
D_{01}, & \text{if } t = 0; \\
D_{01} \cup \{v_{23m_1-1}, v_{23m_1}\}, & \text{if } t = 1; \\
D_{01} \cup \{v_{23m_1}, v_{23m_1+1}\}, & \text{if } 2 \leq t \leq 7 \text{ and } t \neq 5; \\
D_{01} \cup D_{02} \cup \{v_{23m_1+m_2-1}, v_{23m_1+m_2}\}, & \text{if } t = 8, 15, 22; \\
D_{01} \cup D_{02} \cup \{v_{23m_1+m_2}, v_{23m_1+m_2+1}\}, & \text{if } 9 \leq t \leq 21 \text{ and } t \neq 12, 15, 19; \\
D_{01} \cup D_{02} \cup \{v_{23m_1+m_2+1}, v_{23m_1+m_2+4}\}, & \text{if } t = 12, 19. 
\end{cases}$$

For $t = 5$, let $m_3 = \frac{n-51}{23}$ where $n > 51$,

$$D_{03} = \{v_{23i}, v_{23i+4}, v_{23i+10}, v_{23i+11}, v_{23i+17}, v_{23i+21} : 0 \leq i \leq m_3 - 1\},$$
$$D_{04} = \{v_{23m_3+10+i}, v_{23m_3+i+7} : 0 \leq i \leq 4\}$$

and

$$D = \begin{cases} 
\{v_7i, v_7i+1 : 0 \leq i \leq 3\}, & \text{if } n = 28; \\
\{v_7i, v_7i+1 : 0 \leq i \leq 4\} \cup \{v_{35}, v_{39}, v_{41}, v_{45}\}, & \text{if } n = 51; \\
D_{03} \cup D_{04} \cup \{v_{23m_3}, v_{23m_3+4}, v_{n-6}, v_{n-2}\}, & \text{if } n > 51. 
\end{cases}$$

It is not hard to verify that $D$ is a paired-dominating set of $C(n; \{1, 4\})$ with the cardinality equaling to the exact values mentioned in this lemma. 

In Figure 4.1 and Figure 4.2, we show the paired-dominating sets of $C(n; \{1, 4\})$ for $9 \leq n \leq 22$ and $23 \leq n \leq 46$, respectively, where the vertices of paired-dominating sets are in dark.

For convenience, let

$$V'(i, t) = \{v_{i+j} \in V(C(n; \{1, 4\})) : 0 \leq j \leq t - 1\},$$
Lemma 4.3. Suppose to the contrary that there exists \( v \in V(G) \) such that \( \langle \{ 1 \leq i \leq q, v \in N[\{ x_i, y_i \}] \} \rangle - 1 \). For a vertex set \( S \subseteq V(G) \), let

\[
\text{rdd}(S) = \sum_{v \in S} \text{rdd}(v).
\]

Since \( x \) is not adjacent to \( y \) for any two vertices \( x, y \in N(v) \) where \( v \in V(C(n; \{ 1, 4 \})) \), by the definition of \( \text{rdd} \), we have

Observation 4.2. \( \text{rdd}(v) = |N(v) \cap D| - 1 \) for every vertex \( v \in V(C(n; \{ 1, 4 \})) \).

Lemma 4.3. Suppose \( n \geq 23 \). Then \( \text{rdd}(V'(i, 23)) \geq 1 \) for every \( i \in \{0, 1, \ldots, n-1\} \).

Proof. Suppose to the contrary that there exists \( \ell \in \{0, 1, \ldots, n-1\} \) such that

\[
\text{rdd}(V'(\ell, 23)) = 0. \tag{3}
\]

Suppose that there exists \( s \in \{ \ell, \ell + 1, \ldots, \ell + 21 \} \) such that \( (v_s, v_{s+1}) \in D_p \). For \( s \in \{ \ell, \ell + 1, \ldots, \ell + 10 \} \), by (3), we have \( v_{s-1}, v_{s+2}, v_{s+3}, v_{s+4}, v_{s+5}, v_{s+6}, v_{s+8}, v_{s+9} \notin D \). To dominate \( v_{s+3} \), we have \( v_{s+7} \in D \). It follows that \( v_{s+10} \notin D \). Since \( \langle D \rangle \) contains a perfect matching, we have \( v_{s+11} \in D \). It follows that \( v_{s+13} \notin D \) (see Figure 4.3(I) for \( s = \ell \)). Thus, \( v_{s+9} \) would not be dominated by \( D \), a contradiction. For \( s \in \{ \ell + 11, \ell + 12, \ldots, \ell + 21 \} \), by symmetry, we derive a contradiction. Hence, there does not exist \( s \in \{ \ell, \ell + 1, \ldots, \ell + 21 \} \) such that \( (v_s, v_{s+1}) \in D_p \).

To dominate \( v_{\ell+9} \), we have that there exists \( s \in \{ \ell + 1, \ldots, \ell + 13 \} \) such that \( (v_s, v_{s+4}) \in D_p \). By (3), we have \( v_{s-2}, v_{s+1}, v_{s+2}, v_{s+3}, v_{s+6} \notin D \) (see Figure 4.3(II) for \( s = \ell + 1 \)). It follows that \( v_{s+2} \) would not be dominated by \( D \), a contradiction. The lemma follows.

Lemma 4.4. \( \gamma_p(C(n; \{ 1, 4 \})) \geq 2 \left\lceil \frac{3n}{23} \right\rceil \) for \( n \geq 9 \).
Proof. Let $D = \{x_i, y_i : i = 1, 2, \ldots, q\}$ be a minimum paired-dominating set of $C(n; \{1, 4\})$ where $\{x_iy_i : i = 1, 2, \ldots, q\}$ is a perfect matching of $\langle D \rangle$. Since each pair $\{x_i, y_i\}$ dominates exactly 8 vertices, we have $8q - n \geq 0$. It follows that $q \geq \lceil \frac{n}{8} \rceil$.

For $9 \leq n \leq 22$ and $n \neq 16$, since $\left\lceil \frac{n}{8} \right\rceil = \left\lceil \frac{2n}{23} \right\rceil$, we have $\gamma_p(C(n; \{1, 4\})) \geq 2\left\lceil \frac{2n}{23} \right\rceil$.

For $n = 16$, it is easy to verify that two pairs of vertices would not dominate all vertices in $C(n; \{1, 4\})$. Hence, $q \geq 3 = \left\lceil \frac{3n}{23} \right\rceil$, which implies $\gamma_p(C(n; \{1, 4\})) \geq 2\left\lceil \frac{3n}{23} \right\rceil$.

