Ordered semigroups characterized by their $(\in, \in \lor q)$-fuzzy bi-ideals

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Abstract

Using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set, the concept of an $(\alpha, \beta)$-fuzzy bi-ideal in ordered semigroups is introduced, which is a generalization of the concept of a fuzzy bi-ideal in ordered semigroups and some interesting characterizations theorems are obtained. A special attention is given to $(\in, \in \lor q)$-fuzzy bi-ideals.

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1 Introduction

The fundamental concept of a fuzzy set, introduced by L. A. Zadeh, provides a natural frame-work for generalizing several basic notions of algebra. The study of fuzzy sets in semigroups was introduced by Kuroki [19 – 21]. A systematic

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exposition of fuzzy semigroups was given by Mordeson et al., where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik deals with the application of fuzzy approach to the concepts of automata and formal languages. Murali [25] proposed the definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [2, 3] gave the concepts of \((\alpha, \beta)\)-fuzzy subgroups by using the "belongs to" relation \((\in)\) and "quasi-coincident with" relation \((q)\) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an \((\in, \in \vee q)\)-fuzzy subgroup. In [4], \((\in, \in \vee q)\)-fuzzy subrings and ideals are defined. In [5] Davvaz define \((\in, \in \vee q)\)-fuzzy subnearring and ideals of a near ring. In [1] Bhakat define \((\in \vee q)\)-level subset of a fuzzy set. In [6] Jun and Song initiated the study of \((\alpha, \beta)\)-fuzzy interior ideals of a semigroup. In [7] Kazanci and Yamak study \((\in, \in \vee q)\)-fuzzy bi-ideals of a semigroup. The concept of a fuzzy generalized filters was introduced by Ma et al. in [22] and characterized \(R_0\)-algebras in terms of this notion.

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in broader framework of fuzzy setting.

Our aim in this paper is to introduce and study the new sort of fuzzy bi-ideals called \((\alpha, \beta)\)-fuzzy bi-ideals and to study some interesting characterizations of ordered semigroups in terms of \((\alpha, \beta)\)-fuzzy bi-ideals. Special concentration is given to \((\in, \in \vee q)\)-fuzzy bi-ideals and some characterizations of regular and intra-regular ordered semigroups are obtained by using \((\in, \in \vee q)\)-fuzzy bi-ideals.

2 Preliminaries

An ordered semigroup is an ordered set \(S\) at the same time a semigroup such that \(a, b \in S, a \leq b \Rightarrow xa \leq xb\) and \(ax \leq bx\) for all \(x \in S\).

Let \((S, \leq)\) be an ordered semigroup. For \(A \subseteq S\), we denote

\[
(A) := \{t \in S | t \leq h \text{ for some } h \in A\}.
\]

For \(A, B \subseteq S\), we denote, \(AB := \{ab | a \in A, b \in B\}\). Let \(A, B \subseteq S\). Then \(A \subseteq (A), (A)(B) \subseteq (AB)\), and \(((A)) = (A)\).

Let \(S\) be an ordered semigroup and \(0 \neq A \subseteq S\). Then \(A\) is called a subsemigroup of \(S\) if \(A^2 \subseteq A\). A subsemigroup \(A\) of an ordered semigroup \(S\) is called a bi-ideal of \(S\) if (1) \(ASA \subseteq A\) and (2) \((\forall x \in S)(\forall y \in A) (x \leq y \Rightarrow x \in A)\).[9]

**Definition 1** \((\text{cf.}[13])\) An element \(z\) of an ordered semigroup \(S\) is called left (resp. right) zero if for all \(x \in S\), \(zx = x\) (resp. \(xz = z\)). An ordered semigroup \(S\) is called left (resp. right) zero if every element of \(S\) is left (resp. right) zero.
An ordered semigroup $S$ is called regular if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$.

Equivalent definitions:[15, 16]

1. $(\forall a \in S)(a \in (aSa])$.
2. $(\forall A \subseteq S)(A \subseteq (ASA])$.

An ordered semigroup $S$ is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$.

Equivalent definitions:[16]

1. $(\forall a \in S)(a \in (SaS])$.
2. $(\forall A \subseteq S)(A \subseteq (SA^2S])$.

Let $S$ be an ordered semigroup by a fuzzy subset $A$ of $S$, we mean a function $A : S \rightarrow [0, 1]$.

Let $A$ be a fuzzy subset of $S$, then $A$ is called a fuzzy subsemigroup [9] of $S$ if

$$(\forall x, y \in S)(A(xy) \geq \min\{A(x), A(y)\}).$$

**Definition 2** (cf. [9]) Let $S$ be an ordered semigroup and $A$ a fuzzy subset of $S$. Then $A$ is called a fuzzy bi-ideal of $S$ if:

1. $(B_1) (\forall x, y \in S)(x \leq y \Rightarrow A(x) \geq A(y))$.
2. $(B_2) (\forall x, y \in S)(A(xy) \geq \min\{A(x), A(y)\})$.
3. $(B_3) (\forall x, y, z \in S)(A(xyz) \geq \{A(x), A(z)\})$.

Let $S$ be an ordered semigroup and $A$ a fuzzy subset of $S$, then for all $t \in (0, 1]$, the set

$$U(A; t) := \{x \in S | A(x) \geq t\}$$

is called a level subset of $S$.

**Theorem 3** Let $(S, \leq)$ be an ordered semigroup and $A$ a fuzzy subset of $S$. Then $A$ is a fuzzy bi-ideal of $S$ if and only if the level subset $U(A; t)(\neq \emptyset)$ is a bi-ideal of $S$ for all $t \in (0, 1]$.

Let $S$ be an ordered semigroup and $\emptyset \neq B \subseteq S$. Then the characteristic function $\chi_B$ of $B$ is defined as follows:

$$\chi_B : S \rightarrow [0, 1] | x \rightarrow \chi_B(x) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

**Lemma 4** (cf. [9, Theorem 1]). Let $S$ be an ordered semigroup and $B$ a non-empty subset of $S$. Then $B$ is a bi-ideal of $S$ if and only if $\chi_B$ is a fuzzy bi-ideal of $S$. 

3
For $a \in S$, define
\[ A_a := \{(y, z) \in S \times S | a \leq yz\}. \]

For fuzzy subsets $A_1$ and $A_2$ of $S$, define
\[ A_1 \circ A_2 : S \rightarrow [0, 1] | a \rightarrow \begin{cases} \min \{A_1(y), A_2(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset \end{cases} \]

We denote by $F(S)$ the set of all fuzzy subsets of $S$. One can easily see that the multiplication “$\circ$” on $F(S)$ is well defined and associative. The order relation “$\subseteq$” on $F(S)$ is defined as follows:

$A_1 \subseteq A_2$ if and only if $A_1(x) \leq A_2(x)$ for all $x \in S$.

Clearly $(F(S), \circ, \subseteq)$ is an ordered semigroup.

For a nonempty family of fuzzy subsets $\{A_i\}_{i \in I}$, of an ordered semigroup $S$, the fuzzy subsets $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ of $S$ are defined as follows:

$\bigcup_{i \in I} A_i : S \rightarrow [0, 1] | a \rightarrow \left( \bigcup_{i \in I} A_i \right)(a) := \sup_{i \in I} \{A_i(a)\}$ and

$\bigcap_{i \in I} A_i : S \rightarrow [0, 1] | a \rightarrow \left( \bigcap_{i \in I} A_i \right)(a) := \inf_{i \in I} \{A_i(a)\}$.

If $I$ is a finite set, say $I = \{1, 2, ..., n\}$, then clearly
\[ \bigcup_{i \in I} A_i(a) = \max \{A_1(a), A_2(a), ..., A_n(a)\} \] and
\[ \bigcap_{i \in I} A_i(a) = \min \{A_1(a), A_2(a), ..., A_n(a)\}. \]

**Proposition 5** (cf. [9, Proposition 5]). Let $(S, \cdot, \leq)$ be an ordered semigroup and $A, B \subseteq S$. Then

(i) $A \leq B$ if and only if $\chi_A \leq \chi_B$.

