ON GENERALIZED RICCI-RECURRENT $\delta$-LORENTZIAN TRANS-SASAKIAN MANIFOLDS

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Abstract. In this paper we study generalized Ricci-recurrent trans-Sasakian manifolds. It is proved that a generalized Ricci-recurrent $\delta$-Lorentzian cosymplectic manifold is always recurrent. Generalized Ricci-recurrent $\delta$-Lorentzian trans-Sasakian Manifolds of dimension $\geq 5$ are locally classified. We have also proved that if $M$ is one of the $\delta$-Lorentzian Sasakian, $\delta$-Lorentzian $\alpha$-Sasakian, $\delta$-Lorentzian Kenmotsu or $\delta$-Lorentzian $\beta$-Kenmotsu manifolds which is generalized Ricci-recurrent with cyclic Ricci tensor and non-zero $A(\xi)$ everywhere; then $M$ is an Einstein manifold.

1. Introduction

Many authors recently have studied Lorentzian $\alpha$-Sasakian manifolds [1] and Lorentzian $\beta$-Kenmotsu manifolds [9], [5]. In 2011, S.S.Pujar and V.J. Khairnar [12] have initiated the study of Lorentzian Trans-Sasakian manifolds and studied the basic results with some of its properties. Earlier to this, S. S. Pujar [14] has initiated the study of $\delta$-Lorentzian $\alpha$-Sasakian manifolds [5] and $\delta$-Lorentzian $\beta$-Kenmotsu manifolds [12]

In 2010, S.S. Shukla and D.D.Singh [15] have studied $\epsilon$-trans-Sasakian manifolds and its basic results and using these they deduced some of its interesting properties. Earlier to this in 1969 Takahashi [17] had introduced the notion of almost contact metric manifold equipped with pseudo Riemannian metric. In particular, he studied the Sasakian manifolds equipped with Riemannian metric $g$. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as $\epsilon$-almost contact metric manifolds and $\epsilon$-Sasakian manifolds respectively.

Recently [16] and [10], we have observed that there does not exists a light like surface in the $\epsilon$-Sasakian manifolds . On the other hand in almost para contact manifold defined by Motsumoto [7], the semi Riemannian manifold has the index 1 and the structure vector

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field $\xi$ is always a time like. This motivated the Thripathi and others [17] to introduce $\epsilon$-almost para contact structure where the vector field $\xi$ is space like or time like according as $\epsilon = 1$ or $\epsilon = -1$.

A non-flat Riemannian manifold $M$ is called a generalized Ricci-recurrent manifold [18] if its Ricci tensor $S$ satisfies the condition

$$\left(\nabla_X S\right)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z),$$  \hspace{1cm} (1.1)

where $\nabla$ is Levi-Civita connection of the Riemannian metric $g$, and $A$, $B$ are 1-forms on $M$. In particular, if the 1-form $B$ vanishes identically, then $M$ reduces to the well known Ricci-recurrent manifold [8].

In [16], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing $\xi$ is a constant, say $c$. He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $\mathbb{R} \times fC^n$ if $c < 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [8] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [8]. The paper is organized as follows:

In section 2, we introduce notion of $\delta$-Lorentzian trans-Sasakian manifold with an example and some basic results regarding such type of manifolds are also given. In section 3 for generalized Ricci-recurrent $\delta$-Lorentzian trans-Sasakian manifold the relation between 1 forms $A$ & $B$ is established. It is proved that a generalized Ricci-recurrent $\delta$-Lorentzian cosymplectic manifold is always Ricci-recurrent & generalized Ricci-recurrent $\delta$-Lorentzian trans-Sasakian manifolds of dimension $\geq 5$ are also classified. In the last section, an expression for Ricci tensor of a generalized Ricci-recurrent $\delta$-Lorentzian trans-Sasakian manifold with cyclic Ricci tensor is obtained. It is also proved that if $M$ is one of $\delta$-Lorentzian Sasakian, $\delta$-Lorentzian $\alpha$-Sasakian, $\delta$-Lorentzian Kenmotsu or $\delta$-Lorentzian $\beta$-Kenmotsu manifolds which is generalized Ricci-recurrent manifold with cyclic Ricci tensor and non-zero $A(\xi)$ everywhere, then $M$ is an Einstein manifold.