For $n \geq 23$, by Lemma 4.3, we have $8q \geq n + \left\lceil \frac{n}{23} \right\rceil = \left\lceil \frac{24n}{23} \right\rceil$. It follows that $q \geq \left\lceil \frac{1}{8} \times \left\lceil \frac{24n}{23} \right\rceil \right\rceil \geq \left\lceil \frac{1}{8} \times \frac{24n}{23} \right\rceil = \left\lceil \frac{3n}{23} \right\rceil$, which implies $\gamma_p(C(n; \{1, 4\})) \geq 2\left\lceil \frac{3n}{23} \right\rceil$. \hfill \square
By Observation 4.2, we have that

\[ rdd(v) = 2. \]

Proof. Choose arbitrary \( v \in V \) and let \( D \subseteq V \) be a dominating set such that \( |D| = rdd(v) \). By the definition of \( rdd(v) \), we have \( |D| = |\{ v \in V \mid v \notin D \}| = 2 \). Since \( D \) is a dominating set, it follows that \( v \notin D \). Thus, \( rdd(v) = 2 \).

Lemma 4.5. If there exists \( \ell \in \{0, 1, \ldots, n-1\} \) such that \( rdd(v_\ell) \geq 2 \), then \( \mathcal{R} > 24 \).

Proof. By Observation 4.2, we have that \( |N(v_\ell) \cap D| = rdd(v_\ell) + 1 \geq 3 \). Since \( |N(v_\ell) \cap D| = 4 \), we have \( \{v_{\ell+1}, v_{\ell+4}\} \subseteq D \) or \( \{v_{\ell-1}, v_{\ell-4}\} \subseteq D \), say \( \{v_{\ell+1}, v_{\ell+4}\} \subseteq D \). It follows that \( rdd(v_{\ell+5}) \geq 1 \), and thus \( \mathcal{R} \geq \sum_{\ell-1 \leq i \leq \ell} (rdd(V', 23)) - 1 \geq 18 \times (rdd(v_\ell) + rdd(v_{\ell+5}) - 1) \geq 18 \times (2 + 1 - 1) > 24 \). The lemma follows.

In what follows, we admit that \( rdd(v_i) \in \{0, 1\} \) for every \( i \in \{0, 1, \ldots, n-1\} \). Let \( v_1, v_2, \ldots, v_t \) be all the vertices re-dominated once, where \( t = rdd(V(C(n; \{1, 4\}))) \) and \( 0 \leq i_1 < i_2 < \cdots < i_t \leq n - 1 \). We define

\( \Theta_j = i_{j+1} - i_j \)

for \( j = 1, 2, \ldots, t \), where the subscripts are modulo \( t \). Obviously, \( \Theta_1 + \cdots + \Theta_t = n \).

Lemma 4.6. If \( \mathcal{R} \leq 24 \), then \( \Theta_j + \Theta_{j+1} \geq 22 \) for every \( j \in \{1, 2, \ldots, t\} \) where \( t = rdd(V(C(n; \{1, 4\}))) \).

Proof. Choose arbitrary \( \ell \in \{1, 2, \ldots, t\} \). By the definition of \( \mathcal{R} \), we have \( \mathcal{R} = \sum_{i=1}^{t} (23 - \Theta_i) \geq (23 - \Theta_{\ell}) + (23 - \Theta_{\ell+1}) = 46 - (\Theta_{\ell} + \Theta_{\ell+1}) \). Since \( \mathcal{R} \leq 24 \), we have \( 46 - (\Theta_{\ell} + \Theta_{\ell+1}) \leq 24 \). It follows that \( \Theta_{\ell} + \Theta_{\ell+1} \geq 22 \). The lemma follows.

Lemma 4.7. For \( n > 3 \), if there exists \( \ell \in \{0, 1, \ldots, n-1\} \) such that \( v_\ell \in D \) and \( rdd(v_\ell) = 1 \), then \( \mathcal{R} > 24 \).

Proof. Assume to the contrary that \( \mathcal{R} \leq 24 \). By Lemma 4.5, we have that \( rdd(v_i) \in \{0, 1\} \) for every \( i \in \{0, 1, \ldots, n-1\} \). By Observation 4.2, we have \( |N(v_\ell) \cap D| = rdd(v_\ell) + 1 = 2 \). Let \( N(v_\ell) \cap D = \{w_1, w_2\} \). By symmetry, we have \( \{w_1, w_2\} \in \{\{v_{\ell-1}, v_{\ell+1}\}, \{v_{\ell+1}, v_{\ell+4}\}, \{v_{\ell+1}, v_{\ell-4}\}, \{v_{\ell-4}, v_{\ell+4}\}\} \). Since \( D \) contains a perfect matching, we infer that

\[ rdd(w_1) = 1 \text{ or } rdd(w_2) = 1. \]
That is, there exists $j \in \{1, 2, \ldots, t\}$ such that $\Theta_j \leq 4$. By Lemma 4.6, we have that

$$\Theta_{j-1} \geq 18 \text{ and } \Theta_{j+1} \geq 18. \quad (4)$$

From (4), we have $\{w_1, w_2\} \not\in \{\{v_{\ell+1}, v_{\ell+4}\}, \{v_{\ell+1}, v_{\ell-4}\}\}$. If $\{w_1, w_2\} = \{v_{\ell-1}, v_{\ell+1}\}$, by (4), we have $V'(\ell-5, 11) \cap D = \{v_{\ell-1}, v_v, v_{\ell+1}\}$ (see Figure 4.4(I)), which is contradicted with the fact that $D$ contains a perfect matching. If $\{w_1, w_2\} = \{v_{\ell-4}, v_{\ell+4}\}$, by (4), we have $v_{\ell-2}, v_{\ell+2}, v_{\ell+3}, v_{\ell+6} \not\in D$. Since $v_{\ell+1} \not\in D$, we have that $v_{\ell+2}$ would not be dominated by $D$ (see Figure 4.4(II)), a contradiction.

$$\text{Figure 4.4: The graphs for the proof of Lemma 4.7}$$

As an immediate consequence of Lemmas 4.5 and 4.7, we have the following

**Corollary 4.9.** Suppose $(x, y) \in D_p$ and $R \leq 24$. Then $N(x) \cap D = \{y\}$.

**Lemma 4.9.** Suppose $n > 23$ and $R \leq 24$. If there exists $\ell \in \{0, 1, \ldots, n-1\}$ such that $v_\ell \not\in D$ and $rdd(v_\ell) = 1$, then one of the following holds.

(a) $V'(\ell - 5, 11) \cap D = \{v_{\ell-5}, v_{\ell-1}, v_{\ell+1}, v_{\ell+5}\}$;

(b) $V'(\ell - 4, 9) \cap D = \{v_{\ell-4}, v_{\ell-3}, v_{\ell+3}, v_{\ell+4}\}$.

**Proof.** By Lemma 4.5, we have that $rdd(v_i) \in \{0, 1\}$ for every $i \in \{0, 1, \ldots, n-1\}$. By Observation 4.2, we have $|N(v_i) \cap D| = rdd(v_i) + 1 = 2$. By symmetry, we distinguish four cases.

