From the Eisenhart problem to Ricci solitons in $f$-Kenmotsu manifolds

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July 23, 2009
Dedicated to the memory of Neculai Papaghiuc 1947-2008

Abstract

The Eisenhart problem of finding parallel tensors is solved for the symmetric case in the regular $f$-Kenmotsu framework. In this way, the Olszack-Rosca example of Einstein manifolds provided by $f$-Kenmotsu manifolds via locally symmetric Ricci tensors is recovered as well as a case of Killing vector fields. Some other classes of Einstein-Kenmotsu manifolds are presented. Our result is interpreted in terms of Ricci solitons and special quadratic first integrals.

2000 Math. Subject Classification: 53C40; 53C55; 53C12; 53C42. 
Key words: $f$-Kenmotsu manifold; parallel second order covariant tensor field; irreducible metric; Einstein space; Ricci soliton.

Introduction

In 1923, Eisenhart [9] proved that if a positive definite Riemannian manifold $(M, g)$ admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1926, Levy [18] proved that a second order parallel symmetric non-degenerated tensor $\alpha$ in a space form is proportional to the metric tensor. Note that this question can be considered as the dual to the the problem of finding linear connections making parallel a given tensor field; a problem which was considered by Wong in [35]. Also, the former question implies topological restrictions namely if the (pseudo) Riemannian manifold $M$ admits a parallel symmetric $(0,2)$ tensor field then $M$ is locally the direct product of a number of (pseudo) Riemannian manifolds, [36] (cited by [37]). Another situation where the parallelism of $\alpha$ is involved appears in the theory of
totally geodesic maps, namely, as is point out in [22, p. 114], $\nabla \alpha = 0$ is equivalent with the fact that $1 : (M, g) \rightarrow (M, \alpha)$ is a totally geodesic map.

While both Eisenhart and Levy work locally, Ramesh Sharma gives in [26] a global approach based on Ricci identities. In addition to space-forms, Sharma considered this Eisenhart problem in contact geometry [27]-[29], for example for $K$-contact manifolds in [28]. Since then, several other studies appeared in various contact manifolds: nearly Sasakian [33], (para) $P$-Sasakian [32], [6] and [19], $\alpha$-Sasakian [5]. Another framework was that of quasi-constant curvature in [13]. Also, contact metrics with nonvanishing $\xi$-sectional curvature are studied in [10].

Returning to contact geometry, an important class of manifolds are introduced by Kenmotsu in [15] and generalized by Olszack and Rosca in [21]. Recently, there is an increasing flow of papers in this direction e.g. that of our professor N. Papaghiuc [23]-[24] to whom we dedicate this short note.

Motivated by this fact, we studied the case of $f$-Kenmotsu manifolds satisfying a special condition called by us regular and show that a symmetric parallel tensor field of second order must be a constant multiple of the Riemannian metric. There are three remarks regarding our result:

i) it is in agreement with what happens in all previously recalled contact geometries for the symmetric case,

ii) it is obtained in the same manner as in Sharma’s paper [26],

iii) yields a class of Einstein manifolds already indicated by Olszack and Rosca but with a more complicated proof.

Let us point out also that the anti-symmetric case appears without proof in [20].

Our main result is connected with the recent theory of Ricci solitons, a subject included in the Hamilton-Perelman approach (and proof) of Poincaré Conjecture. Ricci solitons in contact geometry were first studied by Ramesh Sharma in [11] and [30]; also the preprint [34] is available in arxiv. In these papers the $K$-contact and $(k, \mu)$-contact (including Sasakian) cases are treated; thus our treatment for the Kenmotsu variant of almost contact geometry seems to be new.

