THE KUNZE-STEIN PHENOMENON ASSOCIATED WITH JACOBI-DUNKL CONVOLUTION

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Abstract. The main purpose of this paper is to establish the endpoint estimate for the Kunze-Stein phenomenon in Lorentz spaces associated with Jacobi-Dunkl convolution.

1. Introduction

Let $\mu$ denote the Haar measure on locally compact group $G$. The convolution of two compactly supported continuous functions $f$ and $g$ is the function $f \ast g$ on $G$ defined by setting

$$f \ast g(x) = \int_G f(y)g(y^{-1}x)d\mu(y), \quad \forall x \in G.$$\hspace{1em}

In 1960, R.A. Kunze and E.M. Stein proved in [8] that, if $G$ is $SL(2, \mathbb{R})$, then the continuous inclusion

$$L^p(G) \ast L^2(G) \subseteq L^2(G)$$

holds whenever $1 \leq p < 2$. A group satisfying (1) for any $1 \leq p < 2$ is called the Kunze-Stein phenomenon group and for such a group (1) is called the Kunze-Stein phenomenon for $G$. In 1978 M. Cowling proved in [2] that every connected real semisimple Lie group with finite center is a Kunze-Stein phenomenon group. After twenty years Cowling, S. Meda and A.G. Setti proved that if $G$ is the group of isometries of a homogeneous tree $H$ (see [4]) or a semisimple Lie group of real rank 1 (see [3]), then $G$ satisfies a more accurate version of the Kunze-Stein phenomenon. By using the Lorentz spaces $L^{p,q}$, they proved that for $1 < p < 2$, the continuous inclusion

$$L^{p,q_1}(G) \ast L^{p,q_2}(G) \subseteq L^{p,q}(G)$$

holds whenever $1 \leq p < 2$.
holds if and only if \( \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2} - 1 \). In 2000, A.D. Ionescu [6] studied the validity of (2) when \( p \to 2 \). He proved that
\[
L^{2,1}(G) \ast L^{2,1}(G) \subseteq L^{2,\infty}(G),
\]
when \( G \) is a semisimple Lie group of rank one. In this paper we establish the endpoint estimate for the Kunze-Stein phenomenon associated with Jacobi-Dunkl convolution similar to (3) and (2).

2. Preliminaries

In this section we recapitulate some results about harmonic analysis associated to Jacobi-Dunkl operators and the Lorentz spaces. For details the reader is referred to ([1], [5]).

Let \( \alpha \geq \beta \geq -\frac{1}{2} \), we consider the Jacobi-Dunkl operator \( \Lambda_{\alpha,\beta} \) defined on \( E(R) \) the space of \( C^\infty \)-functions on \( R \) by:
\[
\Lambda_{\alpha,\beta}(f)(x) = f'(x) + ((2\alpha + 1) \coth(x) + (2\beta + 1) \tanh(x))\left(\frac{f(x) - f(-x)}{2}\right).
\]
For \( \lambda \in \mathbb{C} \), the initial problem
\[
\Lambda_{\alpha,\beta}(f)(x) = i\lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R},
\]
has a unique solution \( \Psi_{\alpha,\beta}^\lambda \) (called the Jacobi-Dunkl kernel) given by
\[
\Psi_{\alpha,\beta}^\lambda(x) = \varphi_{\mu}(x) + \frac{i\lambda}{2(\alpha + 1)} \sinh(x) \cosh(x) \varphi_{\mu+1,\beta+1}(x),
\]
where
\[
\lambda^2 = \mu^2 + \rho^2 \quad \text{with} \quad \rho = \alpha + \beta + 1
\]
and
\[
\varphi_{\mu}(x) = \binom{2F_1}{\beta+i\mu}{\beta-i\mu}{\alpha+1; -\sinh^2 x},
\]
here \( 2F_1 \) denotes the Gaussian hypergeometric function.