**Case 1.** $N(v_i) \cap D = \{v_{\ell-1}, v_{\ell+1}\}$.

By Lemma 4.7, we have $|\{v_{\ell-5}, v_{\ell-2}, v_{\ell+3}\} \cap D| = |\{v_{\ell-3}, v_{\ell+2}, v_{\ell+5}\} \cap D| = 1$. If $v_{\ell-2} \in D$, then $rdd(v_{\ell-2}) = rdd(v_{\ell+2}) = 1$ (see Figure 4.5(I) where the vertices that re-dominated once are in gray). By Lemma 4.6, we derive a contradiction. Hence $v_{\ell-2} \not\in D$. By symmetry, we have $v_{\ell+2} \not\in D$. If $v_{\ell+3} \in D$, then $rdd(v_{\ell+2}) = 1$. Let $i_\ell = \ell$. By Lemma 4.6, we have that $\Theta_{i_\ell} = 2$, $\Theta_{i_\ell-1} \geq 20$ and $\Theta_{i_\ell+1} \geq 20$. It follows that $v_{\ell-3}, v_{\ell+5} \not\in D$ (see Figure 4.5(II)). Since $v_\ell, v_{\ell+2} \not\in D$, we have that $D$ does not contain a perfect matching, a contradiction. Hence $v_{\ell+3} \not\in D$. By symmetry, we have $v_{\ell-3} \not\in D$. Therefore, we conclude that $v_{\ell-5}, v_{\ell+5} \in D$ (see Figure 4.5(III)). Since $v_{\ell-4}, v_{\ell+4} \not\in D$, we have $V'(\ell - 5, 11) \cap D = \{v_{\ell-5}, v_{\ell-1}, v_{\ell+1}, v_{\ell+5}\}$.

**Case 2.** $N(v_i) \cap D = \{v_{\ell+1}, v_{\ell+4}\}$.

Then $rdd(v_{\ell+5}) = 1$. Let $i_\ell = \ell$. By Lemma 4.6, we have that $\Theta_{i_\ell} = 5$, $\Theta_{i_\ell-1} \geq 17$ and $\Theta_{i_\ell+1} \geq 17$. It follows that $v_{\ell-2}, v_{\ell+2}, v_{\ell+3}, v_{\ell+5} \not\in D$. Since $D$ contains a perfect matching, we have $v_{\ell-3} \in D$. It follows that $v_{\ell-5} \not\in D$ (see Figure 4.5(IV)). Thus, $v_{\ell-1}$ would not be dominated by $D$, a contradiction.

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Case 3. \( N(v_i) \cap D = \{v_{i+1}, v_{i-1}\} \).

Then \( rdd(v_{\ell-3}) = 1 \). Let \( i_j = \ell - 3 \). By Lemma 4.6, we have that \( \Theta_j = 3, \Theta_{j-1} \geq 19 \) and \( \Theta_{j+1} \geq 19 \). It follows that \( v_{\ell-6}, v_{\ell-3}, v_{\ell-2}, v_{\ell+3} \notin D \). To dominate \( \{v_{\ell-2}, v_{\ell-1}\} \), we have \( v_{\ell+2}, v_{\ell-5} \in D \). It follows that \( v_{\ell+4}, v_{\ell+5}, v_{\ell+6}, v_{\ell+7} \notin D \). To dominate \( v_{\ell+4} \), we have \( v_{\ell+8} \in D \). It follows that \( v_{\ell+9}, v_{\ell+10}, v_{\ell+11} \notin D \). Since \( D \) contains a perfect matching, we have \( v_{\ell+12} \in D \). It follows that \( v_{\ell+14} \notin D \) (see Figure 4.5(V)). Thus, \( v_{\ell+10} \) would not be dominated by \( D \), a contradiction.

Case 4. \( N(v_i) \cap D = \{v_{i-4}, v_{i+4}\} \).

By Lemma 4.7, we have \(|\{v_{\ell-8}, v_{\ell-5}, v_{\ell-3}\} \cap D| = |\{v_{\ell+3}, v_{\ell+5}, v_{\ell+8}\} \cap D| = 1\).

Suppose \( v_{\ell-8} \in D \). By Lemma 4.6, we have \( v_{\ell-6} \notin D \). By Corollary 4.8, we have \( v_{\ell-7}, v_{\ell-5}, v_{\ell-3} \notin D \). If \( v_{\ell+2} \notin D \), then either \( v_{\ell-2} \) would not be dominated by \( D \) or \( D \) would not contain a perfect matching. Hence \( v_{\ell+2} \in D \). It follows that \( rdd(v_{\ell+3}) = 1 \).

Let \( i_j = \ell \). By Lemma 4.6, we have that \( \Theta_j = 3, \Theta_{j-1} \geq 19 \) and \( \Theta_{j+1} \geq 19 \). It follows that \( v_{\ell-10}, v_{\ell-2} \notin D \) (see Figure 4.5(VI)), and thus \( v_{\ell-6} \) would not be dominated by \( D \), a contradiction. Hence \( v_{\ell-8} \notin D \). By symmetry, we have \( v_{\ell+8} \notin D \).

Suppose \( v_{\ell-5} \in D \). By Corollary 4.8, we have \( v_{\ell-6}, v_{\ell-3} \notin D \). By Lemma 4.6, we have \( v_{\ell-2} \notin D \). Since \( v_{\ell-1} \notin D \), to dominate \( v_{\ell-2} \), we have \( v_{\ell+2} \in D \). It follows that \( rdd(v_{\ell+3}) = 1 \). Let \( i_j = \ell \). By Lemma 4.6, we have that \( \Theta_j = 3, \Theta_{j-1} \geq 19 \) and \( \Theta_{j+1} \geq 19 \). It follows that \( v_{\ell+3}, v_{\ell+6} \notin D \) (see Figure 4.5(VII)). Since \( v_{\ell+1}, v_{\ell-2} \notin D \), we have that \( D \) does not contain a perfect matching, a contradiction. Hence \( v_{\ell-5} \notin D \). By symmetry, we have \( v_{\ell+5} \notin D \).

Therefore, we conclude that \( v_{\ell-3}, v_{\ell+3} \in D \) (see Figure 4.5(VIII)). By Corollary 4.8, we have \( v_{\ell-2}, v_{\ell+2} \notin D \), i.e., \( V(\ell - 4, 9) \cap D = \{v_{\ell-4}, v_{\ell-3}, v_{\ell+3}, v_{\ell+4}\} \).

This completes the proof of Lemma 4.9.

Lemma 4.10. Let \( t = rdd(V(C(n; \{1, 4\})) \). If \( R \leq 24 \), then the following conditions
(a) \( \Theta_i \in \{7, 15, 23\} \) for every \( i \in \{1, 2, \ldots, t\} \);
(b) \(|\{1 \leq i \leq t : \Theta_i = 15\}| \) is even.