Our work is structured as follows. The first section is a very brief review of Kenmotsu geometry and Ricci solitons. The next section is devoted to the (symmetric case of) Eisenhart problem in a $f$-Kenmotsu manifold and several situations yielding Einstein manifolds are derived. Also, the relationship with the Ricci solitons is pointed out. The last section offers a dynamical picture of the subject via Killing vector fields and quadratic first integrals of a special type.
1 \( f \)-Kenmotsu manifolds. Ricci solitons

Let \( M \) be a real \( 2n + 1 \)-dimensional differentiable manifold endowed with an almost contact metric structure \((\varphi, \xi, \eta, g)\):

\[
\begin{align*}
(a) \quad \varphi^2 &= -I + \eta \otimes \xi, \\
(b) \quad \eta(\xi) &= 1, \\
(c) \quad \eta \circ \varphi &= 0, \\
(d) \quad \varphi(\xi) &= 0, \\
(e) \quad \eta(X) &= g(X, \xi), \\
(f) \quad g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y),
\end{align*}
\]

for any vector fields \( X, Y \in \mathcal{X}(M) \) where \( I \) is the identity of the tangent bundle \( TM \), \( \varphi \) is a tensor field of \((1,1)\)-type, \( \eta \) is a 1-form, \( \xi \) is a vector field and \( g \) is a metric tensor field. Throughout the paper all objects are differentiable of class \( C^\infty \).

We say that \((M, \varphi, \xi, \eta, g)\) is an \( f \)-Kenmotsu manifold if the Levi-Civita connection of \( g \) satisfy [20]:

\[
(\nabla_X \varphi)(Y) = f(g(\varphi X, Y)\xi - \varphi(X)\eta(Y)),
\]

where \( f \in C^\infty(M) \) is strictly positive and \( df \land \eta = 0 \) holds. A \( f = \text{constant} \equiv \beta > 0 \) is called \( \beta \)-Kenmotsu manifold with the particular case \( f \equiv 1 \)-Kenmotsu manifold which is a usual Kenmotsu manifold [15].

In a general \( f \)-Kenmotsu manifold we have, [21]:

\[
\nabla_X \xi = f(X - \eta(X)\xi),
\]

and the curvature tensor field:

\[
R(X, Y)\xi = f^2(\eta(X)Y - \eta(Y)X) + Y(f)\varphi^2X - X(f)\varphi^2Y
\]

while the Ricci curvature and Ricci tensor are, [16]:

\[
\begin{align*}
S(\xi, \xi) &= -2n(f^2 + \xi(f)) \\
Q(\xi) &= -2nf^2\xi - \xi(f)\xi - (2n - 1)gradf.
\end{align*}
\]

In the last part of this section we recall the notion of Ricci solitons according to [30, p. 139]. On the manifold \( M \), a Ricci soliton is a triple \((g, V, \lambda)\) with \( g \) a Riemannian metric, \( V \) a vector field and \( \lambda \) a real scalar such that:

\[
\mathcal{L}_V g + 2S + 2\lambda g = 0.
\]

The Ricci soliton is said to be shrinking, steady or expanding according as \( \lambda \) is negative, zero or positive.
2 Parallel symmetric second order tensors and Ricci solitons in $f$-Kenmotsu manifolds

Fix $\alpha$ a symmetric tensor field of (0, 2)-type which we suppose to be parallel with respect to $\nabla$ i.e. $\nabla \alpha = 0$. Applying the Ricci identity

$$\nabla^2 \alpha(X, Y; Z, W) - \nabla^2(X, Y; W, Z) = 0$$

we obtain the relation (1.1) of [26, p. 787]:

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0,$$

which is fundamental in all papers treating this subject. Replacing $Z = W = \xi$ and using (1.4) results in:

$$f^2[\eta(X)\alpha(Y, \xi) - \eta(Y)\alpha(X, \xi)] + Y(f)\alpha(\varphi^2X, \xi) - X(f)\alpha(\varphi^2Y, \xi) = 0,$$

by the symmetry of $\alpha$. With $X = \xi$ we derive:

$$[f^2 + \xi(f)][\alpha(Y, \xi) - \eta(Y)\alpha(\xi, \xi)] = 0$$

and supposing $f^2 + \xi(f) \neq 0$ it results:

$$\alpha(Y, \xi) = \eta(Y)\alpha(\xi, \xi).$$

Let us call a regular $f$-Kenmotsu manifold a $f$-Kenmotsu manifold with $f^2 + \xi(f) \neq 0$ and remark that $\beta$-Kenmotsu manifolds are regular.