2. Preliminaries

A $(2n + 1)$ dimensional manifold $M$, is said to be the $\delta$-almost contact metric manifold if it admits a $(1, 1)$ tensor field $\phi$, a structure tensor field $\xi$, a 1-form $\eta$ and an indefinite metric $g$ such that

$$\phi^2 X = X + \eta(X)\xi, \hspace{1cm} \eta(\xi) = -1,$$  \hspace{1cm} (2.1)

$$g(\xi, \xi) = -\delta, \hspace{1cm} \eta(X) = \delta g(X, \xi),$$  \hspace{1cm} (2.2)

$$g(\phi X, \phi Y) = g(X, Y) + \delta \eta(X)\eta(Y)$$

$$g(X, \phi Y) = g(\phi X, Y)$$  \hspace{1cm} (2.3)
for all vector fields $X$ and $Y$ on $M$, where $\delta$ is such that $\delta^2 = 1$ so The above structure $(\phi, \xi, \eta, G, \delta)$ on $M$ is called the the $\delta-$ Lorentzian structure on $M$. If $\delta = 1$ and this is the usual Lorentzian structure [7] on $M$, the vector field $\xi$ is the time like [1], that is $M$ contains a time like vector field.

From the above equations, one can deduce that
$$\phi \xi = 0, \quad \eta(\phi X) = 0 \quad (2.4)$$

**Example 1.** Let us consider the 3-dimensional manifold $M = \{ (x, y, z) \in \mathbb{R}^3 \}$, where $x, y, z$ are the co-ordinates of a point in $\mathbb{R}^3$. Let $\{ e_1, e_2, e_3 \}$ be the global frames on $M$ given by
$$e_1 = e^z (\frac{\partial}{\partial x} + y \frac{\partial}{\partial z}), \quad e_2 = e^z \frac{\partial}{\partial y}, \quad e_3 = e^z \frac{\partial}{\partial z}$$

Let $g$ be the $\delta$- Lorentzian metric on $M$ defined by
$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0$$
and
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -\delta$$
where $\delta = \pm 1$. Then $\delta$-Lorentzian indefinite metric $g$ on $M$ is in the following form:
$$g = \{ e^{-2z} - \delta y^2 \} (dx^2) + e^{-2z} (dy^2) - \delta e^{-2z} (dz^2) + 2\delta ye^{-2z} dxdy$$

Let $e_3 = \xi$. Let $\eta$ be the 1-form defined by
$$\eta(U) = \delta g(U, e_3)$$
for any vector field $U$ on $M$. Let $\phi$ be the $(1, 1)$ tensor field defined by
$$\phi(e_1) = e_2, \phi(e_2) = e_1, \phi(e_3) = 0$$

Then using linearity of $\phi$ and $g$ and taking $e_3 = \xi$, one obtains
$$\phi(e_1) = e_2, \phi(e_2) = e_1, \phi(e_3) = 0$$
and
$$\eta(U) = \delta g(U, \xi)$$

Putting $W = U = \xi$ in both the above equations respectively, we have
$$g(\xi, \xi) = -\delta, \eta(\xi) = -1$$

Clearly from $g(\phi U, \phi W) = g(U, W) + \delta \eta(U) \eta(W)$, $\phi$ is symmetric. Thus $(\phi, \xi, \eta, g, \delta)$ defines $\delta$- Lorentzian contact metric structure on $M$.

A $\delta$- Lorentzian manifold with structure $(\phi, \xi, \eta, g, \delta)$ is said to be $\delta$- Lorentzian trans-Sasakian manifold $M$ of type $(\alpha, \beta)$ if it satisfies the condition
trans-Sasakian manifold of type \((\alpha, \beta)\) \([12]\). \(\delta\)-Lorentzian trans-Sasakian manifold of type \((0, 0), (0, \beta), (\alpha, 0)\) are the Lorentzian cosympletic, Lorentzian \(\beta\)-Kenmotsu and Lorentzian \(\alpha\)-Sasakian manifolds respectively. In particular if \(\alpha = 1, \beta = 0,\) \(\delta = 1,\) then \(\delta\)-Lorentzian trans-Sasakian manifold reduces to \(\delta\)-Lorentzian Sasakian and \(\delta\)-Lorentzian Kenmotsu manifolds respectively.