**Notation.** For all \( x, y, z \in \mathbb{R} \) and \( \chi \in [0, \pi] \), we put:
- \( \sigma_{x,y,z}^\chi = \begin{cases} \frac{-\cosh z \cos \chi - \cosh x \cos y}{\sinh x \sinh y}, & \text{if } xy \neq 0 \\ 0, & \text{otherwise} \end{cases} \)
- \( g(x,y,z) = 1 - \sigma_{x,y,z}^\chi + \sigma_{z,y,x}^\chi + \sigma_{z,x,y}^\chi \)
- \( I_{x,y} = \left[ -|x| - |y|, -|x| - |y| \right] \cup \left[ |x| - |y|, |x| + |y| \right] \)
- \( K_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta}(\sinh|x| \sinh|y| \sinh|z|)^{-2\alpha - 1} I_{x,y}(z) \times \int_0^\pi g(x,y,z)(g(x,y,z,\chi))^{\alpha-\beta-1} \sin^{2\beta} \chi d\chi \)

where \( g(x,y,z,\chi) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + \cosh x \cosh y \cosh z \cos \chi \). Here
\[
t_+ = \begin{cases} t, & \text{if } t > 0 \\ 0, & \text{otherwise} \end{cases}
\]
and
\[
M_{\alpha,\beta} = \frac{2^{-2\beta} \Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha - \beta) \Gamma(\beta + \frac{3}{2})}.
\]
Proposition 2.1. For \( z \in I_{x,y} \), there exist positive constants \( C \) such that

\[
|K_{\alpha,\beta}(x, y, z)| \leq Ce^{-\rho(|x|+|y|+|z|)} \left( \frac{(1+|x|)(1+|y|)(1+|z|)}{|xyz|} \right)^{2\alpha} \times \left( \frac{(|x|+|y|+|z|)(|x|+|y|)(|z|+|x|+|y|)(|z|-|x|+|y|)}{(1+|x|+|y|+|z|)(1+|x|+|y|+|z|)(|z|-|x|+|y|)(1+|z|+|x|+|y|)} \right)^{\alpha-\frac{1}{2}}
\]

Proof. For \( x, y \in \mathbb{R} \setminus \{0\} \), we have

\[
|K_{\alpha,\beta}(x, y, z)| \leq M_{\alpha,\beta}(\sinh |x| \sinh |y| \sinh |z|)^{-2\alpha} 1_{I_{x,y}}(z)
\]

\[
\times \int_{\Delta_{x,y}} |g^\chi(x, y, z)|(g(x, y, z, \chi))^\alpha-\beta-1 \sin^{2\beta} \chi d\chi,
\]

where \( \Delta_{x,y} = \{ \chi \in [0, \pi] : g(x, y, z, \chi) > 0 \} \), we have

\[
\chi \in \Delta_{x,y} \Leftrightarrow \cosh z \cos \chi > \frac{\cosh^2 x \cosh^2 y \sinh^2 z}{2 \cosh x \cosh y},
\]

then

\[
|\sigma^\chi_{x,y,z}| \leq 1.
\]

Using the fact that

\[
z \in I_{x,y} \Leftrightarrow x \in I_{z,x} \Leftrightarrow y \in I_{z,x},
\]

we get \(|\sigma^\chi_{x,z,y}| \leq 1\) and \(|\sigma^\chi_{y,z,x}| \leq 1\). Then we deduce

\[
|K_{\alpha,\beta}(x, y, z)| \leq M_{\alpha,\beta}(\sinh |x| \sinh |y| \sinh |z|)^{-2\alpha} 1_{I_{x,y}}(z)
\]

\[
\times \int_{\Delta_{x,y}} (g(x, y, z, \chi))^\alpha-\beta-1 \sin^{2\beta} \chi d\chi.
\]

From [7], we get

\[
|K_{\alpha,\beta}(x, y, z)| \leq 2 W_{\alpha,\beta}(|x|, |y|, |z|), \quad |z| \in [\max(|x|, |y|), |x|+|y|],
\]

where \( W_{\alpha,\beta} \) is the Jacobi kernel given by

\[
W_{\alpha,\beta}(|x|, |y|, |z|) = \frac{2^{-2\alpha} \Gamma(\alpha+1)(\cosh x \cosh y \cosh z)^{\alpha-\beta-1}}{\sqrt{\pi} \Gamma(\alpha+1/2)(\sinh |x| \sinh |y| \sinh |z|)^{2\alpha}}
\]

\[
\times \left( 1 - B^2 \right)^{\alpha-\frac{1}{2}} F_1(\alpha+\beta, \alpha-\beta; \alpha+1; \frac{1-B}{2})
\]

and

\[
B = \frac{(\cosh x)^2 + (\cosh y)^2 + (\cosh z)^2 - 1}{2 \cosh x \cosh y \cosh z}.
\]