Proof. (a) Let \( A_1 = \{0 \leq i \leq n - 1 : \text{rdd}(v_i) = 1, V'(i-5, 11) \cap D = \{v_{i-5}, v_{i-1}, v_{i+1}, v_{i+5}\}\) and \( A_2 = \{0 \leq i \leq n - 1 : \text{rdd}(v_i) = 1, V'(i - 4, 9) \cap D = \{v_{i-4}, v_{i-3}, v_{i+3}, v_{i+4}\}\). By Lemma 4.9, we have \( A_1 \cap A_2 = \emptyset \) and

\[
A_1 \cup A_2 = \{0 \leq i \leq n - 1 : \text{rdd}(v_i) = 1\}. \tag{5}
\]

By Lemma 4.3, we have \( \Theta_i \leq 23 \) for every \( i \in \{1, 2, \ldots, t\} \). Let \( \Theta \) be an arbitrary integer of \( \{\Theta_1, \ldots, \Theta_t\} \). That is, there exists \( \ell \in \{0, 1, \ldots, n - 1\} \) such that \( \text{rdd}(v_\ell) = \text{rdd}(v_{\ell + \Theta}) = 1 \) and \( \text{rdd}(v_{\ell + j}) = 0 \) for every \( j \in \{1, 2, \ldots, \Theta - 1\} \). To prove (a), it suffices to show \( \Theta \in \{7, 15, 23\} \).

Case 1. \( \ell \in A_1 \).

By Corollary 4.8, we have \( v_{\ell + 6}, v_{\ell + 9} \notin D \). By Lemma 4.6, we have \( v_{\ell + 7}, v_{\ell + 8}, v_{\ell + 10} \notin D \). To dominate \( \{v_{\ell + 7}, v_{\ell + 8}\} \), we have \( v_{\ell + 11}, v_{\ell + 12} \in D \). It follows from Corollary 4.8 that \( v_{\ell + 13}, v_{\ell + 15}, v_{\ell + 16} \notin D \). By Lemma 4.6, we have \( v_{\ell + 14}, v_{\ell + 17} \notin D \). To dominate \( v_{\ell + 14} \), we have \( v_{\ell + 18} \in D \). Since \( D \) contains a perfect matching, it follows from Corollary 4.8 that \(|\{v_{\ell + 19}, v_{\ell + 22}\} \cap D| = 1\).

If \( v_{\ell + 19} \in D \), then \( \text{rdd}(v_{\ell + 15}) = 1 \) and \( \ell + 15 \in A_2 \) (see Figure 4.6(I) where the vertices that re-dominated once are in gray). Thus, \( \Theta = 15 \). If \( v_{\ell + 22} \in D \), by (5), we have \( v_{\ell + 24}, v_{\ell + 26} \in D \) and \( \text{rdd}(v_{\ell + 23}) = 1 \), i.e., \( \ell + 23 \in A_1 \) (see Figure 4.6(II)). Thus, \( \Theta = 23 \).

Case 2. \( \ell \in A_2 \).

By Corollary 4.8, we have \( v_{\ell + 5}, v_{\ell + 7}, v_{\ell + 8} \notin D \). By Lemma 4.6, we have \( v_{\ell + 6}, v_{\ell + 9} \notin D \). To dominate \( v_{\ell + 6} \), we have \( v_{\ell + 10} \in D \). Since \( D \) contains a perfect matching, it follows from Corollary 4.8 that \(|\{v_{\ell + 11}, v_{\ell + 14}\} \cap D| = 1\).

If \( v_{\ell + 11} \in D \), then \( \text{rdd}(v_{\ell + 7}) = 1 \) and \( \ell + 7 \in A_2 \) (see Figure 4.6(III)). Thus, \( \Theta = 7 \). If \( v_{\ell + 14} \in D \), by (5), we have \( v_{\ell + 16}, v_{\ell + 20} \in D \) and \( \text{rdd}(v_{\ell + 15}) = 1 \), i.e., \( \ell + 15 \in A_1 \) (see Figure 4.6(IV)). Thus, \( \Theta = 15 \).

From the above discuss, we see that \( \Theta_i \in \{7, 15, 23\} \) for every \( i \in \{1, 2, \ldots, t\} \) if \( R \leq 24 \).

(b) Let \( v_{i_1}, v_{i_2}, \ldots, v_{i_t} \) be all the vertices that re-dominated once, where \( 0 \leq i_1 < i_2 < \cdots < i_t \leq n - 1 \). Then \( \Theta_j = i_{j+1} - i_j \) for \( j = 1, 2, \ldots, t \). By the arguments of (a), we conclude that \( \Theta_j = 15 \) if and only if either \( i_j \in A_1 \) and \( i_{j+1} \in A_2 \), or \( i_j \in A_2 \) and \( i_{j+1} \in A_1 \). Note that \( i_{t+1} = i_1 \). We infer that \(|\{1 \leq i \leq t : \Theta_i = 15\}| \) is even. \( \square \)

**Lemma 4.11.** \( \gamma_p(C(n; \{1, 4\})) \geq 2\lceil\frac{2n}{23}\rceil + 2 \) for \( n \equiv 15, 22 \pmod{23} \).

Proof. Suppose to the contrary that \( \gamma_p(C(n; \{1, 4\})) < 2\lceil\frac{2n}{23}\rceil + 2 \), i.e., there exists a paired
for $n = t \in \{x, y\}$ the vertices dominated by the vertex pair $(x, y)$.

Then $rdd$ for $D$ and $rdd(v_{t+1}) = 1$

For $n = 15$, it is not hard to verify that two (three) pairs of vertices would not dominate all vertices in $C(n; \{1, 4\})$. Hence, we need only consider the case for $n > 23$.

Since each pair $\{x_i, y_i\}$ in $C(n; \{1, 4\})$ dominates exactly 8 vertices, we have $8q = rdd(V(C(n; \{1, 4\})))$. By the definition of $\mathfrak{R}$, we have that $23 \times (8q - n) = 23 \times rdd(V(C(n; \{1, 4\}))) = 23 \times \sum_{v \in V(C(n; \{1, 4\}))} rdd(v) = \sum_{0 \leq \ell \leq n-1} rdd(V'(\ell, 23)) = n + \mathfrak{R}$, and thus $q = \frac{3n + 8 + 23}{23}$. By (6), we conclude that $\mathfrak{R} = 8$ for $n \equiv 15 \pmod{23}$ and $\mathfrak{R} = 24$ for $n \equiv 22 \pmod{23}$.

By Lemma 4.5, we have that $rdd(v_i) \in \{0, 1\}$ for every $i \in \{0, 1, \ldots, n-1\}$. Let $t = rdd(V(C(n; \{1, k\})))$. By Lemma 4.10, we have that $\Theta_i \in \{7, 15, 23\}$ for every $i \in \{1, 2, \ldots, t\}$ if $\mathfrak{R} \leq 24$. Let $N_7 = |1 \leq i \leq t : \Theta_i = 7|$ and $N_{15} = |1 \leq i \leq t : \Theta_i = 15|$. Then $\mathfrak{R} = (23 - 23) \times \sum_{0 \leq \ell \leq n-1} rdd(V'(\ell, 23)) = n + \mathfrak{R}$, and thus $q = \frac{3n + 8 + 23}{23}$. By (6), we conclude that $\mathfrak{R} = 8$ for $n \equiv 15 \pmod{23}$ and $\mathfrak{R} = 24$ for $n \equiv 22 \pmod{23}$.