3. \textbf{Generalized Ricci-recurrent \(\delta\)-Lorentzian Trans-Sasakian Manifolds}

Let \(M\) be a \((2n + 1)\) dimensional \(\delta\)-Lorentzian trans-Sasakian manifold. From (2.5), we have

\[\nabla_X \xi = \delta \{- \alpha \phi X - \beta (X + \eta(X)\xi)\}, \quad (3.1)\]

\[\nabla_X \eta(Y) = \alpha g(\phi X, Y) + \beta \{g(X, Y) + \delta \eta(X)\eta(Y)\} \quad (3.2)\]

From equations (2.5), (3.1), (3.2) we have following lemma.

\textbf{Lemma 3.1.} In a \((2n + 1)\) dimensional \(\delta\)-Lorentzian trans-Sasakian manifold, we have

\[R(X, Y)\xi = (\alpha^2 + \beta^2)\eta(Y)X - \eta(X)Y + 2\alpha \beta \{\eta(Y)\phi X - \eta(X)\phi Y\}
+ \delta \{- (X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X\}, \quad (3.3)\]

\[S(X, \xi) = \{2n(\alpha^2 + \beta^2) - \delta(\xi\beta)\}\eta(X) + (2n - 1)\delta(X\beta)
+ \{2\alpha \beta \eta(X) + \delta(X\alpha)\}f + \delta(\phi X)\alpha \quad (3.4)\]

where \(R\) & \(S\) are curvature and Ricci curvature tensors. In particular, we have,

\[S(\xi, \xi) = -2n(\alpha^2 + \beta^2 - \delta(\xi\beta)) \quad (3.5)\]

\[2\alpha \beta - \delta(\xi\alpha) = 0 \quad (3.6)\]

Now we prove the following

\textbf{Theorem 3.2.} Let \(M\) be a \((2n + 1)\) dimensional generalized Ricci-recurrent \(\delta\)-Lorentzian trans-Sasakian manifold. Then the 1-forms \(A\) & \(B\) are related by

\[\delta B(X) = 2n \{X(\alpha^2 + \beta^2 - \delta(\xi\beta)) - (\alpha^2 + \beta^2 - \delta(\xi\beta))A(X)\}
- 2(2n - 1)\{\alpha \phi X + \beta \phi^2 X\}\beta - 2\{\alpha \phi^2 X + \beta \phi X\}\alpha
- 2\{(\alpha \phi X + \beta \phi^2 X)\alpha\}f \quad (3.7)\]

In particular, we get

\[\delta B(\xi) = 2n \{\xi(\alpha^2 + \beta^2 - \delta(\xi\beta)) - (\alpha^2 + \beta^2 - \delta(\xi\beta))A(\xi)\} \quad (3.8)\]
Proof. Using (1.1) in
\[
(\nabla_X \delta S)(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),
\]
we get
\[
A(X)S(Y, Z) + B(X)g(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).
\]  
(3.10)
Putting \( Y = Z = \xi \), in the above equation we obtain
\[
S(\xi, \xi)A(X) + B(X) = XS(\xi, \xi) - 2S(\nabla_X \xi, \xi),
\]
which in view of (3.5), (2.3) & (3.1) yields (3.7). The equation (3.8) is obvious from (3.7).

Let \( A^* \) & \( B^* \) be the associated vector fields of \( A \) & \( B \), that is,
\[
g(X, A^*) = A(X) \text{ and } g(X, B^*) = B(X).
\]

**Corollary 3.3.** In a \((2n+1)\)-dimensional generalized Ricci-recurrent \( \delta \)-Lorentzian \( \alpha - \)Sasakian (resp. \( \delta \)-Lorentzian \( \alpha \)-Sasakian) manifold, we have
\[
\delta B = -2n\alpha^2 A \quad (\text{resp. } \delta B = -2nA)
\]
(3.12)
Thus, the associated vector fields \( A^* \) & \( B^* \) are in opposite direction if \( \delta = 1 \) that is, structure vector field \( \xi \) is space like.

**Proof.** A \( \delta \)-Lorentzian trans-Sasakian manifold of type \((\alpha, 0)\) is \( \delta \)-Lorentzian \( \alpha \)-Sasakian [12]. In this case \( \alpha \) becomes a constant. If \( \alpha = 1 \), then \( \delta \)-Lorentzian \( \alpha \)-Sasakian manifold is \( \delta \)-Lorentzian Sasakian. Thus, from the equation (3.7), the proof follows immediately.