(ii) $\chi_A \cap \chi_B = \chi_{A \cap B}$.

(iii) $\chi_A \circ \chi_B = \chi_{A \cap B}$.

### 3 (α, β)-fuzzy bi-ideals

In what follows let $S$ denote an ordered semigroup and $\alpha, \beta$ denote any one of $\in, \in \cup \cup, \in \cap \cap$ unless otherwise specified.

Every fuzzy bi-ideal of $S$ is an $(\alpha, \beta)$-fuzzy bi-ideal of $S$ as shown in the following Theorem
Theorem 6  For any fuzzy subset $A$ of $S$, the conditions $(B_1)$, $(B_2)$ and $(B_3)$ are equivalent to the conditions $(B_4)$, $(B_5)$ and $(B_6)$, where $(B_4)$, $(B_5)$ and $(B_6)$ are given as follows:

$(B_1)$ $(\forall x, y \in S) (\forall t \in (0, 1])(x \leq y, y_t \in A \Rightarrow x_t \in A)$.

$(B_2)$ $(\forall x, y \in S)(t, r \in (0, 1))(x_t, y_r \in A \Rightarrow (xy)_{\min\{t, r\}} \in A)$.

$(B_3)$ $(\forall x, y, z \in S)(t, r \in (0, 1))(x_t, y_r, z_r \in A \Rightarrow (xyz)_{\min\{t, r\}} \in A)$.

Proof. $(B_1) \implies (B_2)$. Let $x, y \in S$, and $t \in (0, 1]$ be such that $x \leq y$, $y_t \in A$. Then $A(y) \geq t$. Since $x \leq y$, we have $A(x) \geq t$ by $(B_1)$. Hence $x_t \in A$.

$(B_1) \implies (B_2)$. Assume $(B_1)$ is not valid. Then there exist $x, y \in S$ such that $x \leq y$ and $A(x) < A(y)$. Hence $A(x) < t \leq A(y)$ for some $t \in (0, 1]$ and so $y_t \in A$ but $x_t \notin A$, a contradiction. Hence $(B_1)$ is valid.

$(B_2) \implies (B_3)$. Let $x, y \in S$ and $t, r \in (0, 1]$ be such that $x_t, y_r \in A$. Then $A(x) \geq t$ and $A(y) \geq r$. By $(B_2)$, we have $A(xy) \geq \min\{A(x), A(y)\} \geq \min\{t, r\}$, it follows that $(xy)_{\min\{t, r\}} \in A$.

$(B_3) \implies (B_2)$. Let $x, y \in S$. Since $A_A(x) \in A$ and $A_A(y) \in A$. By $(B_3)$ we have $(xy)_{\min\{A_A(x), A_A(y)\}} \in A$, it follows that $A(xy) \geq \min\{A(x), A(y)\}$.

$(B_3) \implies (B_2)$. Let $x, y, z \in S$ and $t, r \in (0, 1]$ be such that $x_t, z_r \in A$. Then $A(x) \geq t$ and $A(z) \geq r$. By $(B_3)$, we have $A(xyz) \geq \min\{A(x), A(z)\} \geq \min\{t, r\}$, it follows that $(xyz)_{\min\{t, r\}} \in A$.

$(B_6) \implies (B_3)$. Let $x, y, z \in S$. Since $A_A(x) \in A$ and $A_A(z) \in A$. By $(B_3)$ we have $(xyz)_{\min\{A_A(x), A_A(z)\}} \in A$, it follows that $A(xyz) \geq \min\{A(x), A(z)\}$. ■

Let $A$ be a fuzzy subset of $S$ such that $A(x) \leq 0.5$ for all $x \in S$. Let $x \in S$ and $t \in (0, 1]$ be such that $x_t \in \land A$. Then $A(x) \geq t$ and $A(x) + t > 1$. It follows that

$$1 < A(x) + t \leq A(x) + A(x) = 2A(x),$$

so that $A(x) > 0.5$. This means that \{x_t | x_t \in \land A\} = \emptyset.$

Definition 7  A fuzzy subset $A$ of $S$ is called an $(\alpha, \beta)$-fuzzy bi-ideal of $S$, where $\alpha \neq \land A$, if it satisfies:

$(B_7)$ $(\forall x, y \in S)(\forall t \in (0, 1])(x \leq y, y_t \alpha A \Rightarrow x_t \beta A)$.

$(B_8)$ $(\forall x, y \in S)(\forall t, r \in (0, 1])(x_t, y_r \alpha A \Rightarrow (xy)_{\min\{t, r\}} \beta A)$.

$(B_9)$ $(\forall x, y, z \in S)(\forall t, r \in (0, 1])(x_t, z_r \alpha A \Rightarrow (xyz)_{\min\{t, r\}} \beta A)$.

Example 8  Consider a set $S = \{a, b, c, d, e\}$ with the following multiplication ".$" and order relation "$\leq$":

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\[ \leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\} \]

Then \((S, \leq)\) is an ordered semigroup (see [17]) and \(\{a\}, \{a, b, e\}\) and \(\{a, b, d, e\}\) are bi-ideals of \(S\). We define a fuzzy subset \(A : S \rightarrow [0, 1]\) by:

\[
A(a) = 0.8, \quad A(b) = 0.7, \quad A(e) = 0.6, \quad A(d) = 0.5, \quad A(c) = 0.3.
\]

Then

\[
U(A; t) := \begin{cases} 
S & \text{if } t \in (0, 0.3], \\
\{a, b, d, e\} & \text{if } t \in (0.3, 0.5], \\
\{a, b, e\} & \text{if } t \in (0.5, 0.6], \\
\{a\} & \text{if } t \in (0.6, 0.8], \\
\emptyset & \text{if } t \in (0.8, 1]
\end{cases}
\]

Clearly \(A\) is an \((\xi, \in \vee q)\)-fuzzy bi-ideal of \(S\). But

(i) \(A\) is not an \((\xi, \in)\)-fuzzy bi-ideal of \(S\), since \(a_{0.78} \in A\) and \(b_{0.66} \in A\) but

\[
(ab)_{\min\{0.78, 0.76\}} = d_{0.76} \overline{\xi} A.
\]

(ii) \(A\) is not a \((q, \xi)\)-fuzzy bi-ideal of \(S\), since \(a_{0.75}qA\) and \(b_{0.65}qA\) but

\[
(ab)_{\min\{0.75, 0.65\}} = d_{0.65} \overline{\xi} A.
\]

(iii) \(A\) is not an \((\xi, q)\)-fuzzy bi-ideal of \(S\), since \(a_{0.30} \in A\) and \(b_{0.20} \in A\) but

\[
(ab)_{\min\{0.30, 0.20\}} = d_{0.20} \overline{\xi} A.
\]

(iv) \(A\) is not an \((q, \xi)\)-fuzzy bi-ideal of \(S\) since \(a_{0.65}qA\) and \(b_{0.55}qA\) but

\[
(ab)_{\min\{0.65, 0.55\}} = d_{0.55} \overline{\xi} qA.
\]

(v) \(A\) is not a \((q, \xi \wedge q)\)-fuzzy bi-ideal of \(S\), since \(a_{0.72}qA\) and \(b_{0.62}qA\) but

\[
(ab)_{\min\{0.72, 0.62\}} = d_{0.62} \overline{\xi \wedge q} A.
\]

(vi) \(A\) is not an \((\xi \vee q, \xi)\)-fuzzy bi-ideal of \(S\), since \(a_{0.64} \in \vee q A\) and \(b_{0.54} \in \vee q A\) but

\[
(ab)_{\min\{0.64, 0.54\}} = d_{0.54} \overline{\xi} A \text{ and so } d_{0.54} \overline{\xi \wedge q} A.
\]