From Lemmas 4.1, 4.4 and 4.11, we have the following

**Theorem 4.12.** For $n \geq 9$,

$$
\gamma_p(C(n; \{1, 4\})) = \begin{cases} 
2 \left\lceil \frac{3n}{23} \right\rceil + 2, & \text{if } n \equiv 15, 22 \pmod{23}; \\
2 \left\lceil \frac{3n}{23} \right\rceil, & \text{otherwise}.
\end{cases}
$$

In the rest of this section, we shall consider the case for $d \geq 2$.

For the readers’ convenience, we shall show the cases for the vertices dominated by a specific vertex pair $(x, y) \in D_p$ in Figure 4.7, where the vertex pair $(x, y)$ are in dark and the vertices dominated by the vertex pair $(x, y)$ are in gray.

Figure 4.6: The graphs for proof of Lemma 4.10
Figure 4.7: The cases for the vertices dominated by a specific vertex pair

Lemma 4.13. For $k = 4$, $n \geq 2k + 1$ and $d \geq 2$,

$$
\gamma^{d}_{dp}(C(n; \{1, k\})) \leq \begin{cases} 
2\lceil \frac{2n}{4kd+1} \rceil + 2, & \text{if } n \equiv 2kd, 4kd - 1, 4kd \pmod{4kd + 1} \\
2\lceil \frac{2n}{4kd+1} \rceil, & \text{otherwise.}
\end{cases}
$$

Proof. It suffices to give a $d$-distance paired-dominating set $D$ of $C(n; \{1, k\})$ for $k = 4$ and $d \geq 2$ with the cardinality equaling to the exact values mentioned in this lemma.

For $9 \leq n \leq 4kd$, let

$$
D = \begin{cases} 
\{v_0, v_1\}, & \text{if } 9 \leq n \leq 2kd - 1; \\
\{v_0, v_1, v_{2kd-2}, v_{2kd-1}\}, & \text{if } n = 2kd; \\
\{v_0, v_1, v_{2kd-1}, v_{2kd}\}, & \text{if } 2kd + 1 \leq n \leq 2kd + 3; \\
\{v_0, v_1, v_{2kd-1}, v_{2kd+3}\}, & \text{if } 2kd + 4 \leq n \leq 4kd - 2; \\
\{v_0, v_1, v_{2kd-1}, v_{2kd+3}, v_{n-2}, v_{n-1}\}, & \text{if } n = 4kd - 1, 4kd.
\end{cases}
$$

For $n \geq 4kd + 1$, let $\alpha = 4kd + 1$, $\beta = 2kd - 1$, $m_1 = \lceil \frac{n}{\alpha} \rceil$ and $t = n \pmod{\alpha}$. Let

$$
D_{01} = \{v_{\alpha i}, v_{\alpha i+1}, v_{\alpha i+\beta}, v_{\alpha i+\beta+4} : 0 \leq i \leq m_1 - 1\},
$$

$$
D_{02} = \{v_{\alpha m_1}, v_{\alpha m_1+1}, v_{\alpha m_1+\beta}, v_{\alpha m_1+\beta+4}\}
$$

and

$$
D = \begin{cases} 
D_{01}, & \text{if } t = 0; \\
D_{01} \cup \{v_{\alpha m_1-1}, v_{\alpha m_1}\}, & \text{if } t = 1; \\
D_{01} \cup \{v_{\alpha m_1}, v_{\alpha m_1+1}\}, & \text{if } 2 \leq t \leq 2kd - 1 \text{ and } t \neq 2kd - 3; \\
D_{01} \cup \{v_{\alpha m_1-5}, v_{\alpha m_1-1}\}, & \text{if } t = 2kd - 3; \\
D_{01} \cup \{v_{\alpha m_1}, v_{\alpha m_1+1}, v_{\alpha m_1+\beta-1}, v_{\alpha m_1+\beta}\}, & \text{if } t = 2kd; \\
D_{01} \cup \{v_{\alpha m_1}, v_{\alpha m_1+1}, v_{\alpha m_1+\beta}, v_{\alpha m_1+\beta+1}\}, & \text{if } 2kd + 1 \leq t \leq 2kd + 3; \\
D_{01} \cup D_{02}, & \text{if } 2kd + 4 \leq t \leq 4kd - 2; \\
D_{01} \cup D_{02} \cup \{v_{n-2}, v_{n-1}\}, & \text{if } t = 4kd - 1, 4kd.
\end{cases}
$$

It is not hard to verify that $D$ is a $d$-distance paired dominating set of $C(n; \{1, k\})$ for $k = 4$ and $d \geq 2$ with the cardinality equaling to the exact values mentioned in this lemma.

For convenience, we give a map $\varphi : \{1, 2, \ldots, q\} \rightarrow \{1, 4\}$ defined by $\varphi(s) = 1$ for $(x_s, y_s) = (v_{i_s}, v_{i_s+1})$ and $\varphi(s) = 4$ for $(x_s, y_s) = (v_{i_s}, v_{i_s+4})$. 


Lemma 4.14. Suppose $k = 4$, $d \geq 2$ and $\ell \in \{1, 2, \ldots, q\}$.

(a) If $\delta_{\ell-1} \geq 2kd + 3$, then $\delta_{\ell} \leq 2$.
(b) If $\varphi(\ell) = 1$, then either $\delta_{\ell-1} \leq 5$ or $\delta_{\ell} \leq 2kd - 1$.
(c) If $\varphi(\ell) = 4$, then either $\delta_{\ell-1} \leq 2$ or $\delta_{\ell} \leq 2kd + 2$.
(d) If $\varphi(\ell) = \varphi(\ell + 1) = 4$ and $2kd \leq \delta_{\ell} \leq 2kd + 2$, then either $\delta_{\ell-1} \leq 2$ or $\delta_{\ell+1} \leq 2$.

Proof. (a) Suppose $\delta_{\ell-1} \geq 2kd + 3$. If $\delta_{\ell} \geq 3$, then $v_{i_{\ell-kd+2}}$ would not be dominated by $D$, a contradiction. Hence $\delta_{\ell} \leq 2$.

(b) Suppose $\varphi(\ell) = 1$. If $\delta_{\ell-1} \geq 6$ and $\delta_{\ell} \geq 2kd$, then $v_{i_{\ell+kd-1}}$ would not be dominated by $D$, a contradiction. Hence either $\delta_{\ell-1} \leq 5$ or $\delta_{\ell} \leq 2kd - 1$.

