Degree Conditions of Fractional ID-\(k\)-factor-critical graphs

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Abstract

We say that a simple graph \(G\) is fractional independent-set-deletable \(k\)-factor-critical, shortly, fractional ID-\(k\)-factor-critical, if \(G - I\) has a fractional \(k\)-factor for every independent set \(I\) of \(G\). Some sufficient conditions for a graph to be fractional ID-\(k\)-factor-critical are studied in this paper. Furthermore, we show that the result is best possible in some sense.

Keywords: fractional \(k\)-factor, independent set, fractional ID-\(k\)-factor-critical

1 Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). The minimum degree of \(G\) is denoted by \(\delta(G)\). For any vertex \(x\) of \(G\), the neighborhood of \(x\) is denoted by \(N_{G}(x)\), the degree of \(x\) is denoted by \(d_{G}(x)\), and we write \(N_{G}[x]\) for \(N_{G}(x) \cup \{x\}\). We use \(G[S]\) and \(G - S\) to denote the subgraph of \(G\) induced by \(S\) and \(V(G) - S\), respectively, for \(S \subseteq V(G)\). The join \(G \vee H\) of disjoint graphs \(G\) and \(H\) is the graph obtained from \(G + H\) by joining each vertex of \(G\) to each vertex of \(H\). Notations and definitions not given in this paper can be found in [1].

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A subset $I$ of $V(G)$ is said to be independent if no two distinct vertices in $I$ are adjacent. A matching in a graph is a set of edges no two of which are adjacent. A matching is perfect if it covers all vertices of the graph. A graph $G$ is factor-critical [5] if $G - v$ has a perfect matching for every vertex $v \in V(G)$. In [7], the concept of factor-critical graph was generalized to the ID-factor-critical graph. We say that $G$ is independent-set-deletable factor-critical (shortly, ID-factor-critical) if for every independent set $I$ of $G$ which has the same parity with $|V(G)|$, $G - I$ has a perfect matching.

Let $h : E(G) \rightarrow [0,1]$ be a function, and let $k \geq 1$ be an integer. If $\sum_{e \ni x} h(e) = k$ holds for each vertex $x \in V(G)$, we call $G[F_h]$ a fractional $k$-factor of $G$ with indicator function $h$ where $F_h = \{ e \in E(G) | h(e) > 0 \}$. A fractional 1-factor is also called a fractional perfect matching [6]. We say that $G$ is fractional ID-$k$-factor-critical if for every independent set $I$ of $G$, $G - I$ has a fractional $k$-factor. When $k = 1$, we say that $G$ is fractional ID-factor-critical if for every independent set $I$ of $G$, $G - I$ has a fractional perfect matching.

Liu and Zhang gave a necessary and sufficient condition for a graph to have fractional $(g,f)$-factor and a $k$-factor in [4] and [8], respectively.

**Lemma 1.1** Let $G$ be a graph. Then $G$ has a fractional $k$-factor if and only if for every subset $S$ of $V(G)$, $\Phi_G(S;k) = k|S| - k|T| + d_{G-S}(T) \geq 0$, where $T = \{ x : x \in V(G) - S, d_{G-S}(x) \leq k - 1 \}$.

**Lemma 1.2** Let $G$ be a graph. Then $G$ has a fractional $k$-factor if and only if for every subset $S$ of $V(G)$, $k|S| - \sum_{i=0}^{k-1} (k-i)p_i(G-S) \geq 0$, where $p_i(G-S) = |\{ x : x \in V(G) - S, d_{G-S}(x) = i \}|$.

The degree condition of ID-factor-critical graphs was studied in [3].

**Lemma 1.3** Let $G$ be a graph with $n$ vertices. Then $G$ is ID-factor-critical if $\delta(G) \geq \frac{2n-1}{3}$.

In this paper, we discuss the degree conditions of fractional ID-$k$-factor-critical graphs. The main results will be given in the next section.
2 Main results

We begin our discussion with a well-known theorem of Dirac [2].

Lemma 2.1 Let $G$ be a graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{2}$. Then $G$ is hamiltonian.

The next result follows easily from Lemma 2.1.

Lemma 2.2 If $G$ is a graph of order $n$ and $\delta(G) \geq \frac{2n}{3}$, then $G$ is fractional ID-$k$-factor-critical when $k = 1, 2$.