(vii) \(A\) is not an \((\xi, \in \vee q)\)-fuzzy bi-ideal of \(S\), since \(a_{0.63} \in \vee q A\) and \(b_{0.53} \in \vee q A\) but

\[
(ab)_{\min\{0.63, 0.53\}} = d_{0.53} \overline{\xi} A.
\]

(viii) \(A\) is not an \((\xi, \in \wedge q)\)-fuzzy bi-ideal of \(S\), since \(a_{0.62} \in A\) and \(b_{0.52} \in A\) but

\[
(ab)_{\min\{0.62, 0.52\}} = d_{0.52} \overline{\xi} A \text{ and so } d_{0.52} \overline{\xi \wedge q} A.
\]

(xi) \(A\) is not a \((q, q)\)-fuzzy bi-ideal of \(S\), since \(a_{0.38}qA\) and \(b_{0.48}qA\) but

\[
(ab)_{\min\{0.38, 0.48\}} = d_{0.38} \overline{\xi} A.
\]
(x) $A$ is not an $(\in \vee q, q)$-fuzzy bi-ideal of $S$, since $a_{0.39} \in \vee qA$ and $b_{0.49} \in \vee qA$ but
\[
(ab)_{\min\{0.39, 0.49\}} = d_{0.39}qA.
\]
(xi) $A$ is not an $(\in \vee q, \in \vee q)$-fuzzy bi-ideal of $S$, $a_{0.68} \in \vee qA$ and $b_{0.58} \in \vee qA$ but
\[
(ab)_{\min\{0.68, 0.58\}} = d_{0.58}qA.
\]

**Theorem 9** Every $(\in, \in)$-fuzzy bi-ideal is an $(\in, \in \vee q)$-fuzzy bi-ideal.

**Proof.** Straightforward. ■

**Theorem 10** Every $(\in \vee q, \in \vee q)$-fuzzy bi-ideal is $(\in, \in \vee q)$-fuzzy bi-ideal.

**Proof.** Let $A$ be an $(\in \vee q, \in \vee q)$-fuzzy bi-ideal of $S$. Let $x, y \in S$, $x \leq y$ and $t \in (0, 1]$ be such that $y_t \in A$. Then $y_t \in \vee qA$. Since $x \leq y$ and $y_t \in \vee qA$ we have $x_t \in \vee qA$. Let $x, y \in S$ and $t, r \in (0, 1]$ be such that $x_t, y_r \in A$. Then $x_t, y_r \in \vee qA$, which implies $(xy)_{\min\{t, r\}} \in \vee qA$. Let now, $x, y, z \in S$ and $t, r \in (0, 1]$ be such that $x_t, z_r \in A$. Then $x_t, z_r \in \vee qA$, which implies $(xyz)_{\min\{t, r\}} \in \vee qA$. ■

**Theorem 11** Let $A$ be a non-zero $(\alpha, \beta)$-fuzzy bi-ideal of $S$. Then the set $A_0 := \{x \in S \mid A(x) > 0\}$ is a bi-ideal of $S$.

**Proof.** Let $x, y \in S$, $x \leq y$. If $y \in A_0$, then $A(y) > 0$. Since $x \leq y$ we have $A(x) \geq A(y)$, then $A(x) > 0$ and so $x \in A_0$. Let $x, y \in A_0$. Then $A(x) > 0$ and $A(y) > 0$. Assume that $A(xy) = 0$. If $\alpha \in \{\in, \in \vee q\}$ then $x_A(x)A$ and $y_A(y)$ $A$ but $(xy)_{\min\{A(x), A(y)\}}A$ for every $\beta \in \{\in, \in \vee q, \in \vee q\}$, a contradiction. Note that $x_AqA$ and $y_AqA$ but $(xy)_{\min\{1, 1\}} = (xy)_{1\vee q}A$ for every $\beta \in \{\in, \in \vee q, \in \vee q\}$, a contradiction. Hence $A(xy) > 0$, that is $xy \in A_0$. Now, let $x, z \in A_0$ and $y \in S$. Then $A(x) > 0$ and $A(z) > 0$. Assume that $A(xy) = 0$. If $\alpha \in \{\in, \in \vee q\}$ then $x_A(x)A$ and $z_A(z)A$ but $(xy)_{\min\{A(x), A(z)\}}A$ for every $\beta \in \{\in, \in \vee q, \in \vee q\}$, a contradiction. Note that $x_AqA$ and $z_AqA$ but $(xy)_{\min\{1, 1\}} = (xy)_{1\vee q}A$ for every $\beta \in \{\in, \in \vee q, \in \vee q\}$, a contradiction. Hence $A(xy) > 0$, that is $xyz \in A_0$. Consequently, $A_0$ is a bi-ideal of $S$. ■

**Theorem 12** Let $S$ be a left(resp. right) zero ordered semigroup and $A$ a non-zero $(q, q)$-fuzzy bi-ideal of $S$. Then $A$ is constant on $A_0$.

**Proof.** Let $a$ be an element of $S$ such that $A(a) = \vee \{A(x) \mid x \in S\}$. Then $a \in A_0$. Suppose that there exists $x \in A_0$ such that $t_x = A(x) \neq A(a) = t_a$. Then $t_x < t_a$. Choose $r, s \in (0, 1]$ such that $1 - t_a < r < 1 - t_x < s$. Then $a_rqA$ and $x_sqA$ but $(ax)_{\min\{r, s\}} = x_{r\vee q}A$ (resp. $(xa)_{\min\{r, s\}} = x_{r\vee q}A$, since $S$ is right(resp. left) zero, a contradiction. Hence $A(x) = A(z)$ for all $x \in A_0$. ■

**Theorem 13** Let $B$ be a bi-ideal and $A$ a fuzzy subset of $S$ such that
(1) \( \forall x \in S \setminus B \) \((A(x) = 0) \)

(2) \( \forall x \in B \) \((A(x) \geq 0.5) \).

Then

(a) \( A \) is a \((q, \in \land q)\)-fuzzy bi-ideal of \( S \).

(b) \( A \) is an \((\in, \in \lor q)\)-fuzzy bi-ideal of \( S \).

**Proof.** (a) Let \( x, y \in S \) \( x \leq y \) and \( t \in (0, 1] \) be such that \( y_t q A \). Then \( y \in B \) and so \( x \in B \). Thus if \( t \leq 0.5 \) then \( A(x) \geq 0.5 \geq t \). Hence \( x_t \in A \). If \( t > 0.5 \), then \( A(x) + t > 0.5 + 0.5 = 1 \) and so \( x_t q A \). It follows that \( x_t \in \lor q A \). Let \( x, y \in S \) and \( r, t \in (0, 1] \) be such that \( x_r q A \) and \( y_t q A \). Then \( x, y \in B \) and we have \( xy \in B \).

If \( \min \{r, t\} \) \( \leq \) \( 0.5 \), then \( A(xy) \geq 0.5 \geq \min \{r, t\} \) and hence \((xy)_{\min \{r, t\}} \in A \). If \( \min \{r, t\} \) \( > \) \( 0.5 \), then

\[
A(xy) + \min \{r, t\} > 0.5 + 0.5 = 1
\]

and so \((xy)_{\min \{r, t\}} q A \). Therefore \((xy)_{\min \{r, t\}} \in \lor q A \). Therefore \( A \) is a \((q, \in \lor q)\)-fuzzy bi-ideal of \( S \).

(b) Let \( x, y \in S \) \( x \leq y \) and \( t \in (0, 1] \) be such that \( y_t \in A \). Then \( A(y) \geq t \).

Thus \( y \in B \) and so \( x \in B \). If \( t \leq 0.5 \) then \( A(x) \geq 0.5 \geq t \). Hence \( x_t \in A \). If \( t > 0.5 \), then \( A(x) + t > 0.5 + 0.5 = 1 \) and so \( x_t q A \). It follows that \( x_r \in \lor q A \).