Differentiating the last equation covariantly with respect to $X$ we have:

$$\alpha(\nabla_X Y, \xi) + f[\alpha(X, Y) - \eta(X)\eta(Y)\alpha(\xi, \xi)] = X(\eta(Y))\alpha(\xi, \xi),$$

which means via (2.3) with $Y \rightarrow \nabla_X Y$:

$$f[\alpha(X, Y) - \eta(X)\eta(Y)\alpha(\xi, \xi)] = [X(g(Y, \xi)) - g(\nabla_X Y, \xi)]\alpha(\xi, \xi) = g(Y, \nabla_X \xi)\alpha(\xi, \xi) = f[g(X, Y) - \eta(X)\eta(Y)]\alpha(\xi, \xi).$$

From the positiveness of $f$ we deduce that:

$$\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y)$$

which together with the standard fact that the parallelism of $\alpha$ implies the $\alpha(\xi, \xi)$ is a constant, via (2.3) yields:

**Theorem** A symmetric parallel second order covariant tensor in a regular $f$-Kenmotsu manifold is a constant multiple of the metric tensor. In other words, a regular $f$-Kenmotsu metric is irreducible which means that
the tangent bundle does not admit a decomposition $TM = E_1 \oplus E_2$ parallel with respect of the Levi-Civita connection of $g$.

**Corollary 1** A locally Ricci symmetric ($\nabla S \equiv 0$) regular $f$-Kenmotsu manifold is an Einstein manifold.

**Remarks**
1) The particular case of dimension three and $\beta$-Kenmotsu of our theorem appears in Theorem 3.1 from [7, p. 2689]. The above corollary has been proved by Olszack and Rosca in another way.
2) In [2] it is shown the equivalence of the following statements for an Kenmotsu manifold:
   i) is Einstein,
   ii) is locally Ricci symmetric,
   iii) is Ricci semi-symmetric i.e. $R \cdot S = 0$ where:
   $$(R(X, Y) \cdot S)(X_1, X_2) = -S(R(X, Y)X_1, X_2) - S(X_1, R(X, Y)X_2).$$

The same implication iii) $\rightarrow$ i) for Kenmotsu manifolds is Theorem 1 from [14, p. 438]. But we have the implication iii) $\rightarrow$ i) in the more general framework of regular $f$-Kenmotsu manifolds since $R \cdot S = 0$ means exactly (2.1) with $\alpha$ replaced by $S$. Every semisymmetric manifold, i. e. $R \cdot R = 0$, is Ricci-semisymmetric but the converse statement is not true. In conclusion:

**Proposition 1** A Ricci-semisymmetric, particularly semisymmetric, regular $f$-Kenmotsu manifold is Einstein.

Another class of spaces related to the Ricci tensor was introduced in [31]; namely a Riemannian manifold is a *special weakly Ricci symmetric space* if there exists a 1-form $\rho$ such that:

$$(\nabla_X S)(Y, Z) = 2\rho(X)S(Y, Z) + \rho(Y)S(Z, X) + \rho(Z)S(X, Y). \quad (2.7)$$

The same condition was sometimes called *generalized pseudo-Ricci symmetric manifold* ([12]) or simply *pseudo-Ricci symmetric manifold* ([4]). By taking $X = Y = Z = \xi$ yields:

$$\xi(S(\xi, \xi)) = 4\rho(\xi)S(\xi, \xi) \quad (2.8)$$

and then for a $\beta$-Kenmotsu manifold we get $\rho(\xi) = 0$. Returning to (2.7) with $Y = Z = \xi$ will result in $\rho(X) = 0$ for every vector field $X$ and thus lead to a generalization of Theorem 3.3. in [1, p. 96]:

**Proposition 2** A $\beta$-Kenmotsu manifold which is special weakly Ricci symmetric is an Einstein space.
We close this section with applications of our Theorem to Ricci solitons:

**Corollary 2** Suppose that on a regular $f$-Kenmotsu manifold the $(0,2)$-type field $\mathcal{L}_V g + 2S$ is parallel where $V$ is a given vector field. Then $(g, V)$ yield a Ricci soliton. In particular, if the given regular $f$-Kenmotsu manifold is Ricci-semisymmetric or semisymmetric with $\mathcal{L}_V g$ parallel, we have the same conclusion.

