ON WEAK SYMMETRIES OF $\delta$-LORENTZIAN $\beta$-KENMOTSU MANIFOLD

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ABSTRACT. The purpose of this paper is to study weakly symmetric and weakly Ricci symmetric $\delta$-Lorentzian $\beta$-Kenmotsu Manifolds. We prove that the sum of the associated 1-forms of weakly symmetric $\delta$-Lorentzian $\beta$-Kenmotsu Manifold and weakly Ricci symmetric $\delta$-Lorentzian $\beta$-Kenmotsu Manifold is nonzero everywhere provided that nonvanishing $\xi$-sectional curvature. The existence of $\delta$-Lorentzian $\beta$-Kenmotsu Manifold is ensured by an example.

1. Introduction

In the year 1987, Chaki [4] establish the proper generalization of pseudosymmetric manifolds. Furthermore, in 1989, Tamassy and Binh [11] introduced the notion of weakly symmetric manifolds. A non-flat Riemannian manifold $(M^n, g)$ $(n > 2)$ is called weakly symmetric if its curvature tensor $\bar{R}$ of the type $(0, 4)$ satisfies the condition

$$\nabla_X \bar{R}(Y, Z, U, V) = A(X) \bar{R}(Y, Z, U, V) + B(Y) \bar{R}(X, Z, U, V) + C(Z) \bar{R}(Y, X, U, V) + D(U) \bar{R}(Y, Z, X, V) + E(V) \bar{R}(Y, Z, U, X)$$

for all vector fields $X, Y, Z, U, V \in X(M^n)$, $A$, $B$, $C$, $D$ and $E$ are 1-forms (not simultaneously zero) and $\nabla$ denotes the operator of covariant differentiation with respect to the Riemannian metric $g$. The 1-Forms are called the associated 1-forms of the manifold and $n$-dimensional manifold of this kind is denoted by $(WS)_n$. If in (1.1) 1-form $A$ is replaced by $2A$ and $E$ is replaced by $A$, then a $(WS)_n$ reduces to the notion of generalized pseudosymmetric manifold by Chaki [5]. Furthermore, in 1999, De and Bandyopadhyay [7] studied a $(WS)_n$ and provided that in such manifold the associated 1-form $B = C$ and $D = E$ and hence the equation (1.1)
reduces as follows
\[ \nabla_X \bar{R}(Y, Z, U, V) = A(X) \bar{R}(Y, Z, U, V) + B(Y) \bar{R}(X, Z, U, V) + B(Z) \bar{R}(Y, X, U, V) + D(U) \bar{R}(Y, Z, X, V) + D(V) \bar{R}(Y, Z, U, X) \] (1.2)

Thereafter, in the year 1993, Tamassy and Binh [12] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold \((M^n, g)(n > 2)\) is called weakly symmetric if its curvature tensor \(\bar{R}\) of the type \((0, 2)\) is not identically zero satisfies the condition
\[ \nabla_X S(Y, Z) = A(X) S(Y, Z) + B(Y) S(X, Z) + C(Z) S(Y, X) \] (1.3)
where \(A, B, C\), are three nonzero 1-forms called the associated 1-forms of the manifold and \(\nabla\) denotes the operator of covariant differentiation with respect to the metric \(g\) and this type of \(n\)-dimensional manifold is denoted by \((WRS)_n\).

As an equivalent notion of \((WRS)_n\), Chaki and Koley [6] introduce the notion of generalized pseudo Ricci symmetric manifold. If in the equation (1.3) the 1-form \(A\) is replaced by \(2A\), then a \((WRS)_n\) reduces to the notion of generalized pseudo Ricci symmetric manifold by Chaki and Koley. Now, if \(A = B = C = 0\) then \((WRS)_n\) reduces to Ricci symmetric manifold and if \(B = C = 0\) then it reduces to Ricci recurrent manifold.

