Degree Conditions of Fractional ID-k-factor-critical graphs*

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Abstract

We say that a simple graph G is fractional independent-set-deletable k-factor-critical, shortly, fractional ID-k-factor-critical, if $G - I$ has a fractional k-factor for every independent set I of G. Some sufficient conditions for a graph to be fractional ID-k-factor-critical are studied in this paper. Furthermore, we show that the result is best possible in some sense.

Keywords: fractional k-factor, independent set, fractional ID-k-factor-critical

1 Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The minimum degree of G is denoted by $\delta(G)$. For any vertex x of G, the neighborhood of x is denoted by $N_G(x)$, the degree of x is denoted by $d_G(x)$, and we write $N_G[x]$ for $N_G(x) \bigcup \{x\}$. We use $G[S]$ and $G - S$ to denote the subgraph of G induced by S and $V(G) - S$, respectively, for $S \subseteq V(G)$. The join $G \vee H$ of disjoint graphs G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H. Notations and definitions not given in this paper can be found in [1].

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A subset I of $V(G)$ is said to be *independent* if no two distinct vertices in I are adjacent. A *matching* in a graph is a set of edges no two of which are adjacent. A matching is *perfect* if it covers all vertices of the graph. A graph G is *factor-critical* [5] if $G-v$ has a perfect matching for every vertex $v \in V(G)$. In [7], the concept of factor-critical graph was generalized to the ID-factor-critical graph. We say that G is *independent-set-deletable factor-critical* (shortly, ID-factor-critical) if for every independent set I of G which has the same parity with $|V(G)|$, $G-I$ has a perfect matching.

Let $h: E(G) \longrightarrow [0,1]$ be a function, and let $k \geq 1$ be an integer. If $\sum_{e \ni x} h(e) = k$ holds for each vertex $x \in V(G)$, we call $G[F_h]$ a *factional k-factor* of G with indicator function h where $F_h = \{e \in E(G) | \ h(e) > 0\}$. A fractional 1-factor is also called a fractional perfect matching [6]. We say that G is *fractional ID-k-factor-critical* if for every independent set I of G, $G-I$ has a fractional k-factor. When $k=1$, we say that G is *fractional ID-factor-critical* if for every independent set I of G, $G-I$ has a fractional perfect matching.

Liu and Zhang gave a necessary and sufficient condition for a graph to have fractional (g,f)-factor and a k-factor in [4] and [8], respectively.

Lemma 1.1 *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$, $\Phi_G(S;k) = k|S| - k|T| + d_{G-S}(T) \geq 0$, where $T = \{x : x \in V(G) - S, d_{G-S}(x) \leq k-1\}$.*

Lemma 1.2 *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$, $k|S| - \sum_{i=0}^{k-1}(k-i)p_i(G-S) \geq 0$, where $p_i(G-S) = |\{x : x \in V(G) - S, d_{G-S}(x) = i\}|$.*

The degree condition of ID-factor-critical graphs was studied in [3].

Lemma 1.3 *Let G be a graph with n vertices.* ***Then G is ID-factor-critical if*** $\delta(G) \geq \frac{2n-1}{3}$ **.**

In this paper, we discuss the degree conditions of fractional ID-k-factor-critical graphs. The main results will be given in the next section.

2 Main results

We begin our discussion with a well-known theorem of Dirac [2].

Lemma 2.1 *Let G be a graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{2}$. Then G is hamiltonian.*

The next result follows easily from Lemma 2.1.

Lemma 2.2 *If G is a graph of order n and $\delta(G) \geq \frac{2n}{3}$, then G is fractional ID-k-factor-critical when $k = 1,\ 2$.*

Proof. Let I be an independent set of G. It is easy to see that $n - |I| \geq \delta(G)$. Hence

$$\begin{aligned} 2\delta(G) - |I| - n &= 2\delta(G) + n - |I| - 2n \\ &\geq 3\delta(G) - 2n \geq 0. \end{aligned}$$

It follows that $\delta(G) - |I| \geq \frac{n-|I|}{2}$.