Naturally, two situations appear regarding the vector field $V$: $V \in \text{span}\xi$ and $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$.

We are interested in expressions for $\mathcal{L}_\xi g + 2S$. A straightforward computation gives:

$$\mathcal{L}_\xi g(X, Y) = 2f(g(X, Y) - \eta(X)\eta(Y)) = 2f g(\varphi X, \varphi Y). \quad (2.9)$$

A general expression of $S$ is known by us only for the the 3-dimensional case and $\eta$-Einstein Kenmotsu manifolds. Let us treat these situations in the following manner:

I) [8, p. 251]:

$$S(X, Y) = \left(\frac{r}{2} + \xi(f) + f^2\right) g(X, Y) -$$

$$- \left(\frac{r}{2} + \xi(f) + 3f^2\right) \eta(X)\eta(Y) - Y(f)\eta(X) - X(f)\eta(Y) \quad (2.10)$$

where $r$ is the scalar curvature. Then, for a 3-dimensional $f$-Kenmotsu manifold we obtain:

$$\alpha := (\mathcal{L}_\xi g + 2S)(X, Y) = (r + 2\xi(f) + 2f + 2f^2)g(X, Y) -$$

$$- (r + 2\xi(f) + 2f + 6f^2)\eta(X)\eta(Y) - 2Y(f)\eta(X) - 2X(f)\eta(Y) \quad (2.11)$$

while, for $\beta$-Kenmotsu:

$$\alpha(X, Y) = (r + 2\beta + 2\beta^2)g(\varphi X, \varphi Y) - 4\beta^2\eta(X)\eta(Y), \quad (2.12)$$

$$(\nabla_Z \alpha)(X, Y) = Z(r)g(\varphi X, \varphi Y) -$$

$$- \beta(r + 2\beta + 6\beta^2)[\eta(X)g(\varphi Y, \varphi Z) + \eta(Y)g(\varphi X, \varphi Z)]. \quad (2.13)$$

Substituting $Z = \xi$, $X = Y \in (\text{span}\xi)^\perp$, and respectively $X = Y = Z \in (\text{span}\xi)^\perp$ in (2.13), we derive that $r$ is a constant, provided $\alpha$ is parallel. Thus, we can state the following:
**Proposition 3** A 3-dimensional $\beta$-Kenmotsu Ricci soliton $(g, \xi, \lambda)$ is expanding and with constant scalar curvature.

**Proof** $\lambda = -\frac{1}{2}\alpha(\xi, \xi) = 2\beta^2$. □

At this point we remark that the Ricci solitons of almost contact geometry studied in [30] and [34] in relationship with the Sasakian case are shrinking and this observation is in accordance with the diagram of Chan that Sasakian and Kenmotsu are opposite sides of the trans-Sasakian moon. Also, the expanding character may be considered as a manifestation of the fact that a $\beta$-Kenmotsu manifold can not be compact.