**Corollary 3.4.** In a \((2n+1)\)-dimensional generalized Ricci-recurrent normal almost \( \delta \)-Lorentzian \( f \)-structure (or \( f \)-Kenmotsu) manifold we have
\[
\delta B(X) = 2n\{X(f^2 - \delta(\xi f)) - (f^2 - \delta(\xi f))A(X)\} - 2(n - 1)f(\phi^2 X)f.
\]
(3.13)

**Proof.** A \( \delta \)-Lorentzian trans-Sasakian structure with \( \alpha = 0 \) and \( \beta = f \) is a normal almost cosymplectic \( \delta \)-Lorentzian \( f \)-structure [12] (or \( \delta \)-Lorentzian \( f \)-Kenmotsu structure [12]). Thus, putting \( \alpha = 0 \) and \( \beta = f \) in the equation (3.7), we get (3.13).

**Corollary 3.5.** For a \((2n+1)\)-dimensional generalized Ricci-recurrent \( \delta \)-Lorentzian \( \beta \)-Kenmotsu (resp. \( \delta \)-Lorentzian Kenmotsu) manifold, we have
\[
\delta B = -2n\beta^2 A \quad (\text{resp. } \delta B = -2nA)
\]
(3.14)
Thus, the associated vector fields \( A^* \) and \( B^* \) are in same direction if \( \delta = -1 \), that is structure vector field \( \xi \) is time like.

**Proof.** A \( \delta \)-Lorentzian trans-Sasakian structure is \( \delta \)-Lorentzian \( \beta \)-Kenmotsu [12] if \( \alpha = 0 \) and \( \beta = \) constant. In particular, \( 1 \)-Kenmotsu structure is a Kenmotsu structure. Putting \( f = \beta = \) constant (resp. \( f = 1 \)) in (3.13), we obtain (3.14).

A \( \delta \)-Lorentzian trans-Sasakian structures of type \((0, 0)\) is cosymplectic [12]. Thus, putting \( \alpha = 0 = \beta \) in (3.7), we get \( B = 0 \). Hence, we have the following theorem:
Theorem 3.6. A generalized Ricci-recurrent $\delta$-Lorentzian cosymplectic manifold $M$ is always Ricci-recurrent.

Now for a generalized Ricci-recurrent $\delta$-Lorentzian trans-Sasakian manifold of dimension $\geq 5$ locally, we give the following classification.

Theorem 3.7. Let $M$ be a generalized Ricci-recurrent $\delta$-Lorentzian trans-Sasakian manifold of dimension $(2n + 1) \geq 5$. Then

1. either $M$ is Ricci-recurrent,
2. or $\delta B + 2n\alpha^2 A = 0$,
3. or $\delta B + 2n\beta^2 A = 0$,

where $\alpha$ & $\beta$ are non-zero constants.

Proof. We know that locally a $\delta$-Lorentzian trans-Sasakian manifold of dimension $\geq 5$ is either $\delta$-Lorentzian cosymplectic, or $\delta$-Lorentzian $\alpha$-Sasakian or $\delta$-Lorentzian $\beta$-Kenmotsu manifold [12]. Hence, in view of Corollaries 1, 3 and Theorem 2, the proof is complete. $\square$

4. Generalized Ricci-recurrent $\delta$-Lorentzian trans-Sasakian manifolds with cyclic Ricci tensor

A Riemannian manifold is said to admit cyclic Ricci tensor if

\[(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0 \quad (4.1)\]

Now we prove the following:

Theorem 4.1. In a $(2n + 1)$-dimensional generalized Ricci-recurrent $\delta$-Lorentzian trans-Sasakian manifold with cyclic Ricci tensor, the Ricci tensor satisfies