We have

\[
1-B^2 = \frac{\sinh(|x|+|y|+|z|) \sinh(|x|+|y|+|z|) \sinh(|x|+|y|+|z|) \sinh(|x|+|y|+|z|) \sinh(|x|+|y|+|z|)}{4(\cosh x \cosh y \cosh z)^2}.
\]

Using the fact that for \( t \geq 0 \), \( \sinh t \sim \frac{1}{1+t} e^t \) and \( \cosh t \sim e^t \) we obtained the result. □
The Jacobi-Dunkl translation operator \( \tau^x_{\alpha,\beta}, x \in \mathbb{R} \) is defined for a continuous function \( f \) on \( \mathbb{R} \), by

\[
\tau^x_{\alpha,\beta} f(y) = \int_{\mathbb{R}} f(z) d\nu^\alpha_{x,y}(z), \quad y \in \mathbb{R},
\]

where \( \nu^\alpha_{x,y} \) are signed measures given by

\[
d\nu^\alpha_{x,y}(z) = \begin{cases} 
K_{\alpha,\beta}(x, y, z)d\mu_{\alpha,\beta}(z), & \text{if } x, y \in \mathbb{R}\setminus\{0\} \\
\partial\delta_x(z), & \text{if } y = 0 \\
\partial\delta_y(z), & \text{if } x = 0.
\end{cases}
\]

Let \( d\mu_{\alpha,\beta}(x) := A_{\alpha,\beta}(x)dx \), where \( A_{\alpha,\beta}(x) = 2^{2\rho}(\sinh |x|)^{2\alpha+1}(\cosh x)^{2\beta+1} \). We denote by \( L^p(\mu_{\alpha,\beta}), p \in [1, \infty) \), the Lebesgue space on \( \mathbb{R} \) with respect to the measure \( \mu_{\alpha,\beta} \). In the following we use the shorter notation \( \|f\|_{p,A} \) instead of \( \|f\|_{L^p(\mu_{\alpha,\beta})} \).

For all \( x \in \mathbb{R} \) and \( f \in L^q(\mu_{\alpha,\beta}), q \in [1, \infty) \), we have \( \|\tau^x_{\alpha,\beta} f\|_{q,A} \leq 4 \|f\|_{q,A} \).

Let \( p, q, r \in [1, \infty] \) such that \( 1/p + 1/q = 1/r + 1 \). The convolution product of \( f \in L^p(\mu_{\alpha,\beta}) \) and \( g \in L^q(\mu_{\alpha,\beta}) \) is defined by

\[
f *_{\alpha,\beta} g(x) = \int_{\mathbb{R}} \tau^x_{\alpha,\beta}(f)(-y) g(y) d\mu_{\alpha,\beta}(y), \quad a.e. \ x
\]

and we have

\[
\|f *_{\alpha,\beta} g\|_{r,A} \leq 4 \|f\|_{p,A} \|g\|_{q,A}.
\]

Now we recall first basic definitions about the Lorentz spaces.

Let \( f \) be a measurable function defined on \( (\mathbb{R}, \mu_{\alpha,\beta}) \). We assume the function \( f \) to be finite almost everywhere and for \( y > 0 \), \( \mu_{\alpha,\beta}(E_y) < \infty \), where

\[
E_y = \{x \in \mathbb{R} : |f(x)| > y\}.
\]

The distribution function of \( f \) is defined by \( \lambda_f(y) = \mu_{\alpha,\beta}(E_y), y > 0 \) and the (nonnegative) rearrangement of \( f \) is defined by

\[
f^*(t) = \inf \{y > 0 : \lambda_f(y) \leq t\} = \sup \{y > 0 : \lambda_f(y) > t\}, \quad t > 0.
\]

The Lorentz space \( L^{(p,q)}(\mathbb{R}, \mu_{\alpha,\beta}) \) (shortly \( L^{(p,q)}(\mu_{\alpha,\beta}) \)) is defined to be vector space of all (equivalence classes) of measurable functions such that \( \|f\|^{(p,q)}_{A} < \infty \) where

\[
\|f\|^{(p,q)}_{A} = \left( \frac{2}{p} \int_{0}^{\infty} \left[ t^{\frac{1}{p}} f^*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad 0 < p, q < \infty,
\]

\[
\|f\|^{(p,\infty)}_{A} = \sup_{t>0} t^{\frac{1}{p}} f^*(t), \quad 0 < p \leq \infty.
\]