(c) Suppose $\varphi(\ell) = 4$. If $\delta_{\ell-1} \geq 3$ and $\delta_{\ell} \geq 2kd + 3$, then $v_{i_{\ell+kd+2}}$ would not be dominated by $D$, a contradiction. Hence either $\delta_{\ell-1} \leq 2$ or $\delta_{\ell+1} \leq 2kd + 2$.

(d) Suppose $\varphi(\ell) = \varphi(\ell + 1) = 4$ and $2kd \leq \delta_{\ell} \leq 2kd + 2$. If $\delta_{\ell-1} \geq 3$ and $\delta_{\ell+1} \geq 3$, then at least one of $\{v_{i_{\ell+kd+2}}, v_{i_{\ell+kd+3}}\}$ would not be dominated by $D$, a contradiction. Hence either $\delta_{\ell-1} \leq 2$ or $\delta_{\ell+1} \leq 2$.

We denote $\Omega_i = \delta_i + \delta_{i+1}$ for $i = 1, 2, \ldots, q$, where the subscripts are taken modulo $q$.

Lemma 4.15. Suppose $k = 4$ and $d \geq 2$. Let $\ell \in \{1, 2, \ldots, q\}$. Then either $\Omega_{\ell} \leq 4kd + 1$, or $\frac{\Omega_{\ell-1} + \Omega_{\ell}}{2} < 4kd + 1$ and $\delta_{\ell-1} \leq 5$.

Proof. Suppose

$$\Omega_{\ell} \geq 4kd + 2.$$  \hfill (7)

By Observation 2.1, we have that $\delta_i \leq 2kd + 5$ for every $i \in \{1, 2, \ldots, q\}$. If $\delta_{\ell} \leq 2kd - 4$ or $\delta_{\ell+1} \leq 2kd - 4$, then $\Omega_{\ell} = \delta_{\ell} + \delta_{\ell+1} \leq (2kd + 5) + (2kd - 4) = 4kd + 1$, a contradiction with (7). Therefore,

$$\delta_{\ell} \geq 2kd - 3 \geq 13$$  \hfill (8)

and

$$\delta_{\ell+1} \geq 2kd - 3 \geq 13.$$  \hfill (9)

It follows from (9) and Lemma 4.14 (a) that

$$\delta_{\ell} \leq 2kd + 2.$$

Case 1. $\varphi(\ell + 1) = 1$.

By (8) and Lemma 4.14 (b), we have $\delta_{\ell+1} \leq 2kd - 1$. It follows that $\Omega_{\ell} = \delta_{\ell} + \delta_{\ell+1} \leq (2kd + 2) + (2kd - 1) = 4kd + 1$, a contradiction with (7).

Case 2. $\varphi(\ell + 1) = 4$.

By (8) and Lemma 4.14 (c), we have $\delta_{\ell+1} \leq 2kd + 2$. 16
Let \( d \) be defined as in Lemma 4.14 (d), we have that either \( \delta_{\ell-1} \leq 5 \) or \( \delta_{\ell} \leq 2kd - 1 \). If \( \delta_{\ell} \leq 2kd - 1 \), then \( \Omega_{\ell} = \delta_{\ell} + \delta_{\ell+1} \leq (2kd - 1) + (2kd + 2) = 4kd + 1 \), a contradiction with (7). Hence \( \delta_{\ell} > 2kd - 1 \), i.e.,
\[
\delta_{\ell-1} \leq 5.
\]
It follows that \( \frac{\Omega_{\ell-1} + \Omega_\ell}{2} = \frac{(\delta_{\ell-1} + \delta_{\ell})(\delta_{\ell} + \delta_{\ell+1})}{2} \leq \frac{5+(2kd+2)+(2kd+2)+(2kd+2)}{2} < 4kd + 1 \).

Suppose \( \varphi(\ell) = 4 \). If \( \delta_{\ell} \leq 2kd - 1 \) or \( \delta_{\ell+1} \leq 2kd - 1 \), then \( \Omega_{\ell} = \delta_{\ell} + \delta_{\ell+1} \leq (2kd - 1) + (2kd + 2) = 4kd + 1 \), a contradiction with (7). Hence \( \delta_{\ell} \geq 2kd \) and \( \delta_{\ell+1} \geq 2kd \).

By Lemma 4.14 (d), we have that
\[
\delta_{\ell-1} \leq 2,
\]
and thus \( \frac{\Omega_{\ell-1} + \Omega_\ell}{2} = \frac{(\delta_{\ell-1} + \delta_{\ell})(\delta_{\ell} + \delta_{\ell+1})}{2} \leq \frac{2+(2kd+2)+(2kd+2)+(2kd+2)}{2} < 4kd + 1 \).

This completes the proof of Lemma 4.15.

**Lemma 4.16.** For \( k = 4, n \geq 2k + 1 \) and \( d \geq 2 \), \( \gamma_p^d(C(n; \{1, k\})) \geq 2\left\lceil \frac{2n}{4kd+1} \right\rceil \).

**Proof.** Let \( S_1 = \{1 \leq i \leq q : \Omega_i \leq 4kd + 1\} \) and \( S_2 = \{1 \leq i \leq q : \Omega_i \geq 4kd + 2\} \). Then \( S_1 \cup S_2 = \{1, 2, \ldots, q\} \). By Lemma 4.15, there exists an injection \( \phi : S_2 \to S_1 \) defined by \( \phi(i) = i - 1 \), where \( i \in S_2 \). Then \( \Omega_i + \Omega_{\phi(i)} < 2(4kd + 1) \) for any \( i \in S_2 \).

It follows that
\[
2n = \sum_{i=1}^{q} \Omega_i = \sum_{i \in S_1} \Omega_i + \sum_{i \in S_2} \Omega_i = \sum_{i \in S_1 \setminus \phi(S_2)} \Omega_i + \sum_{i \in S_2} \Omega_i + \sum_{i \in \phi(S_2)} \Omega_i = \sum_{i \in S_1 \setminus \phi(S_2)} \Omega_i + \sum_{i \in S_2} (\Omega_i + \Omega_{\phi(i)}) \leq (|S_1| - |S_2|) \times (4kd + 1) + |S_2| \times 2(4kd + 1) = (|S_1| + |S_2|) \times (4kd + 1) = q \times (4kd + 1),
\]
which implies \( q \geq \left\lceil \frac{2n}{4kd+1} \right\rceil \), and thus \( \gamma_p^d(C(n; \{1, k\})) \geq 2\left\lceil \frac{2n}{4kd+1} \right\rceil \) for \( k = 4, n \geq 2k + 1 \) and \( d \geq 2 \).

**Lemma 4.17.** For \( k = 4, n \geq 2k + 1 \) and \( d \geq 2 \), suppose \( \delta_i \geq 6 \) for every \( i \in \{1, 2, \ldots, q\} \).

Let \( s \in \{1, 2, \ldots, q\} \).

(a) If \( (\varphi(s), \varphi(s+1)) = (1, 1) \), then \( \delta_s \leq 2kd - 1 \) and \( \delta_s \neq 2kd - 3 \).