Let $H = G - I$. Then $|V(H)| = n - |I|$, and $\delta(H) \geq \delta(G) - |I| \geq \frac{|V(H)|}{2}$. By Lemma 2.1, H has a hamiltonian cycle C. C is also a fractional 2-factor and C also contains a fractional perfect matching. Thus Lemma 2.2 holds. □

Theorem 2.3 *Let k be a positive integer and G be a graph of order n with $n \geq 6k - 8$. If $\delta(G) \geq \frac{2n}{3}$, then G is fractional ID-k-factor-critical.*

Proof. Let X be an independent set of G and $H = G - X$. We have that $|V(H)| = n - |X|$ and $\delta(H) \geq \frac{|V(H)|}{2}$ by the same argument of lemma 2.2. Clearly, Theorem 2.3 holds when $k = 1$ or $k = 2$. Therefore, we may assume $k \geq 3$.

We prove the theorem by contradiction. Suppose H has no fractional k-factor. Then by Lemma 1.1, there exists some subset $S \subseteq V(H)$ such that $\Phi_H(S;k) = k|S| - k|T| + d_{H-S}(T) \leq -1$, where $T = \{x|\ x \in V(H) - S, d_{H-S}(x) \leq k-1\}$. Set $\Psi_H(S;k) = \Phi_H(S;k) + 1$. It follows that $\Psi_H(S;k) \leq 0$.

Let $h_1 = min\{d_{H-S}(x)|\ x \in T\}$. Choose $x_1 \in T$ such that $d_{H-S}(x_1) = h_1$. If $T - N_T[x_1] \neq \emptyset$, let $h_2 = min\{d_{H-S}(x)|\ x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$.

Set $|S| = s,\ |T| = t$, and $|\ N_T[x_1]\ | = p$. We have $p \le h_1 + 1$, $d_{H-S}(T) \ge h_1 p + h_2(t-p)$, and

$$\begin{aligned} 0 \ge \Psi_H(S;k) &= ks - kt + d_{H-S}(T) + 1 \\ &\ge ks - kt + h_1 p + h_2(t-p) + 1. \end{aligned}$$

Set $|V(H)| = m$. Then $m = n - |X| \ge \delta(G) \ge \frac{2n}{3} \ge \frac{12k-16}{3} = 4k - \frac{16}{3}$. Since m is an integer, we have that $m \ge 4k - 5$.

We consider the following cases.

Case 1. $T = N_T[x_1]$.

In this case, we have $t = p \le h_1 + 1,\ 0 \le h_1 \le k-1,\ h_2 = 0$. By $\delta(H) \ge \frac{n-|X|}{2} \ge \frac{n}{3} \ge \frac{6k-8}{3} \ge k$ $(k \ge 3)$ and $d_H(x_1) \le s + h_1$, we have $s \ge k - h_1$ and

$$\begin{aligned} \Psi_H(S;k) &\ge ks - kt + h_1 p + h_2(t-p) + 1 \\ &= ks + (h_1 - k)t + 1 \\ &\ge k(k-h_1) + (h_1 - k)t + 1 \\ &= (k-h_1)(k-t) + 1 \ \ge\ 1. \end{aligned}$$

Then we get a contradiction.

Case 2. $T - N_T[x_1] \neq \emptyset$.

Subcase 2.1. $0 \le h_1 \le 2$.

In this case, we have $t > p$, $0 \le h_1 \le h_2$, $\frac{m}{2} \le d_H(x_1) \le s + h_1$. Then $s \ge \frac{m}{2} - h_1 \ge \frac{4k-5}{2} - h_1 = 2k - \frac{5}{2} - h_1$. Since s is an integer and $m - s - t \ge 0$, we have $s \ge 2k - 2 - h_1$, $t \le m - s \le s + 2h_1$. Then we obtain that

$$\begin{aligned} 0 \ \ge\ \Psi_H(S;k) &\ge ks - kt + h_1 p + h_2(t-p) + 1 \\ &\ge ks - kt + h_1 t + 1 \\ &= ks + (h_1 - k)t + 1 \\ &\ge ks + (h_1 - k)(s + 2h_1) + 1 \\ &= h_1 s + 2h_1^2 - 2h_1 k + 1 \\ &\ge h_1(2k - 2 - h_1) + 2h_1^2 - 2h_1 k + 1 \\ &= h_1^2 - 2h_1 + 1 = (h_1 - 1)^2. \end{aligned}$$