Let \( x, y \in S \) and \( r, t \in (0, 1] \) be such that \( x_r \in A \) and \( y_t \in A \). Then \( x, y \in B \) and we have \( xy \in B \).

If \( \min \{r, t\} \) \( \leq \) \( 0.5 \), then \( A(xy) \geq 0.5 \geq \min \{r, t\} \) and hence \((xy)_{\min \{r, t\}} \in A \). If \( \min \{r, t\} \) \( > \) \( 0.5 \), then

\[
A(xy) + \min \{r, t\} > 0.5 + 0.5 = 1
\]

and so \((xy)_{\min \{r, t\}} q A \). Therefore \((xy)_{\min \{r, t\}} \in \lor q F \). Now let \( x, y, z \in S \) and \( r, t \in (0, 1] \) be such that \( x_r \in A \) and \( z_t \in A \). Then \( x, z \in B \) and we have \( xyz \in B \).

If \( \min \{r, t\} \) \( \leq \) \( 0.5 \), then \( A(xy) \geq 0.5 \geq \min \{r, t\} \) and hence \((xyz)_{\min \{r, t\}} \in A \).

If \( \min \{r, t\} \) \( > \) \( 0.5 \), then

\[
A(xy) + \min \{r, t\} > 0.5 + 0.5 = 1
\]

and so \((xyz)_{\min \{r, t\}} q A \). Therefore \((xyz)_{\min \{r, t\}} \in \lor q F \).

From Example 8, we see that an \((\in, \in \lor q)\)-fuzzy bi-ideal is not a \((q, \in \lor q)\)-fuzzy bi-ideal (Example 8, Part iv).
4 \((\epsilon, \in \vee q)\)-fuzzy bi-ideals

It is well known that the ideal theory plays a fundamental role in the development of ordered semigroups. In [6], Jun et al. introduced the concept of a generalized fuzzy interior ideal of a semigroup. In this section we define the notions of \((\epsilon, \in \vee q)\)-fuzzy bi-ideals of an ordered semigroup and investigate some of their properties in terms of \((\epsilon, \in \vee q)\)-fuzzy bi-ideals.

**Lemma 14** If \(\alpha = \epsilon\) and \(\beta = \epsilon \vee q\) in definition 7. Then \((B_7), (B_8)\) and \((B_9)\) respectively, of definition 7, are equivalent to the following conditions:

\[
(B_{10}) \quad (\forall x, y \in S)(x \leq y \implies A(x) \geq \min\{A(y), 0.5\}).
\]

\[
(B_{11}) \quad (\forall x, y \in S)(A(xy) \geq \min\{A(x), A(y), 0.5\}).
\]

\[
(B_{12}) \quad (\forall x, y, z \in S)(A(xyz) \geq \min\{A(x), A(z), 0.5\}).
\]

**Remark 15** A fuzzy subset \(A\) of an ordered semigroup \(S\) is an \((\epsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\) if and only if it satisfies conditions \((B_{10}), (B_{11})\) and \((B_{12})\) of the above Lemma.

**Remark 16** By the above Remark every fuzzy bi-ideal of an ordered semigroup \(S\) is an \((\epsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\). However, the converse is not true, in general.

**Example 17** Consider the ordered semigroup given in Example 8, and define a fuzzy subset \(A : S \rightarrow [0, 1]\) by:

\[
A(a) = 0.8, \quad A(b) = 0.7, \quad A(c) = 0.6, \quad A(d) = 0.5, \quad A(e) = 0.3.
\]

Clearly \(A\) is an \((\epsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\). But \(A\) is not an \((\alpha, \beta)\)-fuzzy bi-ideal of \(S\) as shown in Example 8.

Using Lemma 13, we have the following characterization of fuzzy bi-ideals of ordered semigroups.

**Proposition 18** Let \((S, \leq)\) be an ordered semigroup and \(\emptyset \neq B \subseteq S\). Then \(B\) is a bi-ideal of \(S\) if and only if the characteristic function \(\chi_B\) of \(B\) is an \((\epsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\).

In the following Theorem we give a condition for an \((\epsilon, \in \vee q)\)-fuzzy bi-ideal to be an \((\epsilon, \in)\)-fuzzy bi-ideal of \(S\).

**Theorem 19** Let \(A\) be an \((\epsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\) such that \(A(x) < 0.5\) for all \(x \in S\). Then \(A\) is an \((\epsilon, \in)\)-fuzzy bi-ideal of \(S\).

**Proof.** Let \(x, y \in S, x \leq y\) and \(t \in (0, 1]\) be such that \(y \epsilon A\). Then \(A(y) \geq t\). By hypothesis,

\[
A(x) \geq \min\{A(y), 0.5\} \geq \min\{t, 0.5\} = t,
\]
so \( x \in A_t \). Let \( x, y \in S \) and \( t, r \in (0, 1] \) be such that \( x_t, y_r \in A \). Then \( A(x) \geq t \) and \( A(y) \geq r \) and so \( A(xy) \geq \min\{A(x), A(y), 0.5\} \geq \min\{t, r, 0.5\} = \min\{t, r\} \) hence \( (xyz)_{\min\{t, r\}} \in A \). Now, let \( x, y, z \in S \) and \( t, r \in (0, 1] \) be such that \( x_t, z_r \in A \). Then \( A(x) \geq t \) and \( A(z) \geq r \) and we have
\[
A(xyz) \geq \min\{A(x), A(z), 0.5\} \geq \min\{t, r, 0.5\},
\]
consequently, \((xyz)_{\min\{t, r\}} \in A\). Therefore \( A \) is an \((\varepsilon, \varepsilon)\)-fuzzy bi-ideal of \( S \). ■

For any fuzzy subset \( A \) of an ordered semigroup \( S \) and \( t \in (0, 1] \), we denote
\[
Q(A; t) := \{x \in S| x_tqA\} \text{ and } [A]_t := \{x \in S| x_t \in \forall qA\}.
\]
Obviously, \([A]_t = U(A; t) \cup Q(A; t)\).

We call \([A]_t\) an \((\varepsilon, \forall q)\)-level bi-ideal of \( A \) and \( Q(A; t) \) a \( q \)-level bi-ideal of \( A \).

We gave a characterization of \((\varepsilon, \varepsilon)\)-fuzzy bi-ideals by using level subsets (see [8]). Now we provide another characterization of \((\varepsilon, \forall q)\)-fuzzy bi-ideals by using the set \([A]_t\).

**Theorem 20** Let \( S \) be an ordered semigroup and \( A \) a fuzzy subset of \( S \). Then \( A \) is an \((\varepsilon, \forall q)\)-fuzzy bi-ideal of \( S \) if and only if \([A]_t\) is a bi-ideal of \( S \) for all \( t \in (0, 1] \).

**Proof.** \( \Rightarrow \). Let \( A \) be an \((\varepsilon, \forall q)\)-fuzzy bi-ideal of \( S \). Let \( x, y \in S, x \leq y \) and \( t \in (0, 1] \) be such that \( y \in [A]_t \). Then \( y_t \in \forall qA \), that is, \( A(y) \geq t \) or \( A(y) + t > 1 \). Since \( A \) is an \((\varepsilon, \forall q)\)-fuzzy bi-ideal of \( S \) and \( x \leq y \), we have \( A(x) \geq \min\{A(y), 0.5\} \). We have the following cases:

Case 1 \( A(y) \geq t \). If \( t > 0.5 \), then \( A(x) \geq \min\{A(y), 0.5\} = 0.5 \) and so
\[
A(x) + t > 0.5 + 0.5 = 1.
\]
Hence \( x_tqA \). If \( t \leq 0.5 \), then \( A(x) \geq \min\{A(y), 0.5\} \geq t \), and hence \( x_t \in A \).