At the same time, in the year 1969, Takahashi [13] has introduced the Sasakian manifolds with Pseudo-Riemannian metric and prove that one can study the Lorentzian Sasakian structure with an indefinite metric. Furthermore, in 1990, K. L. Duggal [8] has initiated the space time manifolds with contact structure and analyzed the paper of Takahashi. T. [13]. In 2009, S. Y. Perktas, E. Kilic, M. M. Tripathi [15] have studied the various properties of Lorentzian \(\beta\)-Kenmotsu manifolds and S.S. Pujar [10] have introduced the notion of \(\delta\) Lorentzian \(\beta\)-Kenmotsu manifolds and its properties. Inspired by these papers and some other papers (see the exhaustive list [1, 9, 11, 14]) we have studied on weak symmetres of \(\delta\) Lorentzian \(\beta\)-Kenmotsu manifolds. In section 2, we consider the \((2n + 1)\) dimensional differentiale manifold \(M\) with Lorentzian almost contact metric structure with indefinite metric \(g\). This section deals with preliminaries of \(\delta\) Lorentzian \(\beta\)-Kenmotsu manifolds. In section 3 of the paper it is proved that the sum of the associated 1-forms of a weakly symmetric \(\delta\) Lorentzian \(\beta\)-Kenmotsu manifolds of non-vanishing \(\xi\)-sectional curvature is nonzero everywhere and hence such a structure exists. In section 4 we study weakly Ricci symmetric \(\delta\)-Lorentzian \(\beta\)-Kenmotsu manifolds and prove that in such a structure, with non-vanishing \(\xi\)-sectional curvature, the sum of the associated 1-forms is non-vanishing everywhere and consequently such a structure exists. Finally section 5 deals with a concrete example of \(\delta\) Lorentzian \(\beta\)-Kenmotsu manifolds.

2. \(\delta\)-Lorentzian \(\beta\)-Kenmotsu manifold

In this section we study \(\delta\)-Lorentzian - \(\beta\)-Kenmotsu manifold. For the manifold almost-Lorentzian contact, we have
\[ \phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1, \eta(X) = g(X, \xi) \]
where \(\phi\) is a tensor field of type \((1, 1)\) and \(\xi\) is a characteristic vector field and \(\eta\) is the 1-form. Therefore, from these conditions one can reduce that \(\phi(\xi) = 0\), \(\eta(\phi(X)) = \)
0 for any vector field $X$ on $M$. It is well known that the Lorentzian contact metric structure [2] or Lorentzian Kenmotsu structure [11] satisfies
\[(\nabla_X \phi)Y = g(\phi(X), Y) + \eta(Y)\phi(X)\]
for any $C^\infty$ vector field $X$ and $Y$ on $M$. More generally, one has the notion of Lorentzian $\beta$-Kenmotsu structure [9] which may be defined by the requirement
\[(\nabla_X \phi)Y = \beta [g(\phi(X), Y) + \eta(Y)\phi(X)] \quad (2.1)\]
for any $C^\infty$ vector field $X$ and $Y$ on $M$ and $\beta$ is a nonzero constant on $M$. Using the equation (2.1), one can reduce the Lorentzian $\beta$-Kenmotsu manifold.
\[(\nabla_X \xi) = \beta[X + \eta(X)]\xi\quad \text{and} \quad (\nabla_X \eta) = \beta [g(X, Y) + \eta(X)\eta(Y)].\]

At this stage, S.S Pujar [10] introducing the notion of $\delta$- Lorentzian $\beta$-Kenmotsu manifold in the following definition.

**Definition 2.1.** A differentiable manifold $M$ of dimension $(2n + 1)$ is called a $\delta$-Lorentzian manifold, if it admits as a one-one tensor field $\phi$ a contravariant vector field $\xi$, a covariant vector field $\eta$ and an indefinite metric $g$ which satisfy

(i) $\phi^2 X = X + \eta(X)\xi$, $\eta(\xi) = -1$, $\eta(\phi(X)) = 0$

(ii) $g(\xi, \xi) = -\delta$, $\eta(X) = \delta g(X, \xi)$

(iii) $g(\phi X, \phi Y) = g(x, Y) + \delta \eta(X)\eta(Y)$

where $\delta$ is such that $\delta^2 = 1$ and for any vector field $X$, $Y$ on $M$. The structure defined above is called a $\delta$-Lorentzian almost contact metric structure. Manifold $M$ together with the structure $(\phi, \xi, \eta, g, \delta)$ is also called a $\delta$ Lorentzian kenmotsu manifold if

\[(\nabla \phi)(Y) = g(\phi(X), Y)\xi + \delta \eta(Y) \phi(X)\]

more generally, S. S. Pujar introduce the definition.