Proof. Let $I$ be an independent set of $G$. It is easy to see that $n - |I| \geq \delta(G)$. Hence

$$2\delta(G) - |I| - n = 2\delta(G) + n - |I| - 2n \geq 3\delta(G) - 2n \geq 0.$$ 

It follows that $\delta(G) - |I| \geq \frac{n - |I|}{2}$.

Let $H = G - I$. Then $|V(H)| = n - |I|$, and $\delta(H) \geq \delta(G) - |I| \geq \frac{|V(H)|}{2}$. By Lemma 2.1, $H$ has a hamiltonian cycle $C$. $C$ is also a fractional 2-factor and $C$ also contains a fractional perfect matching. Thus Lemma 2.2 holds. □

Theorem 2.3 Let $k$ be a positive integer and $G$ be a graph of order $n$ with $n \geq 6k - 8$. If $\delta(G) \geq \frac{2n}{3}$, then $G$ is fractional ID-$k$-factor-critical.

Proof. Let $X$ be an independent set of $G$ and $H = G - X$. We have that $|V(H)| = n - |X|$ and $\delta(H) \geq \frac{|V(H)|}{2}$ by the same argument of lemma 2.2. Clearly, Theorem 2.3 holds when $k = 1$ or $k = 2$. Therefore, we may assume $k \geq 3$.

We prove the theorem by contradiction. Suppose $H$ has no fractional $k$-factor. Then by Lemma 1.1, there exists some subset $S \subseteq V(H)$ such that $\Phi_H(S; k) = k|S| - k|T| + d_{H-S}(T) \leq -1$, where $T = \{x \mid x \in V(H) - S, d_{H-S}(x) \leq k - 1\}$. Set $\Psi_H(S; k) = \Phi_H(S; k) + 1$. It follows that $\Psi_H(S; k) \leq 0$.

Let $h_1 = \min\{d_{H-S}(x) \mid x \in T\}$. Choose $x_1 \in T$ such that $d_{H-S}(x_1) = h_1$. If $T - N_T[x_1] \neq \emptyset$, let $h_2 = \min\{d_{H-S}(x) \mid x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$. 


Set $|S| = s$, $|T| = t$, and $|N_T[x_1]| = p$. We have $p \leq h_1 + 1$, $d_{H-S}(T) \geq h_1 p + h_2(t-p)$, and

$$0 \geq \Psi_H(S; k) = ks - kt + d_{H-S}(T) + 1 \geq ks - kt + h_1 p + h_2(t-p) + 1.$$ 

Set $|V(H)| = m$. Then $m = n - |X| \geq \delta(G) \geq \frac{2n}{3} \geq \frac{12k-16}{3} = 4k - \frac{16}{3}$. Since $m$ is an integer, we have that $m \geq 4k - 5$.

We consider the following cases.

**Case 1.** $T = N_T[x_1]$. 

In this case, we have $t = p \leq h_1 + 1$, $0 \leq h_1 \leq k - 1$, $h_2 = 0$. By $\delta(H) \geq \frac{n-|X|}{2} \geq \frac{n}{3} \geq \frac{6k-8}{3} \geq k$ ($k \geq 3$) and $d_H(x_1) \leq s + h_1$, we have $s \geq k - h_1$ and

$$\Psi_H(S; k) \geq ks - kt + h_1 p + h_2(t-p) + 1 \geq ks + (h_1 - k)t + 1 \geq k(k - h_1) + (h_1 - k)t + 1 = (k - h_1)(k - t) + 1 \geq 1.$$ 

Then we get a contradiction.

**Case 2.** $T - N_T[x_1] \neq \emptyset$.

**Subcase 2.1.** $0 \leq h_1 \leq 2$.

In this case, we have $t > p$, $0 \leq h_1 \leq h_2$, $\frac{m}{2} \leq d_H(x_1) \leq s + h_1$. Then $s \geq \frac{m}{2} - h_1 \geq \frac{4k-5}{2} - h_1 = 2k - \frac{5}{2} - h_1$. Since $s$ is an integer and $m - s - t \geq 0$, we have $s \geq 2k - 2 - h_1$, $t \leq m - s \leq s + 2h_1$. Then we obtain that

$$0 \geq \Psi_H(S; k) \geq ks - kt + h_1 p + h_2(t-p) + 1 \geq ks - kt + h_1 t + 1 = ks + (h_1 - k)t + 1 \geq ks + (h_1 - k)(s + 2h_1) + 1 = h_1 s + 2h_1^2 - 2h_1 k + 1 \geq h_1 (2k - 2 - h_1) + 2h_1^2 - 2h_1 k + 1 = h_1^2 - 2h_1 + 1 = (h_1 - 1)^2.$$
When \( h_1 = 0 \) or \( h_1 = 2 \) (since \( 0 \leq h_1 \leq 2 \) and \( h_1 \) is an integer), we have
\[
0 \geq \Psi_H(S;k) \geq 1,
\]
a contradiction.