(b) If \( (\varphi(s), \varphi(s+1)) = (1, 4) \), then \( \delta_s \leq 2kd - 1 \) and \( \delta_s \notin \{2kd - 3, 2kd - 2\} \).

(c) If \( (\varphi(s), \varphi(s+1)) = (4, 1) \), then \( \delta_s \leq 2kd + 2 \) and \( \delta_s \notin \{2kd, 2kd + 1\} \).

(d) If \( (\varphi(s), \varphi(s+1)) = (4, 4) \), then \( \delta_s \leq 2kd - 1 \).

**Proof.**

(a) Suppose \( (\varphi(s), \varphi(s+1)) = (1, 1) \). If \( \delta_s \geq 2kd \) or \( \delta_s = 2kd - 3 \), then \( v_{i, s + kd - 1} \) would not be dominated by \( D \), a contradiction. Hence \( \delta_s \leq 2kd - 1 \) and \( \delta_s \neq 2kd - 3 \).
Let \( v \equiv 2kd \) or \( \delta_s \in \{2kd - 3, 2kd - 2\} \), then 
\( v_{i+k} \) would not be dominated by \( D \), a contradiction. Hence \( \delta_s \leq 2kd - 1 \) and \( \delta_s \not\in \{2kd - 3, 2kd - 2\} \).

(c) Suppose \((\varphi(s), \varphi(s + 1)) = (4, 1)\). If \( \delta_s \geq 2kd + 3 \) or \( \delta_s = 2kd \), then \( v_{i+k+2} \) would not be dominated by \( D \), a contradiction. If \( \delta_s = 2kd + 1 \), then \( v_{i+k+3} \) would not be dominated by \( D \), a contradiction. Hence \( \delta_s \leq 2kd + 2 \) and \( \delta_s \not\in \{2kd, 2kd + 1\} \).

(d) Suppose \((\varphi(s), \varphi(s + 1)) = (4, 4)\). If \( \delta_s \geq 2kd \), then at least one of \( \{v_{i+k+2}, v_{i+k+3}\} \) would not be dominated by \( D \), a contradiction. Hence \( \delta_s \leq 2kd - 1 \).

From Lemma 4.17, we can easily derive the following result.

**Lemma 4.18.** For \( k = 4 \), \( n \geq 2k + 1 \) and \( d \geq 2 \), suppose \( \delta_i \geq 6 \) for every \( i \in \{1, 2, \ldots, q\} \). Let \( s \in \{1, 2, \ldots, q\} \).

(a) If \((\varphi(s), \varphi(s + 1), \varphi(s + 2)) \in \{(1, 1, 1), (1, 4, 4), (4, 4, 4)\} \), then \( \Omega_s \leq 4kd - 2 \).

(b) If \((\varphi(s), \varphi(s + 1), \varphi(s + 2)) = (1, 1, 4) \), then \( \Omega_s \leq 4kd - 2 \) and \( \Omega_s \not\in 4kd - 4 \).

(c) If \((\varphi(s), \varphi(s + 1), \varphi(s + 2)) \in \{(1, 4, 1), (4, 4, 1)\} \), then \( \Omega_s \not\in \{4kd, 4kd - 1\} \).

(d) If \((\varphi(s), \varphi(s + 1), \varphi(s + 2)) = (4, 1, 1) \), then \( \Omega_s \not\in 4kd - 1 \).

**Lemma 4.19.** Suppose \( k = 4 \), \( n \geq 2k + 1 \) and \( d \geq 2 \). Then \( \gamma_p^d(C(n; \{1, k\})) \geq 2\lfloor \frac{2n}{4kd + 1} \rfloor + 2 \) for \( n \equiv 2kd, 4kd - 1, 4kd \pmod{4kd + 1} \).

**Proof.** Suppose to the contrary that \( \gamma_p^d(C(n; \{1, k\})) < 2\lfloor \frac{2n}{4kd + 1} \rfloor + 2 \), i.e., there exists a \( d \)-distance paired dominating set \( D = \{x_i, y_i : i = 1, 2, \ldots, q\} \) such that

\[
q = \left\lfloor \frac{2n}{4kd + 1} \right\rfloor. \tag{10}
\]

Let \( x \in \mathbb{Z} \) be such that

\[
2n = \sum_{i=1}^{q} \Omega_i = q \times (4kd + 1) - x. \tag{11}
\]

It follows from (10) and (11) that

\[
\left\lfloor \frac{2n}{4kd + 1} \right\rfloor = q = \frac{2n + x}{4kd + 1}. \tag{12}
\]

Since \( 2n \equiv 4kd, 4kd - 1, 4kd - 3 \pmod{4kd + 1} \), by (12), we have

\[
x = 1, 2, 4 \tag{13}
\]

for \( n \equiv 2kd, 4kd, 4kd - 1 \pmod{4kd + 1} \), respectively.

Let \( S_1 = \{1 \leq i \leq q : \Omega_i \leq 4kd + 1\} \) and \( S_2 = \{1 \leq i \leq q : \Omega_i \geq 4kd + 2\} \). Then \( S_1 \cup S_2 = \{1, 2, \ldots, q\} \). By Lemma 4.15, there exists an injection \( \phi : S_2 \to S_1 \) defined by \( \phi(i) = i - 1 \), where \( i \in S_2 \). Then \( \Omega_i + \Omega_{\phi(i)} < 2(4kd + 1) \) for any \( i \in S_2 \).
If there exists \( \ell \in \{1, 2, \ldots, q\} \) such that \( \Omega_{\ell} \geq 4kd + 2 \), by Lemma 4.15, we have \( \delta_{\ell-1} \leq 5 \). It follows from Observation 2.1 that \( \Omega_{\ell-1} = \delta_{\ell-1} + \delta_{\ell} \leq (2kd + 5) \leq (4kd + 1) - 7 \) and \( \Omega_{\ell-2} = \delta_{\ell-2} + \delta_{\ell-1} \leq (2kd + 5) + 5 \leq (4kd + 1) - 7 \), which implies

\[
\ell - 2 \in S_1 \setminus \phi(S_2).
\]

It follows that

\[
\sum_{i=1}^{q} \Omega_i = \sum_{i \in S_1} \Omega_i + \sum_{i \in S_2} \Omega_i
\]

\[
= \sum_{i \in S_1 \setminus \{\phi(S_2) \cup (\ell - 2)\}} \Omega_i + \Omega_{\ell - 2} + \sum_{i \in \phi(S_2)} \Omega_i + \sum_{i \in S_2} \Omega_i
\]

\[
= \sum_{i \in S_1 \setminus \{\phi(S_2) \cup (\ell - 2)\}} \Omega_i + \Omega_{\ell - 2} + \sum_{i \in S_2} \left( \Omega_i + \Omega_{\phi(i)} \right)
\]

\[
\leq (|S_1| - |S_2| - 1) \times (4kd + 1) + ((4kd + 1) - 7) + |S_2| \times 2(4kd + 1)
\]

\[
= (|S_1| + |S_2|) \times (4kd + 1) - 7
\]

By (11), we have \( x \geq 7 \), which is a contradiction with (13). Hence

\[
\Omega_i \leq 4kd + 1 \tag{14}
\]

for every \( i \in \{1, 2, \ldots, q\} \) when \( n \equiv 2kd, 4kd, 4kd - 1 \pmod{4kd + 1} \).