Case 2 \( A(y) + t > 1 \). If \( t > 0.5 \), then
\[
A(x) \geq \min\{A(y), 0.5\} \geq \min\{1 - t, 0.5\} = 1 - t,
\]
that is, \( A(x) + t > 1 \) and thus \( x_tqA \). If \( t \leq 0.5 \), then
\[
A(x) \geq \min\{A(y), 0.5\} \geq \min\{1 - t, 0.5\} = 0.5 \geq t
\]
and so \( x_t \in A \). Thus in both cases, we have \( x_t \in \forall qA \) and so \( x \in [A]_t \). Let \( x, y \in [A]_t \) for \( t \in (0, 1] \). Then \( x_t \in \forall qA \) and \( y_t \in \forall qA \), that is, \( A(x) \geq t \) or \( A(x) + t > 1 \), and \( A(y) \geq t \) or \( A(y) + t > 1 \). Since \( A \) is an \((\varepsilon, \forall q)\)-fuzzy bi-ideal of \( S \), we have,
\[
A(xy) \geq \min\{A(x), A(y), 0.5\}.
\]

Case 1 Let \( A(x) \geq t \) and \( A(y) \geq t \). If \( t > 0.5 \). Then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\} = 0.5
\]

10
and hence \((xy)\alpha A\). If \(t \leq 0.5\). Then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\} \geq t
\]
and so \((xy)\alpha A\). Hence \((xy)\alpha \in \sqrt{qA}\).

Case 2 Let \(A(x) + t > 1\) and \(A(y) + t > 1\). If \(t > 0.5\), then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\}.
\]
If \(t > 0.5\). Then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\} = \min\{t, 1 - t, 0.5\} = t
\]
i.e., \(A(xy) + t > 1\) and thus \((xy)\alpha A\). If \(t \leq 0.5\). Then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\} \geq \min\{t, 1 - t, 0.5\} = t
\]
and so \((xy)\alpha A\). Hence \((xy)\alpha \in \sqrt{qA}\).

Case 3 Let \(A(x) + t > 1\) and \(A(y) \geq t\). If \(t < 0.5\), then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\} \geq \min\{A(x), 0.5\} \geq \min\{1 - t, 0.5\} = 1 - t,
\]
i.e., \(A(xy) + t > 1\) and hence \((xy)\alpha A\). If \(t < 0.5\). Then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\} = \min\{1 - t, t, 0.5\} = t
\]
and so \((xy)\alpha A\). Hence \((xy)\alpha \in \sqrt{qA}\).

Case 4 Let \(A(x) + t > 1\) and \(A(y) + t > 1\). If \(t > 0.5\), then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\} > \min\{1 - t, 0.5\} = 1 - t,
\]
i.e., \(A(xy) + t > 1\) and thus \((xy)\alpha A\). If \(t \leq 0.5\). Then
\[
A(xy) \geq \min\{A(x), A(y), 0.5\} \geq \min\{1 - t, 0.5\} = 0.5 \geq t,
\]
and so \((xy)\alpha A\). Thus in any case, we have \((xy)\alpha \in \sqrt{qA}\). Therefore \(xy \in [A]_{\alpha}\).

Now, let \(x, z \in [A]_{\alpha}\) for \(t \in (0, 1]\) and \(y \in S\). Then \(x_{t} \in \sqrt{qA}\) and \(z_{t} \in \sqrt{qA}\),
that is, \(A(x) \geq t\) or \(A(x) + t > 1\), and \(A(z) \geq t\) or \(A(z) + t > 1\). Since \(A\) is an \((\varepsilon, \in \sqrt{q})\)-fuzzy bi-ideal of \(S\), we have,
\[
A(xyzt) \geq \min\{A(x), A(z), 0.5\}.
\]

Case 1 Let \(A(x) \geq t\) and \(A(z) \geq t\). If \(t > 0.5\). Then
\[
A(xyzt) \geq \min\{A(x), A(z), 0.5\} = 0.5
\]
and hence \((xyz)_t \in qA\). If \(t \leq 0.5\). Then

\[ A(xyz) \geq \min\{A(x), A(z), 0.5\} \geq t \]

and so \((xyz)_t \in A\). Hence \((xyz)_t \in qA\).

Case 2 Let \(A(x) \geq t\) and \(A(z) + t > 1\). If \(t > 0.5\). Then

\[ A(xyz) \geq \min\{A(x), A(z), 0.5\}. \]

If \(t > 0.5\). Then

\[ A(xyz) \geq \min\{A(x), A(z), 0.5\} \]
\[ = \min\{A(z), 0.5\} \]
\[ > \min\{1 - t, 0.5\} = 1 - t, \]

i.e., \(A(xyz) + t > 1\) and thus \((xyz)_t \in qA\). If \(t \leq 0.5\). Then

\[ A(xyz) \geq \min\{A(x), A(z), 0.5\} \]
\[ \geq \min\{t, 1 - t, 0.5\} = t \]

and so \((xyz)_t \in A\). Hence \((xyz)_t \in qA\).

Case 3 Let \(A(x) + t > 1\) and \(A(z) \geq t\). If \(t < 0.5\). Then

\[ A(xyz) \geq \min\{A(x), A(z), 0.5\} \]
\[ \geq \min\{A(x), 0.5\} \]
\[ \geq \min\{1 - t, 0.5\} = 1 - t, \]

i.e., \(A(xyz) + t > 1\) and hence \((xyz)_t \in qA\).

If \(t < 0.5\). Then

\[ A(xyz) \geq \min\{A(x), A(z), 0.5\} \]
\[ \geq \min\{1 - t, 0.5\} = t \]

and so \((xyz)_t \in A\). Hence \((xyz)_t \in qA\).

Case 4 Let \(A(x) + t > 1\) and \(A(z) + t > 1\). If \(t > 0.5\). Then

\[ A(xyz) \geq \min\{A(x), A(z), 0.5\} \]
\[ \geq \min\{A(x), 0.5\} \]
\[ \geq \min\{1 - t, 0.5\} = 1 - t, \]

i.e., \(A(xyz) + t > 1\) and thus \((xyz)_t \in qA\).

If \(t \leq 0.5\). Then

\[ A(xyz) \geq \min\{A(x), A(z), 0.5\} \geq \min\{1 - t, 0.5\} = 0.5 \geq t, \]

and so \((xyz)_t \in A\). Thus in any case, we have \((xyz)_t \in qA\). Therefore \(xyz \in [A]_t\).

Conversely, let \(A\) be a fuzzy subset of \(S\) and \(t \in (0, 1]\) be such that \([A]_t\) is a
Let $x, y \in S$ be such that $A(xy) < t < \min\{A(x), A(y), 0.5\}$ for some $t \in (0, 0.5]$. Then $x, y \in U(A; t) \subseteq [A]_t$, since $x \leq y \in [A]_t$ and $[A]_t$ is a bi-ideal of $S$ we have $x \in [A]_t$. Hence $A(x) \geq t$ or $A(x) + t > 1$, a contradiction. Hence

$$A(x) \geq \min\{A(y), 0.5\} \text{ for all } x \leq y.$$ 

Let $x, y \in S$ be such that $A(xy) < t < \min\{A(x), A(y), 0.5\}$ for some $t \in (0, 0.5]$. Then $x, y \in U(A; t) \subseteq [A]_t$, it implies that $xy \in [A]_t$. Hence $A(xy) \geq t$ or $A(xy) + t > 1$, a contradiction. Hence $A(xy) \geq \min\{A(x), A(y), 0.5\}$ for all $x, y \in S$. Choose $t$ such that $A(xy) < t < \min\{A(x), 0.5\}$. Then $a \in U(A; t) \subseteq [A]_t$. It follows that $xy \in [A]_t$. This implies that $A(xy) \geq t$ or $A(xy) + t > 1$, a contradiction. Hence $A(xy) \geq \min\{A(x), 0.5\}$ for all $a, x, y \in S$. By Theorem 6, it follows that $A$ is an $(\varepsilon, \in \cup q)$-fuzzy bi-ideal of $S$.