When \( h_1 = 1 \), we have \( \Psi_H(S;k) \geq 0 \) and we notice that \( \Psi_H(S;k) = 0 \) holds if and only if \( s = 2k - 2 - h_1 = 2k - 3 \) and \( t = s + 2h_1 = 2k - 1 \). Then \( m \leq 2s + 2h_1 = 4k - 4 \) and \( m \geq s + t = 4k - 4 \), so \( m = 4k - 4 = s + t \). Therefore \( H = G[S \cup T] \) and \( |N_T(x_1)| = p = h_1 + 1 = 2, \ |N_T(x_1)| = 1. \)

So for every vertex \( v \in T, \ |N_T(v)| \geq |N_T(x_1)| \geq 1. \) And \( t = 2k - 1 \) is odd, it follows that there exists a vertex \( u \in T \) such that \( |N_T(u)| \geq 2. \)

\[
0 \geq \Psi_H(S;k) = ks - kt + d_{H-S}(T) + 1
\]
\[
\geq ks - kt + (t - 1) + 2 + 1
\]
\[
= k(2k - 3) - k(2k - 1) + (2k - 1 - 1) + 3
\]
\[
= 1,
\]
a contradiction, too.

**Subcase 2.2.** \( h_1 \geq 3. \)

In this case, \( 3 \leq h_1 \leq k - 1. \) Then \( k - h_2 \geq 1 \) and \( m - s - t \geq 0. \) Thus \( (k - h_2)(m - s - t) \geq 0. \) So
\[
(k - h_2)(m - s - t) \geq \Psi_H(S;k)
\]
\[
\geq ks - kt + h_1p + h_2(t - p) + 1
\]
\[
= ks + (h_1 - k)p + (h_2 - k)(t - p) + 1.
\]
It follows that
\[
(k - h_2)(m - s) - ks \geq (h_1 - h_2)(h_1 + 1) + 1. \quad (1)
\]

Since \( m \geq 4k - 5 \), we have
\[
h_2m \geq h_2(4k - 5). \quad (2)
\]
Furthermore, since \( \frac{m}{2} \leq d_H(x_1) \leq s + h_1 \) and \( \frac{m}{2} \leq d_H(x_2) \leq s + h_2 \), we have \( 2s - m \geq -(h_1 + h_2) \). Then we can obtain that

\[
(2s - m)(2k - h_2) \geq -(h_1 + h_2)(2k - h_2).
\]

By \((2) + (3) + 2 \times (1)\), we get

\[
0 \geq h_2(4k - 5) - (h_1 + h_2)(2k - h_2) + 2(h_1 - h_2)(h_1 + 1) + 2
= 2(h_2 - h_1)k + h_2(h_2 + h_1 - 5) + 2(h_1 - h_2)(h_1 + 1) + 2.
\]

Set \( \Omega(k) = 2(h_2 - h_1)k + h_2(h_2 + h_1 - 5) + 2(h_1 - h_2)(h_1 + 1) + 2 \). Then we obtain

\[
0 \geq \Omega(k).
\]

Since \( k \geq h_2 + 1 \) and \( \Omega(k) \) is a nondecreasing function for \( k \geq h_1 \), then we obtain that \( \Omega(k) \geq \Omega(h_2 + 1) = 3h_2^2 - (3h_1 + 5)h_2 + 2h_1^2 + 2 \). Set \( \Delta = (3h_1 + 5)^2 - 12(2h_1^2 + 2) = -15(h_1 - 1)^2 + 16 \). And \( \Delta < 0 \) when \( h_1 \geq 3 \). It follows that \( \Omega(h_2 + 1) > 0 \) and \( \Omega(k) \geq \Omega(h_2 + 1) > 0 \), which contradicts \((4)\).