When $h_1 = 0$ or $h_1 = 2$ (since $0 \le h_1 \le 2$ and h_1 is an integer), we have

$$0 \ \ge \ \Psi_H(S;k) \ \ \ge \ \ 1,$$

a contradiction.

When $h_1 = 1$, we have $\Psi_H(S;k) \ge 0$ and we notice that $\Psi_H(S;k) = 0$ holds if and only if $s = 2k-2-h_1 = 2k-3$ and $t = s+2h_1 = 2k-1$. Then $m \le 2s+2h_1 = 4k-4$ and $m \ge s+t = 4k-4$, so $m = 4k-4 = s+t$. Therefore $H = G[S \cup T]$ and $\mid N_T[x_1] \mid = p = h_1+1 = 2$, $\mid N_T(x_1) \mid = 1$.

So for every vertex $v \in T$, $\mid N_T(v) \mid \ge \mid N_T(x_1) \mid \ge 1$. And $t = 2k-1$ is odd, it follows that there exists a vertex $u \in T$ such that $\mid N_T(u) \mid \ge 2$.

$$\begin{aligned} 0 \ \ge \ \Psi_H(S;k) \ &= \ ks - kt + d_{H-S}(T) + 1 \\ &\ge \ ks - kt + (t-1) + 2 + 1 \\ &= \ k(2k-3) - k(2k-1) + (2k-1-1) + 3 \\ &= \ 1, \end{aligned}$$

a contradiction, too.

Subcase 2.2. $h_1 \ge 3$.

In this case, $3 \le h_1 \le h_2 \le k-1$. Then $k-h_2 \ge 1$ and $m-s-t \ge 0$. Thus $(k-h_2)(m-s-t) \ge 0$. So

$$\begin{aligned} (k-h_2)(m-s-t) \ &\ge \ \Psi_H(S;k) \\ &\ge \ ks - kt + h_1 p + h_2(t-p) + 1 \\ &= \ ks + (h_1-k)p + (h_2-k)(t-p) + 1. \end{aligned}$$

It follows that

$$(k-h_2)(m-s) - ks \ge (h_1-h_2)(h_1+1) + 1. \qquad (1)$$

Since $m \ge 4k-5$, we have

$$h_2 m \ge h_2(4k-5). \qquad (2)$$

Furthermore, since $\frac{m}{2} \leq d_H(x_1) \leq s + h_1$ and $\frac{m}{2} \leq d_H(x_2) \leq s + h_2$, we have $2s - m \geq -(h_1 + h_2)$. Then we can obtain that

$$(2s - m)(2k - h_2) \geq -(h_1 + h_2)(2k - h_2). \tag{3}$$

By $(2) + (3) + 2 \times (1)$, we get

$$\begin{aligned} 0 &\geq h_2(4k - 5) - (h_1 + h_2)(2k - h_2) + 2(h_1 - h_2)(h_1 + 1) + 2 \\ &= 2(h_2 - h_1)k + h_2(h_2 + h_1 - 5) + 2(h_1 - h_2)(h_1 + 1) + 2. \end{aligned}$$

Set $\Omega(k) = 2(h_2 - h_1)k + h_2(h_2 + h_1 - 5) + 2(h_1 - h_2)(h_1 + 1) + 2$. Then we obtain

$$0 \geq \Omega(k). \tag{4}$$

Since $k \geq h_2+1$ and $\Omega(k)$ is a nondecreasing function for k $(h_2 \geq h_1)$, then we obtain that $\Omega(k) \geq \Omega(h_2+1) = 3h_2^2 - (3h_1+5)h_2 + 2h_1^2 + 2$. Set $\Delta = (3h_1+5)^2 - 12(2h_1^2+2) = -15(h_1 - 1)^2 + 16$. And $\Delta < 0$ when $h_1 \geq 3$. It follows that $\Omega(h_2+1) > 0$ and $\Omega(k) \geq \Omega(h_2+1) > 0$, which contradicts (4).