**Definition 2.2.** A $\delta$-Lorentzian almost contact metric manifold $M$ $(\phi, \xi, \eta, g, \delta)$ is called a Lorentzian $\beta$-kenmotsu manifold if

\[(\nabla \phi)(Y) = \beta \{g(\phi(X), Y)\xi + \delta \eta(Y) \phi(X)\} \quad (2.2)\]

where $\nabla$ is the Levi-Civita connection with respect to $g$. $\beta$ is a smooth function on $M$ and $X, Y$ are vector fields on $M$ and $\delta$ is such that $\delta^2 = 1$ or $\delta = \pm 1$. If $\delta = 1$, then $\delta$-Lorentzian $\beta$-kenmotsu manifold is usual Lorentzian $\beta$-kenmotsu manifold and is called the time like manifold. In this case $\xi$ is called a time like vector field. From (2.2) it follows that

\[\nabla X \xi = \delta \beta \{X + \eta(X)\xi\} \quad (2.3)\]

\[\nabla \eta Y = \beta \{g(X, Y) + \delta \eta(X)\eta(Y)\} \quad (2.4)\]

\[R(X, Y) \xi = \beta^2 \{\eta(X)Y - \eta(X)Y\} + \delta \{(X\beta)\phi^2 Y - (Y\beta)\phi^2 X\} \quad (2.5)\]

\[R(\xi, Y) \xi = \{\beta^2 + \delta (\xi\beta)\} \phi^2 Y, R(\xi, \xi) \xi = 0 \quad (2.6)\]
Theorem 3.2. In a weakly symmetric following. then the manifold is of vanishing $\xi$ provided that $\beta$ is nonzero everywhere.

Definition 3.1. A $\delta$-Lorentzian $\beta$-Kenmotsu manifold $M^{2n+1}, g$ ($n > 1$) is said to be weakly symmetric if its Riemannian curvature tensor $\bar{R}$ of a type (0, 4) satisfies (1.2). Let $e_i : i = 1, 2, ..., (2n + 1)$ be an orthonormal basis of the tangent space $T_p(M)$ at any point $P$ of the manifold. After, setting $Y = V = e_i$ in equation (1.2) and taking summation over $i$, 1 $\leq i \leq 2n + 1$, we get


(3.1)

Now, putting $X = Z = U = \xi$, in equation (3.1) and using (2.5) and (2.9), we get

$$A(\xi) + B(\xi) + D(\xi) = \frac{2\beta(\xi) + \delta(\xi)}{\beta^2 + \delta(\xi)}$$

(3.2)

provided that $\beta^2 + \delta(\xi) \neq 0$. The $\xi$-sectional curvature $K(\xi, X)$ of a $\delta$-Lorentzian $\beta$-Kenmotsu manifold for a unit vector field $X$ orthogonal to $\xi$ is given by $K(\xi, X) = g(R(\xi, X)\xi, X)$. Hence equation (2.6) yields $K(\xi, X) = \beta^2 + \delta(\xi)$. If $\beta^2 + \delta(\xi) = 0$, then the manifold is of vanishing $\xi$-sectional curvature. Hence we can state the following.

Theorem 3.2. In a weakly symmetric $\delta$-Lorentzian $\beta$-Kenmotsu manifold $(M^{2n+1}, g)$ ($n > 1$) of non-vanishing $\xi$-sectional curvature, relation (3.2) holds.

Next, substituting $X$ and $Z$ by $\xi$ in equation (3.1) and then using (2.9) we obtain

$$\nabla_\xi S)(\xi, U) = [A(\xi) + B(\xi)]S(\xi, U) + [\beta^2 + \delta(\xi)](-2n + 1)D(U) + \eta(U)D(\xi)$$

(3.3)
Again, we have
\[
(\nabla_\xi S)(\xi, U) = \nabla_\xi S(\xi, U) - S(\nabla_\xi U) - S(\xi, \nabla_\xi U)
= \nabla_\xi S(\xi, U) - S(\xi, \nabla_\xi U) \quad \text{(using equation (2.8))}
\]
\[
= [4n(\delta(\xi \beta))]\eta(U) - (2n - 1)\delta U(\xi \beta) + \delta U(\xi \beta)
\]
(3.4)

From equations (3.2), (3.3) and (3.4), we get
\[
D(U) = \frac{[4n\beta(\xi \beta) + \delta(\xi \beta)]\eta(U)}{(-2n + 1)(\beta^2 + \delta(\xi \beta))} - \frac{(2n - 1)\delta U(\xi \beta)}{(-2n + 1)((\beta^2 + \delta(\xi \beta))}
\]
\[
+ D(\xi) \left[ \frac{(2n - 1)[\beta^2\eta(U) - \delta(U \beta)]}{(-2n + 1)(\beta^2 + \delta(\xi \beta))} \right]
\]
\[
- \left[ \frac{2\beta(\xi \beta) + \delta(\xi \beta)}{(2n + 1)(\beta^2 + \delta(\xi \beta))} \right] [2n\beta^2\eta(U) - (2n - 1)\delta(U \beta) + \delta U(\xi \beta)]
\]
(3.5)