It is known that \( \|f\|^{(p)}_{A} = \|f\|_A \) and so \( L^{(p,p)}(\mu_{\alpha,\beta}) = L^p(\mu_{\alpha,\beta}) \) and if \( 0 < q_1 \leq q_2 \leq \infty, \) \( 0 < p < \infty \) then \( \|f\|^{(p,q_1)}_{A} \leq \|f\|^{(p,q_2)}_{A} \) holds and hence \( L^{(p,q)}(\mu_{\alpha,\beta}) \subseteq L^{(p,q)}(\mu_{\alpha,\beta}) \).

Notice that if \( \chi_E \) is the characteristic function of a measurable set \( E \subset \mathbb{R} \), we have

\[
\|\chi_E\|_2^{(2,q)} = \mu_{\alpha,\beta}(E)^{\frac{1}{2}} = \|\chi_E\|_{2,A}.
\]

The following lemma is shown in [6]:

**Lemma 2.2.** If \( \delta \neq 0 \) and \( d\mu_1(t) = e^{\delta t} dt, d\mu_2(t) = e^{2\delta t} dt \) are tow measures on \( \mathbb{R} \), then

\[
\|f\|_{L^1(\mathbb{R}, \mu_1)} \leq C \|f\|_{L^{(2,1)}(\mathbb{R}, \mu_2)}.
\]
3. The Kunze-Stein phenomenon

The main purpose of this paper is to prove the following endpoint estimate.

**Theorem 3.1.** The convolution operator defined by (5) satisfies
\[ L^{(2,1)}(\mu_{\alpha,\beta}) *_{\alpha,\beta} L^{(2,1)}(\mu_{\alpha,\beta}) \subseteq L^{(2,\infty)}(\mu_{\alpha,\beta}). \]

**Proof.** In view of the general theory of Lorentz spaces, it suffices to prove that
\[
\left| \int_{\mathbb{R}} (f *_{\alpha,\beta} g)(z) h(z) d\mu_{\alpha,\beta}(z) \right| \leq C \|f\|_{2,1}^{*} \|g\|_{2,1}^{*} \|h\|_{2,1}^{*}, \quad A
\]
whenever \( f, g \) and \( h \) are characteristic functions of open sets with finite measure.

First suppose that one function of \( f, g \) or \( h \) is supported in \([-1, 1]\), say \( f \), we have
\[
\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) \leq \|f *_{\alpha,\beta} g\|_{2,A} \|h\|_{2,A}.
\]
Using (6) and (7), we get (8).

Assume now \( f, g \) and \( h \) are characteristic functions of open sets with finite measure, we can write \( f = f_{0} + f_{1} \), \( g = g_{0} + g_{1} \) and \( h = h_{0} + h_{1} \), where \( f_{0}, g_{0} \) and \( h_{0} \) are supported in \([-1, 1]\) and \( f_{1}, g_{1} \) and \( h_{1} \) are supported in \((-\infty, -1] \cup [1, \infty)\). By the first step, it suffices to prove (8) for \( f = f_{1}, g = g_{1} \) and \( h = h_{1} \).