$U(A; t)$ and $[A]_t$ are bi-ideals of $S$ for all $t \in (0, 1]$, but $Q(A; t)$ is not a bi-ideal of $S$ for all $t \in (0, 1]$, in general. As shown in the following Example.

**Example 21** Consider the ordered semigroup as given in Example 8. Define a fuzzy subset $A$ by

$$A(a) = 0.8, \quad A(b) = 0.7, \quad A(e) = 0.6, \quad A(d) = 0.5, \quad A(c) = 0.3.$$ 

Then $Q(A; t) = \{a, b, d, e\}$ for $0.3 < t \leq 0.5$. Since $c_{0.48} \in A$ and $b_{0.58} \in A$ but $(cb)_{0.48 \cup 0.58} = d_{0.48} \oplus A$. Hence $Q(A; t)$ is not a bi-ideal of $S$ for all $t \in (0.3, 0.5]$.

**Definition 22** Let $(S, .., \leq)$ be an ordered semigroup and $A$ a fuzzy subset of $S$. Then $A$ is called an $(\varepsilon, \in \cup q)$-fuzzy subsemigroup of $S$ if

$$\forall x, y \in S \ (A(xy) \geq \min\{A(x), A(y), 0.5\}).$$ 

**Definition 23** Let $(S, .., \leq)$ be an ordered semigroup and $A$ a fuzzy subset of $S$. Then $A$ is called an $(\varepsilon, \in \cup q)$-fuzzy left (resp. right) ideal of $S$ if the following conditions are satisfied:

(i) $\forall x, y \in S \ (x \leq y \Rightarrow A(x) \geq \min\{A(y), 0.5\})$.

(ii) $\forall x, y \in S \ (A(xy) \geq \min\{A(y), 0.5\})$ (resp. $A(xy) \geq \min\{A(x), 0.5\}$).

**Definition 24** Let $(S, .., \leq)$ be an ordered semigroup and $A, B$ fuzzy subsets of $S$. Then the $0.5$-product of $A$ and $B$ is defined by:

$$A \circ_{0.5} B(a) := \begin{cases} 
\bigvee_{(y, z) \in X_a} \min\{A(y), B(z), 0.5\} & \text{if } X_a \neq \emptyset \\
0 & \text{if } X_a = \emptyset 
\end{cases}$$ 

Let $A$ and $B$ be $(\varepsilon, \in \cup q)$-fuzzy bi-ideals of $S$. Then

$$\forall x \in S \ ((A \cap_{0.5} B)(x) = \{A(x) \wedge B(x) \wedge 0.5\}).$$
**Proposition 25** If \((S, \leq)\) is an ordered semigroup and \(A, B, C, D\) are fuzzy subsets of \(S\) such that \(A \subseteq C\) and \(B \subseteq D\). Then \(A \circ_{0.5} B \subseteq C \circ_{0.5} D\).

**Proposition 26** If \(A\) and \(B\) are \((\varepsilon, \in \vee q)\)-fuzzy bi-ideals of \(S\) then \(A \cap_{0.5} B\) is an \((\varepsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\).

**Lemma 27** Let \(S\) be an ordered semigroup. Then every one-sided \((\varepsilon, \in \vee q)\)-fuzzy ideal is an \((\varepsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\).

**Proof.** Let \(A\) be an \((\varepsilon, \in \vee q)\)-fuzzy left ideal of \(S\) and \(a, b \in S\). Then
\[
A(ab) = \min\{A(b), 0.5\} \geq \min\{A(a), A(b), 0.5\}.
\]

Hence \(A\) is an \((\varepsilon, \in \vee q)\)-fuzzy subsemigroup of \(S\).

Let \(a, b, c \in S\). Then
\[
A(abc) = A((ab)c) = \min\{A(c), 0.5\} \geq \min\{A(a), A(c), 0.5\}.
\]

Let \(a, b \in S\) be such that \(a \leq b\). Then \(A(a) = \min\{A(b), 0.5\}\), since \(A\) is an \((\varepsilon, \in \vee q)\)-fuzzy left ideal of \(S\). Hence \(A\) is an \((\varepsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\). Similarly we can prove that if \(A\) is an \((\varepsilon, \in \vee q)\)-fuzzy right ideal of \(S\) then \(A\) is an \((\varepsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\).

**Definition 28** An \((\varepsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\) is called idempotent if \(A \circ_{0.5} A = A\).

**Proposition 29** Let \(S\) be an ordered semigroup and \(A\) is an \((\varepsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\). Then \(A \circ_{0.5} A \subseteq A\).

**Proof.** Let \(A\) be an \((\varepsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\). Then for each \(a \in S\), we have
\[
(A \circ_{0.5} A)(a) \subseteq A(a).
\]

In fact: If \(X_a = \emptyset\), then \((A \circ_{0.5} A)(a) = 0 \leq A(a)\). If \(X_a \neq \emptyset\) then
\[
(A \circ_{0.5} A)(a) = \bigvee\limits_{(y,z) \in X_a} \min\{A(y), A(z), 0.5\} \\
\leq \bigvee\limits_{(y,z) \in X_a} A(xy) \leq \bigvee\limits_{(y,z) \in X_a} A(a) = A(a).
\]

For an ordered semigroup the fuzzy subset 0 (resp. 1) is defined as follows:

\begin{align*}
(\forall x \in S)(0 : S \rightarrow [0, 1]|x \rightarrow 0(x) = 0) \\
(\forall x \in S)(1 : S \rightarrow [0, 1]|x \rightarrow 1(x) = 1).
\end{align*}

**Lemma 30** Let \(S\) be an ordered semigroup and \(A, B\) are fuzzy subsets of \(S\). Then \(A \circ_{0.5} B \subseteq 1 \circ_{0.5} B\) (resp. \(A \circ_{0.5} B \subseteq A \circ_{0.5} 1\)).
Proof. Follows from Proposition 24. ■

Proposition 31 Let $S$ be an ordered semigroup and $A$ an $(\leq, \in \vee q)$-fuzzy bi-ideal of $S$. Then $A \circ_{0.5} 1 \circ_{0.5} A \subseteq A$.

Proof. Let $a \in S$. If $X_a = \emptyset$. Then $(A \circ_{0.5} 1 \circ_{0.5} A)(a) = 0 \leq A(a)$. If $X_a \neq \emptyset$, then

\[
(A \circ_{0.5} 1 \circ_{0.5} A)(a) = \bigvee_{(y,z) \in X_a} \min\{A(y), (1 \circ_{0.5} A)(z), 0.5\}
\]

\[
= \bigvee_{(y,z) \in X_a} \min\{A(y), \bigvee_{(t,r) \in X_z} \min\{1(t), A(r), 0.5\}, 0.5\}
\]

\[
= \bigvee_{(y,z) \in X_a} \bigvee_{(t,r) \in X_z} \min\{A(y), 1, A(r), 0.5\}
\]

\[
= \bigvee_{(y,z) \in X_a} \bigvee_{(t,r) \in X_z} \min\{A(y), A(r), 0.5\}.
\]

Since $a \leq yz \leq y(tr)$ and $A$ is an $(\leq, \in \vee q)$-fuzzy bi-ideal of $S$, we have $A(a) \geq \min\{A(y), A(r), 0.5\}$. Thus

\[
\bigvee_{(y,z) \in X_a}(t,r) \in X_z \min\{A(y), A(r), 0.5\} \leq \bigvee_{(y,z) \in X_a}(t,r) \in X_z A(a) = A(a),
\]

consequently, $(A \circ_{0.5} 1 \circ_{0.5} A)(a) \leq A(a)$. ■

Theorem 32 An ordered semigroup $S$ is regular if and only if for every $(\leq, \in \vee q)$-fuzzy bi-ideal $A$ of $S$ we have

\[
A \circ_{0.5} 1 \circ_{0.5} A = A \tag{4.1}
\]