II) Recall that the metric $g$ is called $\eta$-Einstein if there exists two real functions $a, b$ such that the Ricci tensor of $g$ is:

$$S = ag + b\eta \otimes \eta.$$ 

For an $\eta$-Einstein Kenmotsu manifold we have, [14, p. 441]:

$$S(X, Y) = \left(\frac{r}{2n} + 1\right) g(X, Y) - \left(\frac{r}{2n} + 2n + 1\right) \eta(X)\eta(Y) \quad (2.14)$$

and then:

$$\alpha(X, Y) = \left(\frac{r}{n} + 4\right) g(X, Y) - \left(\frac{r}{n} + 4 + 4n\right) \eta(X)\eta(Y) \quad (2.15)$$

$$(\nabla_Z \alpha)(X, Y) = \frac{1}{n} Z(r) g(\varphi X, \varphi Y) -$$

$$- \left(\frac{r}{n} + 4n + 4\right) [\eta(Y) g(\varphi X, \varphi Z) + \eta(X) g(\varphi Y, \varphi Z)]. \quad (2.16)$$

**Proposition 4** An $\eta$-Einstein Kenmotsu Ricci soliton $(g, \xi, \lambda)$ is expanding and with constant scalar curvature, thus Einstein.

**Proof** $\lambda = -\frac{1}{2}\alpha(\xi, \xi) = 2n$. The same computation as in Proposition 3 implies constant scalar curvature. □

### 3 The dynamical point of view

We begin this section with a straightforward consequence of the main Theorem, which also appears in the Olzack-Rosca paper, and is related to the last part of Corollary 2:

**Corollary 3** An affine Killing vector field in a $\beta$-Kenmotsu manifold is Killing. As consequence, that scalar provided by the Ricci soliton $(g, V)$ of a Ricci-semisymmetric $\beta$-Kenmotsu manifold is $\lambda = -S(V, V)$.  

7
**Proof** (inspired by [10, p. 504]) Fix \( X \in \mathcal{X}(M) \) an affine Killing vector field: \( \nabla \mathcal{L}_X g = 0 \). From Theorem it follows that \( X \) is *conformal Killing* i.e. \( \mathcal{L}_X g = cg \); more precisely \( X \) is *homothetic* since \( c \) is a constant. Lie differentiating the identity (1.5) along \( X \) and using \( \mathcal{L}_X S = 0 \) (since \( X \) is homothetic) and equation (1.6) we obtain \( g(\mathcal{L}_X \xi, \xi) = 0 \). Hence \( c = (\mathcal{L}_X g)(\xi, \xi) = -2g(\mathcal{L}_X \xi, \xi) = 0 \). Thus \( X \) is Killing. \( \square \)

Let us present another dynamical picture of our results. Let \((M, \nabla)\) be a \( m \)-dimensional manifold endowed with a symmetric linear connection. A *quadratic first integral* (QFI on short) for the geodesics of \( \nabla \) is defined by \( F = a_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \) with a symmetric 2-tensor field \( a = (a_{ij}) \) satisfying the *Killing-type equations*:

\[
a_{ij:k} + a_{jk:i} + a_{ki:j} = 0, \tag{3.1}
\]

where, as usual, the double dot means the covariant derivative with respect to \( \nabla \).

The QFI defined by \( a \) is called *special* (SQFI) if \( a_{ij:k} = 0 \) and the maximum number of linearly independent SQFI a pair \((M, \nabla)\) can admit is \( \frac{m(m+1)}{2} \); a flat space will admit this number. In [17, p. 117] it is shown that a non-flat Riemannian manifold may admit as many as \( M_S(m) = 1 + \frac{(m-2)(m-1)}{2} \) linearly independent SQFI. Therefore, for an almost contact manifold \((m = 2n + 1)\) the maximum number of SQFI is \( M_S(2n+1) = 1 + n(2n-1) > 1 \).

Our main result implies that for a regular \( f \)-Kenmotsu manifold the number of SQFI is exactly 1 and the only SQFI is the *kinetic energy* \( F = g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \). So:

**Proposition 5** There exist almost contact manifolds which does not admit \( M_S(2n+1) \) SQFI.

It remains as an open problem to find examples of almost contact metrics with exactly \( M_S(2n+1) \) SQFI.

**Acknowledgement** Special thanks are offered to Gheorghe Pitis for some useful remarks as well as sending us his book [25], a source of several references. Also, we are very indebted to Marian-Ioan Munteanu, Rosihan M. Ali Dato and the referees who pointed out major improvements.
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