If \( \alpha \geq \frac{1}{2} \), from (5) and (4) we have
\[
\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \tau_{\alpha,\beta}^{*} g(-x) \|h(z) d\mu_{\alpha,\beta}(x) d\mu_{\alpha,\beta}(z)
\]
Using the fact \( K_{\alpha,\beta}(z, -x, y) = K_{\alpha,\beta}(-x, z, y) = K_{\alpha,\beta}(x, y, z) \), we obtain
\[
\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K_{\alpha,\beta}(x, y, z) \|f(x) g(y) h(z) d\mu_{\alpha,\beta}(x) d\mu_{\alpha,\beta}(y) d\mu_{\alpha,\beta}(z)
\]
For \(|x|, |y|, |z| \geq 1\), By Proposition 2.1, there exists a constant \( C \) such that
\[
|K_{\alpha,\beta}(x, y, z)| \leq Ce^{-\rho(|x|+|y|+|z|)}.
\]
Using the fact that for \( t \geq 0 \), \( A_{\alpha,\beta}(t) = 2^{2\rho}(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1} \leq Ce^{2\rho t} \), we get
\[
\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K_{\alpha,\beta}(x, y, z) \|f(x) g(y) h(z) d\mu_{\alpha,\beta}(x) d\mu_{\alpha,\beta}(y) d\mu_{\alpha,\beta}(z)
\]
\[
\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K_{\alpha,\beta}(x, y, z) \|f(x) g(y) h(z) A_{\alpha,\beta}(x) A_{\alpha,\beta}(y) A_{\alpha,\beta}(z) dx dy dz
\]
\[
\leq C \int_{|x| \geq 1} \int_{|y| \geq 1} \int_{|z| \geq 1} f(x) g(y) h(z) e^{\rho(|x|+|y|+|z|)} dx dy dz.
\]
Using Lemma 2.2 we get (8).
If $-\frac{1}{2} < \alpha < \frac{1}{2}$, we only need to prove (8) in which the integral is taken over the domain $D = \{(x, y, z) : |x| \leq |y| \leq |z| \leq |x| + |y|, |x|, |y|, |z| \geq 1\}$. By proposition 2.1, for all $(x, y, z) \in D$, we have

$$|K_{\alpha, \beta}(x, y, z)|A_{\alpha, \beta}(x)A_{\alpha, \beta}(y)A_{\alpha, \beta}(z) \leq Ce^{\rho(|x|+|y|+|z|)}\left(\frac{|x| + |y| - |z|}{1 + |x| + |y| - |z|}\right)^{\alpha - \frac{1}{2}}$$

If $|x| + |y| - |z| \geq 1$, by proceeding as in the analysis of the case $\alpha \geq \frac{1}{2}$, we get (8). If $|x| + |y| - |z| \leq 1$, it suffices to prove (8) for $(x, y, z) \in D_1 = \{(x, y, z) \in D, y \geq 1\}$, for this, make the change of variable $y = u - |x|$, we get

$$\int_{(x, y, z) \in D_1} f(x)g(y)h(z)e^{\rho(|x|+|y|+|z|)}(|x| + y - |z|)^{\alpha - \frac{1}{2}}dxdydz$$

$$\leq \int_{|z| \geq 1} \left(\int_1^\infty \left[\int_{1 \leq |u| \leq |x| \leq |z|} f(x)e^{\rho|x|}g(u-|x|)e^{\rho(u-|x|)}dx\right] (u-|z|)^{\alpha - \frac{1}{2}}du\right)h(z)e^{\rho|z|}dz.$$ 

By Holder’s inequality, we get

$$\int_{1 \leq |x| \leq |u| - |x| \leq z} f(x)g(u - |x|)e^{\rho|x|}e^{\rho(u-|x|)}dx$$

is bounded by $\|f\|_2,A\|g\|_2,A$ uniformly in $u$. Then

$$\int_{(x, y, z) \in D_1} f(x)g(y)h(z)e^{\rho(|x|+|y|+|z|)}(|x| + y - |z|)^{\alpha - \frac{1}{2}}dxdydz$$

$$\leq C\|f\|_2,A\|g\|_2,A\int_1^\infty \left(\int_{|z|}^{|z|+1} (u-|z|)^{\alpha - \frac{1}{2}}du\right)h(z)e^{\rho|z|}dz,$$

since $\alpha - \frac{1}{2} > -1$, then

$$\int_{(x, y, z) \in D_1} f(x)g(y)h(z)e^{\rho(|x|+|y|+|z|)}(|x| + y - |z|)^{\alpha - \frac{1}{2}}dxdydz \leq C\|f\|_{(2,1),A}\|g\|_{(2,1),A}\|h\|_{(2,1),A}.$$

This completes the proof of the theorem. \(\square\)

From the last Theorem and the bilinear interpolation theorem (see [3], Theorem 1.2), we deduce the following result:

**Theorem 3.2.** Let $1 < p < 2$ and $(q_1, q_1, q_2) \in [1, \infty]^3$ such that $1 + \frac{1}{q_1} = \frac{1}{q_1} + \frac{1}{q_2}$, then

$$L^{(p, q_1)}(\mu_{\alpha, \beta}) \ast_{\alpha, \beta} L^{(p, q_2)}(\mu_{\alpha, \beta}) \subseteq L^{(p, q)}(\mu_{\alpha, \beta}).$$

**References**


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