\[
\delta A(\xi)S(X, Y) = 2n\{(\alpha^2 + \beta^2 - \delta(\xi\beta))A(\xi) - \xi(\alpha^2 + \beta^2 - \delta(\xi\beta))\}g(X, Y) \\
- (2n - 1)\delta\{A(X)Y\beta + A(Y)X\beta\} \\
- (2n - 1)\delta(\xi\beta)\{\eta(Y)A(X) + \eta(X)A(Y)\} \\
- \delta\{A(X)(\phi Y)\alpha + A(Y)(\phi X)\alpha\} \\
- 2\alpha\beta\{\eta(Y)A(X) + \eta(X)A(Y)\}f \\
- \delta\{A(X)Y\alpha + A(Y)X\alpha\}f \\
- 2n(\eta(X)Y(\alpha^2 + \beta^2 - \delta(\xi\beta) + \eta(Y)X(\alpha^2 + \beta^2 - \delta(\xi\beta)) \\
+ 2(2n - 1)(\eta(X)(\alpha\phi Y + \beta\phi^2 Y)\beta + \eta(Y)(\alpha\phi X + \beta\phi^2 X)\beta) \\
+ 2\eta(X)(\alpha\phi^2 Y + \beta\phi Y)\alpha + \eta(Y)(\alpha\phi^2 X + \beta\phi X)\alpha) \\
+ 2\{(\alpha\phi X + \beta\phi^2 X)\alpha\}f\eta(Y) + 2\{(\alpha\phi Y + \beta\phi^2 Y)\alpha\}f\eta(X) \\
\]
Proof. Suppose that $M$ is a generalized Ricci symmetric manifold admitting cyclic Ricci tensor. Then in view of (1.1) and (4.1), we get,

$$
\begin{align*}
0 &= A(X)S(Y, Z) + B(X)g(Y, Z) + A(Y)S(Z, X) \\
&\quad + B(Y)g(Z, X) + A(Z)S(X, Y) + B(Z)g(X, Y)
\end{align*}
$$

(4.3)

Put $Z = \xi$ in the above equation, we get

$$
\begin{align*}
A(\xi)S(X, Y) &= -B(\xi)g(X, Y) - A(\xi)S(Y, \xi) - A(Y)S(X, \xi) \\
&\quad - B(X)g(Y, \xi) + B(Y)S(X, \xi)
\end{align*}
$$

(4.4)

Using (3.8) & (3.4) in (4.4), we get (4.2).

\[\square\]

Corollary 4.2. For a $(2n + 1)$-dimensional generalized Ricci-recurrent manifold $M$ with cyclic Ricci tensor, we have the following results:

1. If $M$ is an $\delta$–Lorentzian $\alpha$–Sasakian manifold, then

$$
\delta A(\xi)S(X, Y) = 2n\alpha^2 A(\xi)g(X, Y)
$$

2. If $M$ is an $\delta$–Lorentzian Sasakian manifold, then

$$
\delta A(\xi)S(X, Y) = 2nA(\xi)g(X, Y)
$$

3. If $M$ is a $\delta$–Lorentzian $f$–Kenmotsu manifold, then

$$
\begin{align*}
\delta A(\xi)S(X, Y) &= 2n\{A(\xi)(f^2 - \delta(\xi f)) - \xi(f^2 - \delta(\xi f))\} g(X, Y) \\
&\quad - (2n - 1)\delta \{A(X)Y f + A(Y)X f\} \\
&\quad - (2n - 1)\delta(\xi f)\{\eta(Y)A(X) + \eta(X)A(Y)\} \\
&\quad - 2n\{\eta(X)(f^2 - \delta(\xi f)) + \eta(Y)(f^2 - \delta(\xi f))\} \\
&\quad + 2(2n - 1)\{\eta(X)(f\phi^2 Y f + \eta(Y)(f\phi^2)X f\}
\end{align*}
$$

4. If $M$ is a $\delta$–Lorentzian $\beta$–Kenmotsu manifold, then

$$
\delta A(\xi)S(X, Y) = 2n\beta^2 A(\xi)g(X, Y)
$$

5. If $M$ is a $\delta$–Lorentzian Kenmotsu manifold, then

$$
\delta A(\xi)S(X, Y) = 2nA(\xi)g(X, Y)
$$

6. If $M$ is a $\delta$–Lorentzian cosymplectic manifold, then

$$
\delta A(\xi)S(X, Y) = 0
$$

Since $\delta \neq 0$, we have

$$
A(\xi)S(X, Y) = 0
$$

As we know that a Riemannian manifold is Einstein if

$$
S(X, Y) = \rho g(X, Y).
$$

Therefore, in view of corollary (4), we have the following theorem;

Theorem 4.3. Let $M$ be generalized Ricci-recurrent manifold with cyclic Ricci tensor. If $M$ is one of $\delta$–Lorentzian $\alpha$–Sasakian, $\delta$–Lorentzian Sasakian, $\delta$–Lorentzian Kenmotsu and $\delta$–Lorentzian $\beta$–Kenmotsu manifolds with non-zero $A(\xi)$ everywhere, then $M$ is Einstein.
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References


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