Proof. $\implies$. Let $S$ be a regular ordered semigroup and let $a \in S$. Since $S$ is regular there exists $x \in S$ such that $a \leq axa = ax(axa) = a(xa)$. Then $(a, xaxa) \in X_a$, and we have

\[
(A \circ_{0.5} 1 \circ_{0.5} A)(a) = \bigvee_{(y,z) \in X_a} \min\{A(y), (1 \circ_{0.5} A)(z), 0.5\}
\]

\[
\geq \min\{A(a), (1 \circ_{0.5} A)(xa), 0.5\}
\]

\[
= \min\{A(a), \bigvee_{(t,r) \in X_{xa}} \min\{1(t), A(r), 0.5\}, 0.5\}
\]

\[
\geq \min\{A(a), \min\{1(xax), A(a), 0.5\}, 0.5\}
\]

\[
= \min\{A(a), \min\{1, A(a), 0.5\}, 0.5\}
\]

\[
= \min\{A(a), 0.5\} = A(a).
\]

Hence $A(a) \leq (A \circ_{0.5} 1 \circ_{0.5} A)(a)$. On the other hand, by Proposition 30, we have $(A \circ_{0.5} 1 \circ_{0.5} A)(a) \leq A(a)$. Therefore $(A \circ_{0.5} 1 \circ_{0.5} A)(a) = A(a)$.

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\[
\iffalse

\leq.
\fi
\text{Let } A \text{ be an } (\epsilon, \in \mathcal{V}) \text{-fuzzy bi-ideal of } S \text{ such that (4.1), is satisfied.}
\text{To prove that } S \text{ is regular, we will prove that } (BSB) = B \text{ for all bi-ideals } B \text{ of } S. \text{ Let } b \in B, \text{ the by Proposition 17, } \chi_B \text{ is an } (\epsilon, \in \mathcal{V}) \text{-fuzzy bi-ideal of } S. \text{ By hypothesis}
\[
(\chi_B \circ \chi_B)(b) = \chi_B(b).
\]
Since \( b \in B \), then \( \chi_B(b) = 1 \) and we have \( (\chi_B \circ \chi_B)(b) = 1 \). By Proposition 5 (iii), we have \( \chi_B \circ \chi_B = \chi_{BSB} \) and hence \( \chi_{BSB}(b) = 1 \implies b \in (BSB). \text{ Thus } B \subseteq (BSB). \text{ Since } B \text{ is a bi-ideal of } S, \text{ we } (BSB) \subseteq (B) = B. \text{ Therefore } (BSB) = B. \]

\textbf{Lemma 33} \textit{Let } A \text{ and } B \text{ be } (\epsilon, \in \mathcal{V}) \text{-fuzzy bi-ideals of } S. \text{ Then } A \circ B \text{ is also an } (\epsilon, \in \mathcal{V}) \text{-fuzzy bi-ideal of } S.

\textbf{Proof.} \text{Let } A \text{ and } B \text{ be } (\epsilon, \in \mathcal{V}) \text{-fuzzy bi-ideals of } S. \text{ Let } a \in S. \text{ If } X_a = \emptyset \text{ then}
\[
((A \circ B) \circ (A \circ B))(a) = 0 \leq (A \circ B)(a).
\]
If \( X_a \neq \emptyset \) then
\[
((A \circ B) \circ (A \circ B))(a) = \bigvee_{(y,z) \in X_a} \{(A \circ B)(y) \wedge (A \circ B)(z) \wedge 0.5\}
\]
\[
= \bigvee_{(y,z) \in X_a} \left( \bigvee_{(p_1,q_1) \in X_a} \left( \bigvee_{(p_2,q_2) \in X_a} \left( A(p_1) \wedge B(q_1) \wedge 0.5 \right) \right) \right) \wedge \left( \bigvee_{(p_2,q_2) \in X_a} \left( A(p_2) \wedge B(q_2) \wedge 0.5 \right) \right)
\]
\[
= \bigvee_{(y,z) \in X_a} \left( \bigvee_{(p_1,q_1) \in X_a} \left( \bigvee_{(p_2,q_2) \in X_a} \left( A(p_1) \wedge A(p_2) \wedge B(q_1) \wedge B(q_2) \wedge 0.5 \right) \right) \right)
\]
\[
\leq \bigvee_{(y,z) \in X_a} \left( \bigvee_{(p_1,q_1) \in X_a} \left( \bigvee_{(p_2,q_2) \in X_a} \left( A(p_1) \wedge A(p_2) \wedge B(q_2) \wedge 0.5 \right) \right) \right).
\]
Since \( a \leq yz, y \leq p_1q_1 \) and \( z \leq p_2q_2. \text{ Then } a \leq (p_1,q_1)(p_2,q_2) = p_1(p_2,q_2) \) and we have \((p_1,q_1,p_2,q_2) \in X_a. \text{ Then}
\[
\bigvee_{(y,z) \in X_a} \left( \bigvee_{(p_1,q_1) \in X_a} \left( \bigvee_{(p_2,q_2) \in X_a} \left( A(p_1) \wedge A(p_2) \wedge B(q_2) \wedge 0.5 \right) \right) \right)
\]
\[
\leq \bigvee_{(p_1,q_1,p_2,q_2) \in X_a} \left[ A(p_1) \wedge A(p_2) \wedge B(q_2) \wedge 0.5 \right].
\]
Since \( A \text{ is an } (\epsilon, \in \mathcal{V}) \text{-fuzzy bi-ideal of } S \text{ we have}
\[
A(p_1,q_1,p_2) \geq \{A(p_1) \wedge A(p_2) \wedge 0.5\}.
\]

Then
\[
\bigvee_{(p_1,q_1,p_2,q_2) \in X_a} \left[ \{A(p_1) \wedge A(p_2) \wedge B(q_2) \wedge 0.5\} \right]
\]
\[
\leq \bigvee_{(p_1,q_1,p_2,q_2) \in X_a} \left[ \{A(p_1,q_1) \wedge B(q_2) \wedge 0.5\} \right]
\]
\[
\leq \bigvee_{(p,q) \in X_a} \left[ \{A(p) \wedge B(q) \wedge 0.5\} \right] = (A \circ_{0.5} B)(a).
\]

Therefore \(((A \circ_{0.5} B) \circ_{0.5} (A \circ_{0.5} B))(a) \leq (A \circ_{0.5} B)(a)\), and \(A \circ_{0.5} B\) is an \((\varepsilon, \in \vee q)\)-fuzzy subsemigroup of \(S\). Let \(a, b, c \in S\). Then
\[
(A \circ_{0.5} B)(a) \wedge (A \circ_{0.5} B)(c) = \left[ \bigvee_{(p,q) \in X_a} \{A(p) \wedge B(q) \wedge 0.5\} \right]
\]
\[
\wedge \left[ \bigvee_{(r,s) \in X_c} \{A(r) \wedge B(s) \wedge 0.5\} \right]
\]
\[
= \bigvee_{(p,q) \in X_a} \bigvee_{(r,s) \in X_c} \left[ \{A(p) \wedge B(q) \wedge 0.5\} \wedge \{A(r) \wedge B(s) \wedge 0.5\} \right]
\]
\[
= \bigvee_{(p,q) \in X_a} \bigvee_{(r,s) \in X_c} \left[ \{A(p) \wedge A(r) \wedge B(q)\} \wedge B(s) \wedge 0.5\right]
\]
\[
\leq \bigvee_{(p,q) \in X_a} \bigvee_{(r,s) \in X_c} \left[ \{A(p) \wedge A(r) \wedge B(s) \wedge 0.5\} \right].
\]