for any vector field \(U\), provided that \(\beta^2 + \delta(\xi \beta) \neq 0\). Next, setting \(X = U = \xi\) in equation (3.1) and proceeding in a similar manner as above we get
\[
B(Z) = \frac{[4n\beta(\xi \beta) + \delta(\xi \beta)]\eta(Z)}{(-2n + 1)(\beta^2 + \delta(\xi \beta))} - \frac{(2n - 1)\delta Z(\xi \beta)}{(-2n + 1)((\beta^2 + \delta(\xi \beta))}
\]
\[
+ D(\xi) \left[ \frac{(2n - 1)[\beta^2\eta(Z) - \delta(Z \beta)]}{(-2n + 1)(\beta^2 + \delta(\xi \beta))} \right]
\]
\[
- \left[ \frac{2\beta(\xi \beta) + \delta(\xi \beta)}{(2n + 1)(\beta^2 + \delta(\xi \beta))} \right] [2n\beta^2\eta(Z) - (2n - 1)\delta(Z \beta) + \delta U(\xi \beta)]
\]
(3.6)

for any vector field \(Z\), provided that \(\beta^2 + \delta(\xi \beta) \neq 0\). This leads to the following:

**Theorem 3.3.** In a weakly symmetric \(\delta\)-Lorentzian \(\beta\)-Kenmotsu manifold \((M^{2n+1}, g)\) \((n > 1)\) of non-vanishing \(\xi\)-sectional curvature, the associated 1-forms \(D\) and \(B\) are given by relation (3.5) and (3.6), respectively.

Again, setting \(Z = U = \xi\) in equation (3.1) we get
\[
(\nabla_X S)(\xi, \xi) = A(X)S(\xi, \xi) + [B(\xi) + D(\xi)]S(X, \xi)
+ B(R(X, \xi)\xi) + D(R(X, \xi)\xi)
\]
\[
= -2n(\beta^2 + \delta(\xi \beta))A(X) + [B(\xi) + D(\xi)]S(X, \xi)
- (\beta^2 + \delta(\xi \beta))\eta(X)B(\xi) + D(\xi) + B(X) + D(X)
\]
(3.7)

Now we have
\[
(\nabla_X S)(\xi, \xi) = \nabla_X S(\xi, \xi) - 2S(\nabla_X \xi, \xi),
\]
which yields by using equations (2.3) and (2.8), that
\[
(\nabla_X S)(\xi, \xi) = -2\beta(X \beta) - 2n\delta(X \xi \beta).
\]
(3.8)
In view of equations (3.5), (3.6), (3.7) and (3.8) yields
\[ A(X) + B(X) + D(X) = \frac{2n\delta X(\xi\beta)}{\beta^2 + \delta(\xi\beta)} \]
\[ - \frac{4n\beta(\xi\beta) + \delta\xi(\xi\beta)}{2n(\beta^2 + \delta(\xi\beta))} \eta(X) \]
\[ + \left\{ \frac{(2n-1)\delta X(\xi\beta) + \beta(X\beta)}{2n(\beta^2 + \delta(\xi\beta))} \right\} \]
\[ + \left[ \frac{2\beta(\xi\beta) + \delta\xi(\xi\beta)}{2n(\beta^2 + \delta(\xi\beta))^2} \right] [2n\beta^2\eta(X) - (2n-1)\delta(X\beta) + \delta\eta(X)(\xi\beta)] \]
(3.9)

for any vector field \( X \), provided that \( \beta^2 + \delta(\xi\beta) \neq 0 \). This leads to the following:

**Theorem 3.4.** In a weakly symmetric \( \delta \)- Lorentzian \( \beta \)-kenmotsu manifold \( (M^{(2n+1)}, g) \) \( (n > 1) \) of non-vanishing \( \xi \)- sectional curvature, the sum of the associated 1-forms is given by relation (3.9).

In particular, if \( \phi(\text{grad}\alpha) = \text{grad}\beta \) then \( (\xi\beta) = 0 \) and hence relation (3.9) to the following form
\[ A(X) + B(X) + D(X) = \frac{\beta(X\beta)}{n\beta^2} \]
(3.10)
for any vector field \( X \), provided that \( \beta^2 \neq 0 \).