The above arguments yield that \( H \) has a fractional \( k \)-factor and \( G \) is fractional ID-\( k \)-factor-critical. The proof is completed.

In [8] we have the following result about fractional \( k \)-factors.

**Theorem 2.4** If \( G \) has fractional \( k \)-factors, then \( G \) has fractional \( m \)-factor for \( 1 \leq m \leq k \).

Theorem 2.4 implies immediately the following result.

**Theorem 2.5** If a graph \( G \) is fractional ID-\( k \)-factor-critical, then \( G \) is fractional ID-\( m \)-factor-critical for \( 1 \leq m \leq k \).

### 3 The sharpness of the bounds in Theorem 2.3.

In this section we show that the conditions in Theorem 2.3 are best possible.

Let \( G = (2k - 4)K_1 \lor (2k - 3)K_1 \lor (k - 1)K_2 \). Then we have \( n = |V(G)| = 6k - 9 \) and \( \delta(G) = 4k - 6 \geq \frac{2m}{n} \). Clearly, \( A = (2k - 3)K_1 \) is an independent set of \( G \). Let
\[ H = G - A = (2k-4)K_1 \vee (k-1)K_2, \text{ Choose } S = (2k-4)K_1. \text{ Then } \sum_{i=0}^{k-1} (k-i)p_i(H - S) = (k-1)(2k-2) = k(2k-4) + 2 > k(2k-4) = k|S|. \text{ Therefore, by Lemma 1.2, } H \text{ has no fractional } k\text{-factor. Hence } G \text{ is not fractional ID-}k\text{-factor-critical. In this sense, the bound of } n \text{ is best possible.} \]

This bound of \( \delta(G) \) is sharp indeed. To see this, we construct a graph \( G \) with \( \delta(G) = \lceil \frac{2n}{3} \rceil - 1 \) which is not fractional ID-\( k \)-factor-critical as follows.

**Case 1.** \( n = 3m \).

In this case, let \( G = (m-1)K_1 \vee mK_1 \vee (m+1)K_1, n = |V(G)| = 3m \) and \( \delta(G) = 2m - 1 = \lceil \frac{2n}{3} \rceil - 1. \)

Clearly, \( A = (m-1)K_1 \) is an independent set of \( G \). Let \( H = G - A = mK_1 \vee (m+1)K_1. \) Choose \( S = mK_1. \text{ Then } \sum_{i=0}^{k-1} (k-i)p_i(H - S) = k(m+1) > km = k|S|. \text{ By Lemma 1.2, } H \text{ has no fractional } k\text{-factor. So } G \text{ is not fractional ID-}k\text{-factor-critical.} \)

**Case 2.** \( n = 3m + 1. \)

In this case, let \( G = mK_1 \vee mK_1 \vee (m+1)K_1, n = |V(G)| = 3m + 1 \) and \( \delta(G) = 2m = \lceil \frac{2n}{3} \rceil - 1. \text{ Clearly, } A = mK_1 \) is an independent set of \( G \). Let \( H = G - A = mk_1 \vee (m+1)K_1. \) By the same argument as above, \( H \) has no fractional \( k\text{-factor}. \text{ Thus } G \text{ is not fractional ID-}k\text{-factor-critical.} \)

**Case 3.** \( n = 3m + 2. \)

In this case, let \( G = (m+1)K_1 \vee mK_1 \vee (m+1)K_1, n = |V(G)| = 3m + 2 \) and \( \delta(G) = 2m + 1 = \lceil \frac{2n}{3} \rceil - 1. \text{ Clearly, } A = (m+1)K_1 \) is an independent set of \( G \). We obtain that \( G \) is not fractional ID-\( k\text{-factor-critical} \) by the same argument as above.

When \( k = 1 \), let \( G \) be a graph and let \( I \) be an arbitrary independent set of \( G \). If \( I \) has the same parity with \( |V(G)| \), we have known that if \( \delta(G) \geq \frac{2m-1}{3} \), then \( G \) is ID-factor-critical, that is, \( G - I \) has a perfect matching \([3]\). Obviously, \( G - I \) has a fractional perfect matching. If \( I \) does not have the same parity with \( |V(G)| \), we have known that if \( \delta(G) \geq \frac{2n}{3} \), \( G \) is fractional ID-factor-critical, that is, \( G - I \) has a fractional perfect matching. Hence the bound of \( \delta(G) \) is sharp by the above argument.
References