The above arguments yield that H has a fractional k-factor and G is fractional ID-k-factor-critical. The proof is completed. □

In [8] we have the following result about fractional k-factors.

Theorem 2.4 *If G has fractional k-factors, then G has fractional m-factor for $1 \leq m \leq k$.*

Theorem 2.4 implies immediately the following result.

Theorem 2.5 *If a graph G is fractional ID-k-factor-critical, then G is fractional ID-m-factor-critical for $1 \leq m \leq k$.*

3 The sharpness of the bounds in Theorem 2.3.

In this section we show that the conditions in Theorem 2.3 are best possible.

Let $G = (2k-4)K_1 \vee (2k-3)K_1 \vee (k-1)K_2$. Then we have $n = |V(G)| = 6k - 9$ and $\delta(G) = 4k - 6 \geq \frac{2n}{3}$. Clearly, $A = (2k-3)K_1$ is an independent set of G. Let

$H = G - A = (2k-4)K_1 \vee (k-1)K_2$. Choose $S = (2k-4)K_1$. Then $\sum_{i=0}^{k-1}(k-i)p_i(H-S) = (k-1)(2k-2) = k(2k-4)+2 > k(2k-4) = k|S|$. Therefore, by Lemma 1.2, H has no fractional k-factor. Hence G is not fractional ID-k-factor-critical. In this sense, the bound of n is best possible.

This bound of $\delta(G)$ is sharp indeed. To see this, we construct a graph G with $\delta(G) = \lceil \frac{2n}{3} \rceil - 1$ which is not fractional ID-k-factor-critical as follows.

Case 1. $n = 3m$.

In this case, let $G = (m-1)K_1 \vee mK_1 \vee (m+1)K_1$, $n = |V(G)| = 3m$ and $\delta(G) = 2m - 1 = \lceil \frac{2n}{3} \rceil - 1$.

Clearly, $A = (m-1)K_1$ is an independent set of G. Let $H = G - A = mK_1 \vee (m+1)K_1$. Choose $S = mK_1$. Then $\sum_{i=0}^{k-1}(k-i)p_i(H-S) = k(m+1) > km = k|S|$. By Lemma 1.2, H has no fractional k-factor. So G is not fractional ID-k-factor-critical.

Case 2. $n = 3m+1$.

In this case, let $G = mK_1 \vee mK_1 \vee (m+1)K_1$, $n = |V(G)| = 3m+1$ and $\delta(G) = 2m = \lceil \frac{2n}{3} \rceil - 1$. Clearly, $A = mK_1$ is an independent set of G. Let $H = G - A = mk_1 \vee (m+1)K_1$. By the same argument as above, H has no fractional k-factor. Thus G is not fractional ID-k-factor-critical.

Case 3. $n = 3m+2$.

In this case, let $G = (m+1)K_1 \vee mK_1 \vee (m+1)K_1$, $n = |V(G)| = 3m+2$ and $\delta(G) = 2m+1 = \lceil \frac{2n}{3} \rceil - 1$. Clearly, $A = (m+1)K_1$ is an independent set of G. We obtain that G is not fractional ID-k-factor-critical by the same argument as above.

When $k = 1$, let G be a graph and let I be an arbitrary independent set of G. If I has the same parity with $|V(G)|$, we have known that if $\delta(G) \geq \frac{2n-1}{3}$, then G is ID-factor-critical, that is, $G - I$ has a perfect matching [3]. Obviously, $G - I$ has a fractional perfect matching. If I does not have the same parity with $|V(G)|$, we have known that if $\delta(G) \geq \frac{2n}{3}$, G is fractional ID-factor-critical, that is, $G - I$ has a fractional perfect matching. Hence the bound of $\delta(G)$ is sharp by the above argument.

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