**Corollary 3.5.** If a weakly symmetric \( \beta \neq 0 \), \( \delta \)- Lorentzian \( \beta \)-kenmotsu manifold \( (M^{(2n+1)}, g) \) \( (n > 1) \) satisfies the condition \( \phi(\text{grad}\alpha) = \text{grad}\beta \), then the sum of the associated 1-forms is given by relation (3.10).

If \( \beta = 1 \) then equation (3.9) yields
\[ A(X) + B(X) + D(X) = \frac{2n\delta X(\xi)}{1 + \delta(\xi)} \]
\[ - \frac{4n(\xi) + \delta\xi(\xi)}{2n(1 + \delta(\xi))} \eta(X) \]
\[ + \left\{ \frac{(2n-1)\delta X(\xi) + X}{2n(1 + \delta(\xi))} \right\} \]
\[ + \left[ \frac{2(\xi) + \delta\xi(\xi)}{2n(1 + \delta(\xi))^2} \right] [2n\eta(X) - (2n-1)\delta(X) + \delta\eta(X)(\xi)] \]
(3.11)

**Corollary 3.6.** There is no weakly symmetric \( \delta \)- Lorentzian \( \beta \)-kenmotsu manifold \( (M^{(2n+1)}, g) \) \( (n > 1) \), unless the sum of the associated 1-forms is given by relation (3.11).

If \( \beta = 0 \), then (3.9) yields
\[ A(X) + B(X) + D(X) = 0 \]
(3.12)
for all \( X \). This leads to the following:

**Corollary 3.7.** There is no weakly symmetric cosympletic \( \delta \)- Lorentzian \( \beta \)-kenmotsu manifold \( (M^{(2n+1)}, g) \) \( (n > 1) \), unless the sum of the associated 1-forms is everywhere zero.
In the next section, we prove the sum of the associated 1-forms Weakly Ricci Symmetric \( \delta \)-Lorentzian \( \beta \)-Kenmotsu manifold of non-vanishing \( \xi \)-sectional curvature is nonzero everywhere.

4. **Weakly Ricci Symmetric \( \delta \)-Lorentzian \( \beta \)-Kenmotsu manifolds**

**Definition 4.1.** A \( \delta \)-Lorentzian \( \beta \)-Kenmotsu manifold \( (M^{2n+1}, g) \) \((n \geq 1)\) is said to be weakly Ricci symmetric if its Ricci tensor of type \((0, 2)\) is not identically zero and satisfies relation \[(1.3)\].

**Theorem 4.2.** In a weakly Ricci symmetric \( \delta \)-Lorentzian \( \beta \)-Kenmotsu manifold \( (M^{2n+1}, g) \) \((n \geq 1)\) of non-vanishing \( \xi \)-sectional curvature, the following relations hold:

\[
A(\xi) + B(\xi) + C(\xi) = \frac{2\beta(\xi\beta) + \delta(\xi\beta)}{\beta^2 + \delta(\xi\beta)} \tag{4.1}
\]

\[
[r - 2n\beta^2 - \delta(\xi\beta)] [A(\xi) + B(\xi)] = \frac{r(3\beta(\xi\beta) + \delta(\xi\beta) + \delta\beta^3)}{\beta^2 + \delta(\xi\beta)}
- (6n + (2n + 1)\delta - 1)\beta(\xi\beta) - 2n(2n + 1)\beta^3
+ (2n - 1)\delta[\text{div}(\text{grad} \beta) - (\rho_1 \beta) - (\rho_2 \beta)] \tag{4.2}
\]

where \( r \) is the scalar curvature of the manifold, \( \text{div} \) denotes the divergence, \( \rho_1, \rho_2 \) being the associated vector fields corresponding to the 1-form \( A \) and \( B \), respectively.

**Proof.** From equation \[(1.3)\] it follows that

\[
(\nabla_X S)(Y, \xi) = A(X)S(Y, \xi) + B(Y)S(X, \xi) + C(\xi)S(Y, X) \tag{4.3}
\]