For \( n = 2kd \), i.e., \( q = 1 \), we may assume \((x_1, y_1) \in \{(v_0, v_1), (v_0, v_4)\}\). Then \( v_{kd+2} \) would not be dominated by \( D \), a contradiction.

For \( n = 4kd - 1, 4kd \), i.e., \( q = 2 \), by Observation 2.1, we have \( \delta_j \leq 2kd + 5 \) for \( j = 1, 2 \). It follows that \( \delta_j \geq (4kd - 1) - (2kd + 5) = 2kd - 6 \geq 6 \) for \( j = 1, 2 \). If \((\varphi(1), \varphi(2)) \in \{(1, 1), (4, 4)\}\), by Lemma 4.17 (a) and (d), we have \( n = \delta_1 + \delta_2 \leq (2kd - 1) + (2kd - 1) = 4kd - 2 \), a contradiction. If \((\varphi(1), \varphi(2)) \in \{(1, 4), (4, 1)\}\), by Lemma 4.17 (b) and (c), we have \( n = \delta_1 + \delta_2 \neq 4kd, 4kd - 1 \), a contradiction. Therefore, it remains to consider the case for \( n \notin \{2kd, 4kd - 1, 4kd\} \), i.e., \( q \geq 3 \).

**Case 1.** \( n \equiv 2kd, 4kd \pmod{4kd + 1} \).

Then \( x = 1, 2 \). It follows from (11) and (14) that \( 4kd - 1 \leq \Omega_i \leq 4kd + 1 \) for every \( i \in \{1, 2, \ldots, q\} \), and there exists \( \ell \in \{1, 2, \ldots, q\} \) such that \( \Omega_{\ell} < 4kd + 1 \). By Observation 2.1, we have that \( \delta_i = \Omega_i - \delta_{i+1} \geq (4kd - 1) - (2kd + 5) = 2kd - 6 \geq 6 \) for every \( i \in \{1, 2, \ldots, q\} \). By Lemma 4.18 (a) and (b), we conclude that for any \( i \in \{1, 2, \ldots, q\} \), \( \varphi(i) \neq \varphi(i+1) \). Since \( q \geq 3 \), by Lemma 4.18 (e), we derive a contradiction.

**Case 2.** \( n \equiv 4kd - 1 \pmod{4kd + 1} \).

Then \( x = 4 \). It follows from (11) and (14) that \( 4kd - 3 \leq \Omega_i \leq 4kd + 1 \) for every \( i \in \{1, 2, \ldots, q\} \), and there exists \( \ell \in \{1, 2, \ldots, q\} \) such that \( \Omega_{\ell} < 4kd + 1 \).

By Observation 2.1, we have that \( \delta_i = \Omega_i - \delta_{i+1} \geq (4kd - 1) - (2kd + 5) = 2kd - 6 \geq 6 \) for every \( i \in \{1, 2, \ldots, q\} \). If \( \Omega_i \geq 4kd - 1 \) for every \( i \in \{1, 2, \ldots, q\} \), by Lemma 4.18 (a) and (b), we conclude that for any \( i \in \{1, 2, \ldots, q\} \), \( \varphi(i) \neq \varphi(i+1) \). Since \( q \geq 3 \), by Lemma 4.18 (e), we derive a contradiction.
4.18 (c), we have that $\Omega_i = 4kd + 1$ for every $i \in \{1, 2, \ldots, q\}$, which is a contradiction. Hence, there exists $s \in \{1, 2, \ldots, q\}$ such that $\Omega_s \in \{4kd - 2, 4kd - 3\}$.

**Case 2.1** Suppose $\Omega_s = 4kd - 3$.

By (11) and (14), we have that $\Omega_i = 4kd + 1$ for every $i \in \{1, 2, \ldots, q\} \setminus \{s\}$. It follows that either $\delta_s \leq 2kd - 2$ or $\delta_{s+1} \leq 2kd - 2$. If $\delta_s \leq 2kd - 2$, by Lemma 4.17, then $\Omega_{s-1} = \delta_{s-1} + \delta_s \leq (2kd + 2) + (2kd - 2) = 4kd$, a contradiction. If $\delta_{s+1} \leq 2kd - 2$, by Lemma 4.17, then $\Omega_{s+1} = \delta_{s+1} + \delta_{s+2} \leq (2kd - 2) + (2kd + 2) = 4kd$, a contradiction.

**Case 2.2** Suppose $\Omega_s = 4kd - 2$.

By (11) and (14), there exists $t \in \{1, 2, \ldots, q\} \setminus \{s\}$ such that $\Omega_t = 4kd$ and $\Omega_i = 4kd + 1$ for every $i \in \{1, 2, \ldots, q\} \setminus \{s, t\}$. By Lemma 4.18, we conclude that $(\varphi(t), \varphi(t + 1), \varphi(t + 2)) \in \{(4, 1, 1), (4, 4, 1)\}$.

Suppose $(\varphi(t), \varphi(t + 1), \varphi(t + 2)) = (4, 1, 1)$. By Lemma 4.17 (a) and (c), we have that $\delta_t = 2kd + 2$ and $\delta_{t+1} = 2kd - 2$. By Lemma 4.17 (a) and (b), we have that $\Omega_{t+1} = \delta_{t+1} + \delta_{t+2} \leq (2kd - 2) + (2kd - 1) = 4kd - 3$, a contradiction.

Suppose $(\varphi(t), \varphi(t + 1), \varphi(t + 2)) = (4, 4, 1)$. By Lemma 4.17 (a) and (c), we have that $\delta_{t+1} = 2kd + 2$ and $\delta_t = 2kd - 2$. By Lemma 4.17 (b) and (d), we have that $\Omega_{t-1} = \delta_{t-1} + \delta_t \leq (2kd - 1) + (2kd - 2) = 4kd - 3$, a contradiction. $\square$

From Lemmas 4.13, 4.16 and 4.19, we have the following

**Theorem 4.20.** For $k = 4$, $n \geq 2k + 1$ and $d \geq 2$,

$$\gamma_p^d(C(n; \{1, k\})) = \begin{cases} 2\left\lceil \frac{2n}{4kd+1} \right\rceil + 2, & \text{if } n \equiv 2kd, 4kd - 1, 4kd \pmod{4kd + 1} \\ 2\left\lceil \frac{2n}{4kd+1} \right\rceil, & \text{otherwise.} \end{cases}$$

**References**