Since \(a \leq pq\), and \(c \leq rs\). Then \(abc \leq (pq)(rs) = (pq)(r)s\) and we have \((p(qb)r, s) \in X_{abc}\). Thus
\[
\bigvee_{(p,q) \in X_a} \bigvee_{(r,s) \in X_c} \left[ \{A(p) \wedge A(r) \wedge B(s) \wedge 0.5\} \right]
\]
\[
\leq \bigvee_{(p(qb)r, s) \in X_{abc}} \left[ \{A(p) \wedge A(r) \wedge B(s) \wedge 0.5\} \right].
\]

Since \(A\) is an \((\varepsilon, \in \vee q)\)-fuzzy bi-ideal of \(S\), we have
\[
A(p(qb)r) \supseteq \{A(p) \wedge A(r) \wedge 0.5\}.
\]

Hence
\[
\bigvee_{(p(qb)r, s) \in X_{abc}} \left[ \{A(p) \wedge A(r) \wedge B(s) \wedge 0.5\} \right]
\]
\[
\leq \bigvee_{(p(qb)r, s) \in X_{abc}} \left[ \{A(p(qb)r) \wedge B(s) \wedge 0.5\} \right]
\]
\[
\leq \bigvee_{(x,y) \in X_{abc}} \left[ \{A(x) \wedge B(y) \wedge 0.5\} \right] = (A \circ_{0.5} B)(abc).
\]
Theorem 34

Let \( (A \circ_{0.5} B)(abc) \geq (A \circ_{0.5} B)(a) \wedge (A \circ_{0.5} B)(c) \).

Let \( x, y \in S \) be such that \( x \leq y \). If \( (p, q) \in X_y \) then \( y \leq pq \) and so \( x \leq pq \implies (p, q) \in X_x \implies X_y \subseteq X_x \). If \( X_x = \emptyset \), then \( X_y = \emptyset \) and we have \( (A \circ_{0.5} B)(x) = 0 = (A \circ_{0.5} B)(y) \). If \( X_x \neq \emptyset \), then \( X_y \neq \emptyset \) and we have

\[
(A \circ_{0.5} B)(y) = \bigvee_{(p, q) \in X_y} \{A(p) \wedge B(q) \wedge 0.5\} \\
\leq \bigvee_{(c, d) \in X_x} \{A(c) \wedge B(d) \wedge 0.5\} \\
= (A \circ_{0.5} B)(x).
\]

Therefore \( (A \circ_{0.5} B)(x) \geq (A \circ_{0.5} B)(y) \), consequently \( A \circ_{0.5} B \) is an \( (\varepsilon, \varepsilon \vee q) \)-fuzzy bi-ideal of \( S \). ■

Theorem 34 Let \( S \) be an ordered semigroup. The following are equivalent:

(i) \( S \) is both regular and intra-regular.
(ii) \( A \circ_{0.5} A = A \) for every \( (\varepsilon, \varepsilon \vee q) \)-fuzzy bi-ideal \( A \) of \( S \).
(iii) \( A \cap_{0.5} B = A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A \) for all \( (\varepsilon, \varepsilon \vee q) \)-fuzzy bi-ideals \( A \) and \( B \) of \( S \).

Proof. (i) \( \implies \) (ii). Let \( A \) be an \( (\varepsilon, \varepsilon \vee q) \)-fuzzy bi-ideal of \( S \) and \( a \in S \). Since \( S \) is regular and intra-regular there exists \( x, y, z \in S \) such that \( a \leq axa \leq axza \), and \( a \leq ya^2z \). Then \( a \leq axa \leq ax(ya^2z)za = (axya)(azxa) \) and hence \( (axya, azxa) \in X_a \). Then

\[
(A \circ_{0.5} A)(a) = \bigvee_{(p, q) \in X_a} \{A(p) \wedge A(q) \wedge 0.5\} \\
\geq \{A(axya) \wedge A(azxa) \wedge 0.5\} \\
\geq \{A(a) \wedge A(a) \wedge 0.5\} \cap \{A(a) \wedge A(a) \wedge 0.5\} \wedge 0.5 \\
= \{A(a) \wedge 0.5\} = A(a).
\]

On the other hand, by Proposition 28, we have \( (A \circ_{0.5} A)(a) \leq A(a) \).

(ii) \( \implies \) (iii). Let \( A \) and \( B \) be \( (\varepsilon, \varepsilon \vee q) \)-fuzzy bi-ideals of \( S \). Then \( A \cap_{0.5} B \) is an \( (\varepsilon, \varepsilon \vee q) \)-fuzzy bi-ideal of \( S \). By (ii)

\[
A \cap_{0.5} B = (A \cap_{0.5} B) \circ_{0.5} (A \cap_{0.5} B) \\
\subseteq A \circ_{0.5} B.
\]

Similarly, \( A \cap_{0.5} B \leq B \circ_{0.5} A \). Thus

\[
A \cap_{0.5} B \leq A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A.
\]

On the other hand, \( A \circ_{0.5} B \) and \( B \circ_{0.5} A \) are \( (\varepsilon, \varepsilon \vee q) \)-fuzzy bi-ideals of \( S \) by Lemma 33. Hence \( A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A \) is an \( (\varepsilon, \varepsilon \vee q) \)-fuzzy bi-ideal of \( S \). By
(ii)  
\[ A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A \]
\[ = (A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A) \circ_{0.5} (A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A) \]
\[ \subseteq (A \circ_{0.5} B) \circ_{0.5} (B \circ_{0.5} A) = A \circ_{0.5} (B \circ_{0.5} B) \circ_{0.5} A \]
\[ = A \circ_{0.5} B \circ_{0.5} A \text{ (as } B \circ_{0.5} B = B \text{ by (i) above}) \]
\[ \subseteq A \circ_{0.5} 1 \circ_{0.5} A \]
\[ = A \text{ (as } A \circ_{0.5} 1 \circ_{0.5} A = A \text{ by Theorem 31).} \]

By a similar way we can prove that \( A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A \subseteq B \). Consequently,

\[ A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A \subseteq A \cap_{0.5} B. \]

Therefore \( A \cap_{0.5} B = A \circ_{0.5} B \cap_{0.5} B \circ_{0.5} A \).

(iii) \( \implies \) (i). To prove that \( S \) is regular we prove that \( P \cap Q = (PQ) \cap (QP) \) for every bi-ideal \( P \) and \( Q \) of \( S \). Let \( b \in P \cap Q \). By Proposition 17, \( \chi_P \) and \( \chi_Q \) are \((\in, \vee_q)\)-fuzzy bi-ideals of \( S \). By (iii) \( (\chi_P \cap_{0.5} \chi_Q)(b) = (\chi_P \circ_{0.5} \chi_Q \cap_{0.5} \chi_Q \circ_{0.5} \chi_P)(b) \) since \( b \in P \) and \( b \in Q \), then \( \chi_P(b) = 1 \) and \( \chi_Q(b) = 1 \). Then \( (\chi_P \cap_{0.5} \chi_Q)(b) = 1 \) and hence \( (\chi_P \circ_{0.5} \chi_Q)(b) = 0.5 \). Since \( b \in P \cap Q \), then \( \chi_P(\cap_{0.5} \chi_Q)(b) = 0.5 \) and \( \chi_Q(\cap_{0.5} \chi_P)(b) = 0.5 \) and hence \( \chi_{(PQ) \cap (QP)}(b) = 0.5 \implies b \in (PQ) \cap (QP) \). On the other hand, if \( b \in (PQ) \cap (QP) \), then

\[
1 = \chi_{(PQ) \cap (QP)}(b) = (\chi_P \circ_{0.5} \chi_Q \cap_{0.5} \chi_Q \circ_{0.5} \chi_P)(b) = (\chi_P \cap_{0.5} \chi_Q \circ_{0.5} \chi_P)(b) \text{ (by (iii))} = \chi_{(PQ) \cap (QP)}(b)
\]

hence \( b \in P \cap Q \). Therefore \( P \cap Q = (PQ) \cap (QP) \), consequently, \( S \) is both regular and intra-regular. This completes the proof. \( \blacksquare \)

References