In view of \[(2.8)\] we obtain from \[(4.3)\]

\[
A(X)[2n\beta^2\eta(Y) - (2n - 1)\delta(Y\beta) + \delta\eta(Y)(\xi\beta)]
+ B(Y)[2n\beta^2\eta(X) - (2n - 1)\delta(X\beta) + \delta\eta(X)(\xi\beta)] + C(\xi)S(Y, X)
= 4n\beta(X\beta)\eta(Y) - (2n - 1)X(Y\beta)\delta + \delta X(\xi\beta)\eta(Y) + [2n\beta^3 + \delta(\xi\beta)]g(X, Y)
+ (2n - 1)[(\nabla_X Y)\beta + \beta(Y\beta)\eta(X)] - \delta\beta S(Y, X) \tag{4.4}
\]

where \[(2.9)\] has been used. Setting \( X = Y = \xi \) in \[(4.4)\] and then using \[(2.9)\] we obtain relation \[(4.1)\]. Let \( e_i, i = 1, 2,..., (2n + 1) \) be an orthonormal basis of the tangent space \( T_p M \) at any point of the manifold, then setting \( X = Y = e_i \) in \[(4.4)\] and taking summation over \( i, 1 \leq i \leq 2n + 1 \) and then using \[(2.8)\] we obtain

\[
[A(\xi) + B(\xi)][2n\beta^2 + \delta(\xi\beta)] - (2n - 1)\delta[(\rho_1 \beta) + (\rho_2 \beta)] + rC(\xi)
= (6n + (2n + 1)\delta - 1)\beta(\xi\beta) + \delta(\xi\beta) + 2n(2n + 1)\beta^3
- (2n - 1)\delta[\text{div}(\text{grad} \beta)\delta - \delta r] \tag{4.5}
\]

where \( r = \sum_{i=1}^{2n+1} S(e_i, e_i) \) eliminating \( C(\xi) \) from \[(4.1)\] and \[(4.5)\] we obtain \[(4.2)\]. This proves the theorem. \( \square \)
5. Example of $\delta$-Lorentzian $\beta$-Kenmotsu manifolds

We consider the 3-dim. manifold $M = (x, y, z) \in \mathbb{R}^3 : Z \neq 0$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Let $e_1, e_2, e_3$ be a linearly independent global frame on $M$ given by

$$e_1 = e^{-z} \frac{\partial}{\partial y}, \quad e_2 = e^{-z} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_3 = \beta \frac{\partial}{\partial z}$$

Let $g$ be the an indefinite metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -\delta$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0$$

and the $\delta$-Lorentzian metric $g$ is thus given by

$$g = g_{11}(dx)^2 + g_{22}(dy)^2 + g_{33}(dz)^2 + 2g_{12}dx \wedge dy$$
$$= 2e^{2z}(dx)^2 + e^{2z}(dy)^2 - \frac{\delta}{\beta^2}(dz)^2 - 2e^{2z}dx \wedge dy$$

$$\begin{pmatrix}
  2e^{2z} & -2e^{2z} & 0 \\
 -e^{2z} & e^{2z} & 0 \\
 0 & 0 & \frac{\delta}{\beta^2}
\end{pmatrix}$$

where $\delta = \pm 1$. If $\delta = -1$, then $\delta$-Lorentzian metric $g$ becomes a Riemannian positive definite metric on $M$ so that in this case the characteristic vector field $\xi$ becomes aspace like and if $\delta = 1$, then it becomes a light like. Let $\eta$ be the 1-form defined by

$$\eta(X) = \delta g(X, \xi)$$

for any vector field $X$ on $M^3$. Let $\phi$ be the tensor field of type $(1, 1)$ defined by

$$\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0$$

using the linearity property of $g$ and $\phi$, one can deduce

$$\phi^2X = X + \eta(X)\xi, \quad \eta(X) = -1, \quad g(\xi, \xi) = -\delta$$
$$g(\phi X, \phi Y) = g(X, Y) + \delta \eta(X)\eta(Y).$$

Also, $\eta(e_1) = 0, \eta(e_2) = 0, \eta(e_3) = -1$ for any vector field $X$ and $Y$ on $M$. Let $\nabla$ be the Levi-Civita connection with respect to $g$. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \delta \beta e_1, \quad [e_2, e_3] = \delta \beta e_2$$

Using Koszule’s formula for Levi-Civita connection $\nabla$ with respect to $g$, that is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$$
one can easily calculate
\[ \nabla e_1 e_3 = \delta \beta e_1, \quad \nabla e_3 e_3 = 0, \quad \nabla e_2 e_3 = \delta \beta e_2 \]
\[ \nabla e_2 e_2 = -\delta \beta e_3, \quad \nabla e_1 e_2 = 0, \quad \nabla e_2 e_1 = 0 \]

with these information the structure \((\phi, \xi, \eta, g, \delta)\) satisfies (2.2) and (2.3). Hence \(M^3(\phi, \xi, \eta, g, \delta)\) defines a \(\delta\)-Lorentzian \(\beta\)-Kenmotsu manifold.

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